THE GROTHENDIECK RING OF VARIETIES IS NOT A DOMAIN

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Abstract. If \( k \) is a field, the ring \( K_0(V_k) \) is defined as the free abelian group generated by the isomorphism classes of geometrically reduced \( k \)-varieties modulo the set of relations of the form \( [X - Y] = [X] - [Y] \) whenever \( Y \) is a closed subvariety of \( X \). The multiplication is defined using the product operation on varieties. We prove that if the characteristic of \( k \) is zero, then \( K_0(V_k) \) is not a domain.

1. The Grothendieck ring of varieties

Let \( k \) be a field. By a \( k \)-variety we mean a geometrically reduced, separated scheme of finite type over \( k \). Let \( V_k \) denote the category of \( k \)-varieties. Let \( K_0(V_k) \) denote the free abelian group generated by the isomorphism classes of \( k \)-varieties, modulo all relations of the form \( [X - Y] = [X] - [Y] \) where \( Y \) is a closed \( k \)-subvariety of a \( k \)-variety \( X \). Here, and from now on, \([X]\) denotes the class of \( X \) in \( K_0(V_k) \). The operation \( [X] \cdot [Y] := [X \times_k Y] \) is well-defined, and makes \( K_0(V_k) \) a commutative ring with 1. It is known as the Grothendieck ring of \( k \)-varieties. A completed localization of \( K_0(V_k) \) is needed for the theory of motivic integration, which has many applications: see [Loo00] for a survey.

Our main result is the following.

**Theorem 1.** Suppose that \( k \) is a field of characteristic zero. Then \( K_0(V_k) \) is not a domain.

**Remark.** We conjecture that the result holds also for fields \( k \) of characteristic \( p \). But we use a result whose proof relies on resolution of singularities and weak factorization of birational maps, which are known only in characteristic zero.

2. Abelian varieties of \( GL_2 \)-type

If \( A \) is an abelian variety over a field \( k_0 \), and \( k \) is a field extension of \( k_0 \), then \( \text{End}_k(A) \) denotes the endomorphism ring of the base extension \( A_k := A \times_{k_0} k \), that is, the ring of endomorphisms defined over \( k \).
Lemma 2. Let $k$ be a field of characteristic zero, and let $\overline{k}$ denote an algebraic closure. There exists an abelian variety $A$ over $k$ such that $\text{End}_k(A) = \text{End}_{\overline{k}}(A) \simeq \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of a number field of class number 2.

Let us precede the proof of Lemma 2 with a few paragraphs of motivation. Our strategy will be to find a single abelian variety $A$ over $\mathbb{Q}$ such that the base extension $A_k$ works over $k$.

Let $A$ be a simple abelian variety over $\mathbb{Q}$. Let $E = \text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$. Since $A$ is simple, $E$ is a division algebra. The Lie algebra $\text{Lie} A$ is a nonzero left $E$-vector space, so $[E : \mathbb{Q}] \leq \dim_{\mathbb{Q}} \text{Lie} A = \dim A$. If equality holds and $E$ is commutative (hence a number field), then $A$ is said to be of $\text{GL}_2$-type. (The terminology is due to the following: If $A$ is of $\text{GL}_2$-type, then the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a Tate module $V_{\ell} A$ can be viewed as a representation $\rho_{\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E \otimes \mathbb{Q}_{\ell}).$)

Because $\mathbb{Q}$ has class number 1, we must take $[E : \mathbb{Q}] \geq 2$ to find an $A$ over $\mathbb{Q}$ as in Lemma 2. The inequality $\dim A \geq [E : \mathbb{Q}]$ then forces $\dim A \geq 2$. Moreover, if we want $\dim A = 2$, then $A$ must be of $\text{GL}_2$-type.

