ON THE MINIMAL NUMBER OF CRITICAL POINTS OF FUNCTIONS ON $h$-COBORDISMS

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Abstract. Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism (i.e., the Whitehead torsion of $(W, M_0)$ is non-zero) with $W$ compact, connected and $\dim W \geq 6$. We prove that every smooth function $f : W \to [0, 1]$, $f(M_0) = 0$, $f(M_1) = 1$ has at least 2 critical points. This estimate is sharp: $W$ possesses a function as above with precisely two critical points.

Introduction

Let $(W, M_0, M_1)$ be an $h$-cobordism, [3]. Here $W$ is always assumed to be smooth, connected and compact and $M_i, i = 0, 1$ is always assumed to be closed. Recall that an $h$-cobordism $(W, M_0, M_1)$ is called trivial if there is a diffeomorphism $(W, M_0, M_1) \cong (M \times [0, 1], M_0, M_0)$. We say that a function (not necessarily Morse) $f : W \to [0, 1]$ is regular if $f^{-1}(M_0) = 0$, $f^{-1}(M_1) = 1$ and both values 0 and 1 are regular values of $f$. It is well known that an $h$-cobordism $(W, M_0, M_1)$ is trivial if and only if $W$ possesses a regular function without critical points. In this note we prove the following theorem.

Theorem. Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism with $\dim W \geq 6$. Then every regular function on $W$ has at least two critical points. Moreover, this estimate is sharp: $W$ possesses a regular function with precisely two critical points.

We denote by $I$ the closed interval $[0, 1]$.

1. Preliminaries

Let $f : W \to I$ be a regular Morse function on an $h$-cobordism $(W, M_0, M_1)$. Choose a Riemannian metric on $W$ and consider integral trajectories for the vector field $-\nabla f$, the so-called anti-gradient trajectories. We say that an anti-gradient trajectory $a = a(t)$ is a special trajectory from $p$ to $q$ if $\lim_{t \to -\infty} a(t) = p$ and $\lim_{t \to +\infty} a(t) = q$ where $p$ and $q$ are critical points of $f$ such that the index of $p$ is one more than the index of $q$. We can and shall assume that the number of special trajectories is finite (this is true for generic function and metric).

For every critical point of $f$ we fix orientations of unstable disks (left-hand disks in terminology of [3]). Then every unstable sphere (in a certain level)
gets an orientation. Moreover, every stable sphere gets a coorientation, i.e., an orientation of its normal bundle in the corresponding level set. Now, for every special trajectory \( a \) from \( p \) to \( q \) we define the number \( \varepsilon(a) = \pm 1 \) as follows. Take \( c \in ]f(q), f(p)[ \). Then our trajectory \( a \) meets the level \( f^{-1}(c) \) in a certain point \( x \), which is a point of transversal intersection of the corresponding stable and unstable spheres. We define \( \varepsilon(a) \) to be the intersection index at \( x \).

2. Whitehead torsion

Given a ring \( R \), we define a based \( R \)-module to be a free finite generated left \( R \)-module \( M \) with a fixed \( R \)-free basis.

Recall the definition of the Whitehead torsion of an \( h \)-cobordism \((W, M_0, M_1)\). Given a group \( \pi \), let \( A = A(\pi) \) denote the set of long exact sequences

\[
\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0
\]

such that each \( C_i \) is a based \( \mathbb{Z}[\pi] \)-module and all but finite number of modules \( C_i \) are zero modules. Furthermore, each \( \partial_i \) is a \( \mathbb{Z}[\pi] \)-module homomorphism. Let us call the exact sequence of based \( \mathbb{Z}[\pi] \)-modules trivial if it has only two non-zero terms and the corresponding isomorphism is given by the identity matrix. The term-wise direct sum operation converts \( A \) into an abelian semigroup. Let \( R \) be the equivalence relation on \( A \) generated by the following operations:

- interchanging of the elements;
- replacement of a basis element by the sum of this element with the multiple of another basis element;
- addition of the trivial exact sequence;
- multiplication of any basis element by the element \( \pm g, g \in \pi \).

The above mentioned operation in \( A \) induces a group structure in \( A/R \). This groups is called the Whitehead group of \( \pi \) and is denoted by \( \text{Wh}(\pi) \), [4]. It turns out to be that \( \text{Wh}(\pi) \) is a functor of \( \pi \). In particular, every homomorphism \( \varphi : \pi \rightarrow G \) induces a homomorphism \( \text{Wh}(\varphi) : \text{Wh}(\pi) \rightarrow \text{Wh}(G) \). Namely, the homomorphism \( \varphi \) yields the homomorphism \( \mathbb{Z}[\varphi] : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[G] \) of group rings, which turns \( \mathbb{Z}[G] \) into the right \( \mathbb{Z}[\pi] \)-module \( \mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \). Now, for every based \( \mathbb{Z}[\pi] \)-module \( C \) we can form the based \( \mathbb{Z}[G] \)-module \( \mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C \). The sequence \( \{ \mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C_n \} \) turns out to be exact because all the \( C_n \)'s are free, etc.

