INTERPOLATION BY PROPER HOLOMORPHIC EMBEDDINGS OF THE DISC INTO $\mathbb{C}^2$

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Dedicated to the memory of my mother

1. The result

Let $\Delta$ be the open unit disc in $\mathbb{C}$. A map $f: \Delta \to \mathbb{C}^2$ is called a proper holomorphic embedding if it is a holomorphic immersion which is one to one and such that the preimage of every compact set is compact. If $f: \Delta \to \mathbb{C}^2$ is a proper holomorphic embedding then $f(\Delta)$ is a closed submanifold of $\mathbb{C}^2$ which is, via $f$, biholomorphically equivalent to $\Delta$.

It is not trivial to prove that there are proper holomorphic embeddings from $\Delta$ to $\mathbb{C}^2$ [St, A, GS]. It is known that given a discrete set $E \subset \mathbb{C}^2$ there is a proper holomorphic embedding $f: \Delta \to \mathbb{C}^2$ such that $E \subset f(\Delta)$ [FGS]. In the present paper we prove a stronger result:

**Theorem 1.1** Given a discrete set $S \subset \Delta$ and a proper injection $\varphi: S \to \mathbb{C}^2$ there is a proper holomorphic embedding $f: \Delta \to \mathbb{C}^2$ that extends $\varphi$.

In other words, given an injective sequence $\{\zeta_j\} \subset \Delta$ such that $|\zeta_j| \to 1$ and an injective sequence $\{w_j\} \subset \mathbb{C}^2$ such that $|w_j| \to +\infty$ there is a proper holomorphic embedding $f: \Delta \to \mathbb{C}^2$ such that $f(\zeta_j) = w_j$ ($j \in \mathbb{N}$).

The proof of the Carleman approximation theorem of Buzzard and Forstnerič [BFo] can be adapted to prove such a result for proper holomorphic embeddings $f: \mathbb{C} \to \mathbb{C}^2$. In the proof there one uses the fact that $\mathbb{C}$ admits particularly simple embeddings into $\mathbb{C}^2$ of the form $\zeta \to (\zeta, a(\zeta))$ where $a$ is an entire function. There are no such embeddings for $\Delta$ so a different proof is necessary in our case. In the induction step of our proof we use simultaneous composition by automorphisms on the left and on the right, a novelty introduced by Buzzard and Forstnerič.

2. The scheme of the proof

Suppose that $S \subset \Delta$ is a discrete set and let $\varphi: S \to \mathbb{C}^2$ be a proper injection. With no loss of generality assume that $S$ is infinite.

Denote by $B$ the open unit ball in $\mathbb{C}^2$. We shall construct inductively a sequence $K_n$ of compact subsets of $\Delta$, such that $bK_n$ is a smooth Jordan curve.
for each \( n \in \mathbb{N} \) and such that \( K_n \subset \subset K_{n+1} \) \((n \in \mathbb{N})\), \( \cap_{n=1}^\infty K_n = \Delta \), an increasing sequence \( r_n \) of positive numbers converging to \(+\infty\), a decreasing sequence \( \epsilon_n \) of positive numbers and a sequence \( f_n \) of holomorphic maps from \( \Delta \) to \( \mathbb{C}^2 \) which are one to one and regular and such that the following hold:

(i) \( \varphi((\Delta \setminus K_n) \cap S) \subset \mathbb{C}^2 \setminus r_n \mathbb{B} \)

(ii) \( f_n(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_n \mathbb{B} \)

(iii) \( f_{n+1}(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_{n-1} \mathbb{B} \)

(iv) \( f_n|K_n \cap S = \varphi|K_n \cap S \)

(v) \( |f_{n+1} - f_n| < \epsilon_n/2^n \) on \( K_n \)

(vi) If \( h \) is a holomorphic map on \( \text{Int} K_n \) that satisfies \( |h - f_n| < \epsilon_n \) on \( \text{Int} K_n \), then \( h \) is one to one and regular on \( K_{n-1} \)

(vii) \((1 - 1/n) \Delta \subset K_n \)

Suppose for a moment that we have done this. By (v) and (vii) \( f_n \) converges, uniformly on compacta in \( \Delta \), to a holomorphic map \( f \). By (v), \( |f_n - f| \leq \sum_{j=0}^{\infty} |f_{j+1} - f_j| \leq \sum_{j=0}^{\infty} \epsilon_j/2^j \leq \epsilon_n \) on \( K_n \) which implies by (vi) that \( f \) is regular and one to one on \( K_{n-1} \). As this holds for every \( n \) it follows that \( f \) is regular and one to one on \( \Delta \). By (iv), \( f \) extends \( \varphi \). Let \( \zeta \in K_{n+1} \setminus K_n \). By (v), \( |f_{j+1}(\zeta) - f_j(\zeta)| < \epsilon_j/2^j \) \((j > n + 1)\) which, by (iii) implies that \( |f(\zeta)| \geq |f_{n+1}(\zeta)| - \sum_{j=n+1}^{\infty} |f_{j+1}(\zeta) - f_j(\zeta)| \geq r_{n-1} - \sum_{j=n+1}^{\infty} \epsilon_j/2^j \geq r_{n-1} - \epsilon_{n+1} \).