Abelian varieties of $\text{GL}_2$-type are closely connected to modular forms. For each $N \geq 1$, let $\Gamma_1(N)$ denote the classical modular group, let $X_1(N)$ denote the corresponding modular curve over $\mathbb{Q}$, and let $J_1(N)$ be the Jacobian of $X_1(N)$. G. Shimura, in Theorem 1 of [Shi71], attached to each weight-2 newform $f$ on $\Gamma_1(N)$ an abelian variety quotient $A_f$ of $J_1(N)$. (Previously, in Theorem 7.14 of [Shi71], he had attached to $f$ an abelian subvariety of $J_1(N)$.) Let $E_f$ be the number field generated over $\mathbb{Q}$ by the Fourier coefficients of $f$. Theorem 1 of [Shi73] shows also that $\dim A_f = [E_f : \mathbb{Q}]$, and that there is an injective $\mathbb{Q}$-algebra homomorphism $\theta : E_f \hookrightarrow E := \text{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$ mapping each Fourier coefficient to the endomorphism of $A_f$ induced by the associated Hecke correspondence on $X_1(N)$. Corollary 4.2 of [Rib80] proves that $\theta$ is an isomorphism. It follows that $A_f$ is of $\text{GL}_2$-type.

Conversely, it is conjectured that each simple abelian variety over $\mathbb{Q}$ of $\text{GL}_2$-type is $\mathbb{Q}$-isogenous to some $A_f$. See [Rib92] for more details. The $\dim A = 1$ case of this conjecture is the statement that elliptic curves over $\mathbb{Q}$ are modular, which is known [BCDT01].

Therefore we are led to consider $A_f$ of dimension 2, where $f$ is a newform as above.

Proof of Lemma 2. Tables [Ste] show that there exists a weight-2 newform $f = \sum_{n=1}^{\infty} a_n q^n$ on $\Gamma_0(590)$ (hence also on $\Gamma_1(590)$) such that $E_f = \mathbb{Q}(\sqrt{10})$ and $a_3 = \sqrt{10}$. Let $A = A_f$ be the corresponding abelian variety over $\mathbb{Q}$. Then $\dim A = [E_f : \mathbb{Q}] = 2$. But $\text{End}_{\mathbb{Q}}(A)$ is an order of $E = E_f$ containing $a_3 = \sqrt{10}$, so $\text{End}_{\mathbb{Q}}(A)$ is the maximal order $\mathbb{Z}[\sqrt{10}]$ of $E$. Since 590 is squarefree, $A$ is semistable over $\mathbb{Q}$ by Theorem 6.9 of [DR73], and then Corollary 1.4(a) of [Rib75] shows that all endomorphisms of $A$ over any field extension $k$ of $\mathbb{Q}$ are defined over $\mathbb{Q}$. Finally, the class number of $\mathbb{Z}[\sqrt{10}]$ is 2. \qed
Remarks.
1. After one knows that \( \text{End}_\mathbb{Q}(A) = \mathbb{Z}[\sqrt{10}] \), another way to prove \( \text{End}_\mathbb{Q}(A) = \mathbb{Z}[\sqrt{10}] \) is to use the fact that \( \text{End}_\mathbb{Q}(A) \) injects into the endomorphism ring of the reduction \( A_p \) over \( \overline{\mathbb{F}}_p \) for any prime \( p \) not dividing 590. The latter endomorphism rings can be computed using Eichler-Shimura theory and Honda-Tate theory. Combining the information from a few primes \( p \) yields the result.
2. The smallest \( N \) for which there exists a newform \( f \) on \( \Gamma_0(N) \) with \( E_f \) of class number 2 is 276. The advantage of 590 is that it is squarefree. (In fact, our original proof applied the technique in the previous remark at level 276.)
3. The case \( k = \mathbb{C} \) of Lemma 2 has an easy proof: let \( A \) be an elliptic curve over \( \mathbb{C} \) with complex multiplication by \( \mathbb{Z}[\sqrt{-5}] \).

3. Abelian varieties and projective modules

Let \( A \) be an abelian variety over a field \( k \), and let \( \mathcal{O} = \text{End}_k(A) \). Given a finite-rank projective right \( \mathcal{O} \)-module \( M \), we define an abelian variety \( M \otimes \mathcal{O}A \) as follows: choose a finite presentation \( \mathcal{O}^m \rightarrow \mathcal{O}^n \rightarrow M \rightarrow 0 \), and let \( M \otimes \mathcal{O}A \) be the cokernel of the homomorphism \( A^m \rightarrow A^n \) defined by the matrix that gives \( \mathcal{O}^m \rightarrow \mathcal{O}^n \). It is straightforward to check that this is independent of the presentation, and that \( M \mapsto M \otimes \mathcal{O}A \) defines a fully faithful functor \( T \) from the category of finite-rank projective right \( \mathcal{O} \)-modules to the category of abelian varieties over \( k \). (Essentially the same construction is discussed in the appendix by J.-P. Serre in [Lau01].)