For every \( h \)-cobordism \((W, M_0, M_1)\) with \( \pi_1(W) = \pi \) the Whitehead torsion \( \tau(W, M_0, M_1) \in \text{Wh}(\pi) \) is defined as follows. Consider a regular Morse function \( f : W \rightarrow I \), Riemannian metric, etc. as in §1. Fix a point \( x_0 \in W \) and, for every critical point \( p \) of \( f \), choose a path \( u(p) \) from \( x_0 \) to \( p \). Every special trajectory \( a \) from \( p \) to \( q \) gives us a map \( a : \mathbb{R} \rightarrow W \) which is well defined up to shift of \( t \in \mathbb{R} \). We define a path \( v = v_a : I \rightarrow W \) as follows. Let \( \lambda(t) : ]0,1[ \rightarrow \mathbb{R} \) be a function such that

\[
\lim \lambda(t)_{t \rightarrow 0} = -\infty, \quad \lim \lambda(t)_{t \rightarrow 1} = +\infty.
\]
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We set \( v(0) = p, \ v(1) = q, \ v(t) = a(\lambda(t)) \). Now, consider the loop \( u(p) \circ v_0(u(q))^{-1} \) (where \( \circ \) denotes the product of paths) and define \( g(a) \in \pi = \pi_1(W) \) as the based homotopy class of the loop constructed.

Let \( p_1, \ldots, p_k \) be all the critical points of the index \( n \). Define \( C_n \) to be the free \( \mathbb{Z}[\pi] \)-module generated by symbols \( [p_1], \ldots, [p_k] \). In other words, \( C_n \) consists of formal linear combinations

\[
\sum_{i=1}^{k} \alpha_i [p_i], \quad \alpha_i \in \mathbb{Z}[\pi].
\]

We define the differential \( \partial_n : C_n \to C_{n-1} \) to be a \( \mathbb{Z}[\pi] \)-module homomorphism such that

\[
\partial_n[p] = \sum_{q} \sum_{a \in T(p,q)} \varepsilon(a) g(a)[q]
\]

where \( q \) runs over all critical points of the index \( n - 1 \) and \( T(p,q) \) is the set of special trajectories from \( p \) to \( q \).

It follows from the Morse theory that \( H_*(\{C_n, \partial_n\}) = H_*(\widetilde{W}, \widetilde{M}_0) \) where \( (\widetilde{W}, \widetilde{M}_0) \) is the universal covering of the pair \( (W, M_0) \). Since \( M_0 \) is a deformation retract of \( W \), we conclude that \( \widetilde{M}_0 \) is a deformation retract of \( \widetilde{W} \), and therefore the complex \( \{C_n, \partial_n\} \) is acyclic, i.e. the sequence

\[
\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0
\]

is exact. Thus, the above sequence determines a certain element \( \tau = \tau(W, M_0) \in \text{Wh}(\pi) \), the so-called Whitehead torsion of the h-cobordism \( (W, M_0, M_1) \).

According to well-known Barden–Mazur–Stallings Theorem, \( [1, 2, 5] \), an h-cobordism \( (W, M_0, M_1) \) with \( \text{dim} \ W \geq 6 \) is trivial if and only if \( \tau(W, M_0) = 0 \).

2.1. Lemma. Suppose that an h-cobordism \( (W, M_0, M_1) \) possesses a regular Morse function \( f \) such that all the critical points and special trajectories of \( f \) are contained in a simply connected domain \( U \) of \( W \). Then \( \tau(W, M_0) = 0 \).

Proof. Since \( \tau(W, M_0) \) does not depend on the choice of the based point \( x_0 \) and the paths \( u(p) \), we can assume that \( x_0 \in U \) and every path \( u(p) \) belongs to \( U \). Then, for every special trajectory \( a, g(a) \) is the neutral element of \( \pi = \pi_1(W) \). Thus,

\[
\tau(W, M_0) \in \text{Im} \{ \text{Wh}(j) : \text{Wh}\{e\} \to \text{Wh}(\pi) \}
\]

where \( j : \{e\} \to \pi \) is the inclusion of the trivial subgroup. But it follows from the elementary linear algebra that \( \text{Wh}\{e\} = 0 \), see e.g. \([4]\). Thus, \( \tau(W, M_0) = 0 \).

3. Proof of the theorem

Let \( f : M \to \mathbb{R} \) be a smooth function (not necessarily Morse) on a Riemannian manifold \( M \). Let \( U \) be an open ball in \( M \) and suppose that \( U \) contains precisely one critical point \( o \).
3.1. Lemma. There exists a regular Morse function $g$ which is $C^\infty$-close to $f$ in the Whitney topology and such that every special $g$-trajectory is contained in $U$ whenever its ends are contained in $U$.