This holds for every \( n \). Since \( r_n \) increase to \(+\infty\) and since \( \epsilon_n \) are decreasing it follows that the map \( f \) is proper. Thus, \( f \) has all the required properties.

In the process we shall also construct two sequences \( S_n, T_n \) of positive numbers such that \( S_{n+1} = S_n \) for even \( n \) and \( T_{n+1} = T_n \) for odd \( n \). Each map \( f_n \) will be of the form \( f_n = A_n \circ g_n \) where \( A_n \) is a holomorphic automorphism of \( \mathbb{C}^2 \) and \( g_n \) is a one to one and regular holomorphic map from an open neighbourhood \( U_n \) of \( \Delta \) to \( \mathbb{C}^2 \) which, for even \( n \) is transverse to \( \{ (z, w) : |z| = S_n \} \) and satisfies \( g_n^{-1}(\{|z| = S_n\}) = b\Delta \), and for odd \( n \), is transverse to \( \{ (z, w) : |w| = T_n \} \) and satisfies \( g_n^{-1}(\{|w| = T_n\}) = b\Delta \).

With no loss of generality assume that \( 0 \notin S \). To begin the induction, let \( f_1(\zeta) = (0, \zeta) \) and let \( r_1 > r_1 < 1/2 \) be such that \( 2r_1 \mathbb{B} \) contains no point of \( S \). Put \( K_0 = r_1 \Delta, K_1 = 2r_1 \Delta \). Then (i), (ii) and (vii) are satisfied for \( n = 1 \) and (iv) is vacuously satisfied for \( n = 1 \). Put \( S_1 = T_1 = 1 \) and \( A_1 = \text{Id} \) so that \( f_1 = A_1 \circ g_1 \) where \( g_1(\zeta) = (0, \zeta) \) and \( U_1 = \mathbb{C} \). Clearly \( g_1 \) is transverse to \( \{|w| = T_1\} \) and \( g_1^{-1}(\{|w| = T_1\}) = b\Delta \). Put \( r_0 = r_1/2 \). Then \( A_1(\{|w| > T_1/2\}) \) misses \( 2r_0 \mathbb{B} \). Put \( \epsilon_0 = \min\{1, r_1/2\} \).

Given \( f_n = A_n \circ g_n \) we shall have \( f_{n+1} = A_{n+1} \circ g_{n+1} \) with \( A_{n+1} = \Psi_{n+1} \circ \Theta_{n+1} \circ A_n \) where \( \Theta_{n+1} \) and \( \Psi_{n+1} \) are holomorphic automorphisms of \( \mathbb{C}^2 \) and with \( g_{n+1} = G_{n+1} \circ g_n \circ p_{n+1} \) where \( p_{n+1} \) is a conformal map from a neighbourhood \( U_{n+1} \) of \( \Delta \) to \( p_{n+1}(U_{n+1}) \subset \mathbb{C} \) which is a slight perturbation of the identity on \( \Delta \) and \( G_{n+1} \) is an automorphism of \( \mathbb{C}^2 \) of the form

\[
G_{n+1}(z, w) = \left(z + S_{n+1}\left(\frac{w}{T_n}\right)^{M_{n+1}}, w\right) \quad \text{if } n \text{ is odd},
\]
\[(2.1') G_{n+1}(z, w) = \left( z, w + T_{n+1} \left( \frac{z}{S_n} \right)^{M_{n+1}} \right) \text{ if } n \text{ is even.} \]