Lemma 3. Let \( k \) be a field of characteristic zero. There exist abelian varieties \( A \) and \( B \) over \( k \) such that \( A \times A \cong B \times B \) but \( A_k \not\cong B_k \).

Proof. Let \( A \) and \( \mathcal{O} \) be as in Lemma 2. Let \( I \) be a nonprincipal ideal of \( \mathcal{O} \). Since \( \mathcal{O} \) is a Dedekind domain, the isomorphism type of a direct sum of fractional ideals \( I_1 \oplus \ldots \oplus I_n \) is determined exactly by the nonnegative integer \( n \) and the product of the classes of the \( I_i \) in the class group \( \text{Pic}(\mathcal{O}) \). Since \( \text{Pic}(\mathcal{O}) \cong \mathbb{Z}/2 \), we have \( \mathcal{O} \oplus \mathcal{O} \cong I \oplus I \) as \( \mathcal{O} \)-modules. Applying the functor \( T \) yields \( A \times A \cong B \times B \), where \( B := I \otimes \mathcal{O}A \). Since \( \text{End}_{\overline{k}}(A) \) also equals \( \mathcal{O} \), we have \( B_{\overline{k}} = I \otimes \mathcal{O}A_{\overline{k}} \). Since \( T \) for \( \overline{k} \) is fully faithful, \( A_{\overline{k}} \not\cong B_{\overline{k}} \).

4. Stable birational classes and Albanese varieties

For any extension of fields \( k \subseteq k' \), there is a ring homomorphism \( K_0(V_k) \rightarrow K_0(V_{k'}) \) mapping \( [X] \) to \( [X_{k'}] \).

Let \( k \) be a field of characteristic zero. Smooth, projective, geometrically integral \( k \)-varieties \( X \) and \( Y \) are called \textit{stably birational} if \( X \times \mathbb{P}^m \) is birational to \( Y \times \mathbb{P}^n \) for some integers \( m, n \geq 0 \). The set \( \text{SB}_k \) of equivalence classes of this relation is a monoid under product of varieties over \( k \). Let \( \mathbb{Z}[\text{SB}_k] \) denote the corresponding monoid ring.
When \(k = \mathbb{C}\), there is a unique ring homomorphism \(K_0(V_k) \to \mathbb{Z}[SB_k]\) mapping the class of any smooth projective integral variety to its stable birational class [LL01]. (In fact, this homomorphism is surjective, and its kernel is the ideal generated by \(\mathbb{L} := [A^1]\).) The proof in [LL01] requires resolution of singularities and weak factorization of birational maps [AKMW00, Theorem 0.1.1], [Wlo01, Conjecture 0.0.1]. The same proof works over any algebraically closed field of characteristic zero.

The set \(AV_k\) of isomorphism classes of abelian varieties over \(k\) is a monoid. The Albanese functor mapping a smooth, projective, geometrically integral variety to its Albanese variety induces a homomorphism of monoids \(SB_k \to AV_k\), since the Albanese variety is a birational invariant, since formation of the Albanese variety commutes with products, and since the Albanese variety of \(\mathbb{P}^n\) is trivial. Therefore we obtain a ring homomorphism \(\mathbb{Z}[SB_k] \to \mathbb{Z}[AV_k]\).

5. Zerodivisors

Proof of Theorem 1. Let \(A\) and \(B\) be as in Lemma 3. Then \(([A] + [B])([A] - [B]) = 0\) in \(K_0(V_k)\). On the other hand, \([A] + [B]\) and \([A] - [B]\) are nonzero, because their images under the composition

\[
K_0(V_k) \to K_0(V_k) \to \mathbb{Z}[SB_k] \to \mathbb{Z}[AV_k]
\]

are nonzero. (The Albanese variety of an abelian variety is itself.)

\(\square\)

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References


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