Proof. Let $D(r) = \{ m \in M \mid d(m, o) < r \}$ where $d$ is the distance function on $M$. We can and shall assume that the injectivity radius at $o$ is at least one and that $U = D(1)$. Then there are positive constants $C$ and $E$ such that, for every function $g$ which is $C^\infty$-close to $f$, the following estimates hold in $D(1) \setminus D(1/2)$:

$$|\text{grad } g| \geq E, \quad |L_{\text{grad } g} d(m, o)| \leq C.$$ 

Choose a function $g$ close to $f$. Let $p$ and $q$ be two critical points of $g$ which belong to $U$. Suppose that there is a special trajectory $a(t)$ from $p$ to $q$ which meets the boundary of $D(3/4)$. We claim that in this case

$$g(p) - g(q) \geq \frac{E^2}{4C}.$$ 

Indeed, since $L_{\text{grad } g} d(m, o)| \leq C$, we conclude that

$$a \left[ t - \frac{1}{4C}, t + \frac{1}{4C} \right]$$

does not meet $D(1/2)$ whenever $a(t) \notin D(3/4)$. So, if $a(t_0) \notin D(3/4)$ then

$$g(p) - g(q) \geq \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} |\text{grad } g|^2 dt \geq \frac{E^2}{4C}.$$ 

Now we can finish the proof as follows. Since $f$ has only one critical point, there exists $g$ close to $f$ and such that $g(p) - g(q)$ is small enough for all critical points $p$ and $q$ of $g$. This is a contradiction. \hfill \Box

3.2. Corollary. If an $h$-cobordism $(W, M_0, M_1)$ possesses a regular function $f$ with one critical point $p$, then $\tau(W, M_0) = 0$. In particular, if $\dim W \geq 6$ then the $h$-cobordism is trivial.

Proof. Because of Lemma 3.1, we can perturb the function $f$ in a small neighborhood of the critical point and get a function $f_1$ such that all its critical points and special trajectories belong to a disk neighborhood of $p$. Now the result follows from Lemma 2.1. \hfill \Box

3.3. Proposition. Every $h$-cobordism $(W, M_0, M_1)$, $\dim W \geq 6$ possesses a regular function with at most 2 critical points.

Proof. Consider a regular Morse function $f : W \to I$. Asserting as in [1, Lemme 1] and [3, §4], we can modify $f$ and to get a regular Morse function which has at most two critical levels $a, b$, $a < b$ and index of each of critical points is equal to 2 or 3. Because of this, every critical level is path connected. Now, following [6, Th. 2.7 and Prop.2.9], we can contract the critical points in each of the levels and get a regular function with at most 2 critical points. \hfill \Box

Clearly, Corollary 3.2 and Proposition 3.3 together imply the Theorem.
3.4. Remarks. 1. Asserting as in 3.2, one can show that, for every regular function \( f \) on a non-trivial \( h \)-cobordism, the number of critical levels of \( f \) is at least 2 provided that all the critical points of \( f \) are isolated.

2. Every \( h \)-cobordism \((W,M_0,M_1)\) possesses a regular function with 1 critical level. Namely, choose collars of the boundary components and define \( f \) to be constant on complements of collars and depending on the “vertical” coordinate only for collars. In greater detail, consider a smooth function

\[
\varphi : I \to I, \quad \varphi(t) = \begin{cases} 
    t & \text{if } 0 \leq t \leq \varepsilon/4 \text{ or } 1 - \varepsilon/4 \leq t \leq 1, \\
    1/2 & \text{if } \varepsilon/2 \leq t \leq 1 - \varepsilon/2
\end{cases}
\]

for \( \varepsilon > 0 \) small enough. Choose collars \( M_0 \times [0, \varepsilon] \) and \( M_1 \times [1 - \varepsilon, 1] \) and define \( f : W \to I \) by setting

\[
 f(x) = \begin{cases} 
    \varphi(t) & \text{if } x = (m, t) \in M_0 \times [0, \varepsilon], \\
    \varphi(t) & \text{if } x = (m, t) \in M_1 \times [1 - \varepsilon], \\
    1/2 & \text{else.}
\end{cases}
\]

3. Every trivial \( h \)-cobordism \((M \times I, M, M)\) possesses a regular function with 1 critical point. Indeed, consider a function \( \varphi : M \to I \) such that \( \varphi^{-1}(1) \) is a point \( m_0 \) (and therefore \( m_0 \) is a critical point of \( \varphi \)) and define

\[
 f : M \times I \to I, \quad f(m, t) = (t - 1/2)(1 - \varphi(m)) + \varphi(m)(t - 1/2)^3.
\]

It is easy to see that \( f \) has just one critical point \((m_0, 1/2)\).

4. Notice that, for every \( h \)-cobordism \((W,M_0,M_1)\), the relative Lusternik–Schnirelmann category \( \text{cat}(W,M_0) = 0 \), while every regular function on any non-trivial \( h \)-cobordism \((W,M_0,M_1)\) has at least two critical points.

5. It is easy to see that, because of the collar theorem, the regularity condition for \( f \) in the Theorem can be weaken as follows: \( f(M_0) = 0 \) and \( f(M_1) = 1 \).

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