3. The induction step, Part 1

Suppose for a moment that we have constructed \( f_n = A_n \circ g_n, K_n, S_n, T_n, r_n \) and \( \varepsilon_{n-1} \). We want to show how to obtain \( \varepsilon_n, K_{n+1}, S_{n+1}, T_{n+1}, r_{n+1} \) and \( f_{n+1} = A_{n+1} \circ g_{n+1} \). Suppose that \( n \) is odd so that \( g_n: U_n \to \mathbb{C}^2 \) is transverse to \( \{(z, w): |w| = T_n\} \) and satisfies \( g_n^{-1}(\{|w| = T_n\}) = b\Delta \). Put \( T_{n+1} = T_n \). Since \( g_n \) is transverse to \( \{|w| = T_n\} \) and \( S \) is discrete one can, after shrinking \( U_n \) if necessary, choose \( T_{n1}, T_{n2}, T_{n3} \) such that

\[
\frac{T_n}{2} < T_{n3} < T_{n2} < T_{n1} < T_n
\]

where \( T_{n3} \) is so close to \( T_n \) that for all \( T, T_{n3} \leq T \leq T_n, g_n \) is transverse to \( \{|w| = T\} \) and \( g_n^{-1}(\{|w| = T\}) \) is a smooth Jordan curve, that

\[
\mathbb{D} \subset g_n^{-1}(\{|w| < T_{n2}\}) \subset \mathbb{D}
\]

that \( g_n^{-1}(\{T_{n3} \leq |w| \leq T_{n2}\}) \) contains no point of \( S \), and that \( g_n^{-1}(\{|w| < T_{n1}\}) \) contains a point in \( S \) that does not belong to \( \mathbb{D} \). Put

\[
P_{n+1} = g_n^{-1}(\{|w| \leq T_{n3}\}), \quad Q_{n+1} = g_n^{-1}(\{|w| \leq T_{n2}\}), \quad K_{n+1} = g_n^{-1}(\{|w| \leq T_{n1}\})
\]

With no loss of generality assume that \( T_{n3} \) has been chosen so close to \( T_n \) that

\[ (vii) \]

holds with \( n \) replaced by \( n + 1 \). We have

\[
\mathbb{D} \subset P_{n+1} \subset Q_{n+1} \subset K_{n+1}.
\]

Clearly \( bK_{n+1} \) is a smooth Jordan curve.

By (i), \( r_n < \min\{|\varphi(w)|: w \in (\mathbb{D} \setminus K_n) \cap S\} \). Thus, one can choose \( r_{n+1} > r_n \) such that

\[
(3.1) \quad \min\{|\varphi(w)|: w \in (\mathbb{D} \setminus K_{n+1}) \cap S\} - 1 < r_{n+1} < \min\{|\varphi(w)|: w \in (\mathbb{D} \setminus K_{n+1}) \cap S\}.
\]

Then (i) is satisfied with \( n \) replaced by \( n + 1 \). Choose \( \varepsilon_n, 0 < \varepsilon_n < \varepsilon_{n-1} \), such that

\[
(3.2) \quad \varepsilon_n < r_n - r_{n-1}, \quad \varepsilon_n < r_{n-1},
\]

and such that (vi) holds. Since \( f_n \) is one to one and regular on \( \mathbb{D} \) this is possible by a lemma of Narasimhan [Na, p. 926].

Choose \( R, R > 2r_{n+1}, R > 2r_n + \varepsilon_n \), so large that \( f_n(K_n) + \mathbb{B} \subset R\mathbb{B} \) and that \( \varphi(K_{n+1} \cap S) \subset R\mathbb{B} \). We need the following lemma.

**Lemma 3.1.** Let \( R > 0 \) and let \( w_1, w_2, \ldots, w_n \in R\mathbb{B}, \ w_i \neq w_j \ (i \neq j) \). Given \( \gamma > 0 \) there is a \( \delta > 0 \) such that whenever \( q_1, q_2, \ldots, q_n \in \mathbb{C}^2 \) satisfy \( |q_i - w_i| < \delta, 1 \leq i \leq n \), there is a holomorphic automorphism \( \Psi \) of \( \mathbb{C}^2 \) such that:

(i) \( \Psi(q_i) = w_i \ (1 \leq i \leq n) \)

(ii) \( |\Psi(w) - w| < \gamma \ (w \in R\mathbb{B}). \)
Lemma 3.1 provides a \( \theta_n, 0 < \theta_n < \frac{\varepsilon_n}{2^{n+2}} \), such that

\[
\text{whenever } \psi: K_{n+1} \cap S \to \mathbb{C}^2 \text{ satisfies } |\psi - \varphi| < 3\theta_n \text{ on } K_{n+1} \cap S \text{ there is a holomorphic automorphism } \Psi \text{ of } \mathbb{C}^2 \text{ such that } \\
\Psi \circ \psi = \varphi|_{K_{n+1}} \text{ and such that } |\Psi - \text{Id}| < \varepsilon_n/2^{n+1} \text{ on } R\mathbb{B}.
\]

By (3.2) we may assume that

\[
r_n - 3\theta_n > r_{n-1} + \varepsilon_n + \theta_n, \quad 2r_{n-1} - \theta_n > r_{n-1} + \varepsilon_n + \theta_n.
\]

4. Proof of Lemma 3.1

**Sublemma 4.1** Suppose that \( R > 0 \) and let \( \alpha_1, \ldots, \alpha_n \in R\Delta, \alpha_i \neq \alpha_j \) (\( i \neq j \)). There are \( \eta > 0 \) and \( L < \infty \) such that whenever \( \beta_1, \ldots, \beta_n \) satisfy \( |\beta_i - \alpha_i| < \eta \), \( 1 \leq i \leq n \), then for every \( j, 1 \leq j \leq n \), there is a polynomial \( Q_j \) such that

(i) \( Q_j(\beta_i) = \delta_{ij} \) (\( 1 \leq i, j \leq n \))

(ii) \( |Q_j(\zeta)| \leq L \) (\( \zeta \in 2R\Delta \)).

**Proof.** Choose \( \eta > 0 \) so small that \( \alpha_i + \eta \Delta \subset R\Delta \) (\( 1 \leq i \leq n \)) and let \( |\beta_i - \alpha_i| < \eta (1 \leq i \leq n) \). For each \( j, 1 \leq j \leq n \), the polynomial

\[
Q_j(\zeta) = \prod_{k=1, k \neq j}^{n} \frac{\zeta - \beta_k}{\beta_j - \beta_k}
\]

satisfies (i). If \( |\zeta| < 2R \) then

\[
|Q_j(\zeta)| \leq \frac{(3R)^{n-1}}{(\min_{j \neq k} |\beta_j - \beta_k|)^{n-1}}.
\]

Now, let \( \gamma = \min_{j \neq k} |\alpha_j - \alpha_k| \). Passing to a smaller \( \eta \) we may assume that

\( 0 < \eta < \gamma/2 \). If \( |\alpha_i - \beta_i| < \eta \), \( 1 \leq i \leq n \), then \( \min_{j \neq k} |\beta_j - \beta_k| \geq \gamma - 2\eta > 0 \) so \( Q_j \) satisfies (ii) with \( L = [3R/(\gamma - 2\eta)]^{n-1} \). This completes the proof. \( \Box \)

**Proof of Lemma 3.1.** Choose a coordinate system in \( \mathbb{C}^2 \) such that if \( w_i = (w_i^1, w_i^2) \) then \( w_i^1 \neq w_j^1, w_i^2 \neq w_j^2 \) if \( i \neq j, 1 \leq i, j \leq n \). By Sublemma 4.1 there are \( \eta > 0 \) and \( L < \infty \) such that whenever \( \beta_i^1 \) satisfy \( |\beta_i^1 - w_i^1| < \eta \) and \( \beta_i^2 \) satisfy \( |\beta_i^2 - w_i^2| < \eta \), \( 1 \leq i \leq n \), then for each \( j, 1 \leq j \leq n \), there are polynomials \( Q_j^1 \) and \( Q_j^2 \) such that \( Q_j^1(\beta_i^1) = 1, Q_j^1(\beta_i^1) = 0 (i \neq j), Q_j^2(\beta_i^2) = 1, Q_j^2(\beta_i^2) = 0 (i \neq j) \) and \( |Q_j^1| < L, |Q_j^2| < L \) on \( 2R\Delta \). Let \( |z_j - w_j| < \eta, 1 \leq j \leq n \). Our map \( \Phi \) will be of the form \( \Phi = T \circ S \) where \( T, S \) are the automorphisms of \( \mathbb{C}^2 \)

\[
T(\xi, \zeta) = (\xi, \zeta + Q_1(\xi)), \quad S(\xi, \zeta) = (\xi + Q_2(\zeta), \zeta)
\]

such that

\[
S(R\Delta \times R\Delta) \subset (2R\Delta) \times (R\Delta),
\]

\[
|S(\xi, \zeta) - (\xi, \zeta)| < \gamma/2 \quad ((\xi, \zeta) \in (R\Delta)^2),
\]

\[
|T(\xi, \zeta) - (\xi, \zeta)| < \gamma/2 \quad ((\xi, \zeta) \in (2R\Delta) \times (R\Delta)),
\]
and
\[(4.4) \quad S(z^1_i, z^2_i) = (w^1_i, z^2_i), \quad T(w^1_i, z^2_i) = (w^1_i, w^2_i) \quad (1 \leq i \leq n).\]

By (4.1)-(4.4) the map $\Phi$ satisfies (i) and (ii) in Lemma 3.1. To construct $S$, put $\beta_j = z^2_j$, $1 \leq j \leq n$, and let $Q^2_j$, $1 \leq j \leq n$, be as above. In particular, $Q^2_j(z^2_i) = \delta_{ji}$, $1 \leq i, j \leq n$. Put
\[Q_2(\zeta) = \sum_{j=1}^n (w^1_j - z^1_j)Q^2_j(\zeta).\]

We have
\[Q_2(z^2_j) = \sum_{i=1}^n (w^1_i - z^1_i)Q^2_i(z^2_j) = w^1_j - z^1_j\]
and so $S(z^1_i, z^2_i) = (z^1_i + w^1_i - z^1_i, z^2_i) = (w^1_i, z^2_i)$. We have
\[|Q_2(\zeta)| \leq n \cdot \max_{1 \leq j \leq n} |w^1_j - z^1_j| \cdot L, \quad (|\zeta| < R)\]
which implies that
\[|S(\xi, \zeta) - (\xi, \zeta)| = |(Q_2(\zeta), 0)| \leq n \cdot L \cdot \max_{1 \leq j \leq n} |w_j - z_j|, \quad (|\zeta| < R).\]

In particular, if $\eta > 0$ is small enough then $|Q_2(\zeta)| < R$, $(|\zeta| < R)$, so that (4.1) and (4.2) hold. To construct $T$, put $\beta^1_j = w^1_j$, $1 \leq j \leq n$, and let $Q^1_j$, $1 \leq j \leq n$, be as above. Put
\[Q_1(\zeta) = \sum_{j=1}^n (w^2_j - z^2_j)Q^1_j(\zeta).\]

We have $Q_1(w^1_j) = w^2_j - z^2_j$ $(1 \leq j \leq n)$, so $T(w^1_i, z^2_i) = (w^1_i, z^2_i + w^2_i - z^2_i) = (w^1_i, w^2_i)$, $(1 \leq i \leq n)$. Again, $|Q_1(\zeta)| \leq n \cdot \max_{1 \leq j \leq n} |w^2_j - z^2_j| \cdot L$, $(|\zeta| < 2R)$, which implies that
\[|T(\xi, \zeta) - (\xi, \zeta)| = |(0, Q_1(\zeta))| \leq n \cdot \max_{1 \leq j \leq n} |w_j - z_j| \cdot L, \quad (|\xi| < 2R).\]

In particular, if $\delta = \eta$ is small enough then (4.3) holds. The equality (4.4) is clear. This completes the proof.

**Remark.** Lemma 3.1 holds for $\mathbb{C}^N$, $N \geq 2$. The proof is an easy elaboration of the proof above.

**5. The induction step, Part 2**

We need the following:

**Lemma 5.1** Let $r > 0$ and let $\Phi \colon \mathbb{C} \rightarrow \mathbb{C}^2$ be a proper holomorphic embedding. Let $\Sigma \subset \mathbb{C}$ be a domain bounded by a smooth Jordan curve and assume that $\Phi(b\Sigma) \subset \mathbb{C}^2 \setminus r\mathbb{R}$. Then the set $(r\mathbb{R}) \cup \Phi(\Sigma)$ is polynomially convex.
Proof. Since $\Sigma$ is a Jordan domain with smooth boundary it is easy to see that if $K \subset \mathbb{C} \setminus \Sigma$ is a compact set, if $a, b \in (\mathbb{C} \setminus \Sigma) \setminus K$, and if $p$ is a path in $\mathbb{C} \setminus K$ joining $a$ and $b$ then there is a path $\tilde{p} \in (\mathbb{C} \setminus \Sigma) \setminus K$ joining $a$ and $b$. Let $K = \{ \zeta \in \mathbb{C} \setminus \Sigma: |\Phi(\zeta)| \leq r \}$. Since $\Phi(b \Sigma) \subset \mathbb{C}^2 \setminus r \overline{B}$ and since $|\Phi(\zeta)| \to +\infty$ as $|\zeta| \to +\infty$, the set $K$ is compact. Suppose for a moment that $(\mathbb{C} \setminus \Sigma) \setminus K$ is not connected. The preceding discussion implies that $\{ \zeta \in \mathbb{C}: |\Phi(\zeta)| > r \}$ has a bounded component which contradicts the maximum principle. Thus, $(\mathbb{C} \setminus \Sigma) \setminus K$ is connected which implies that for each $q \in \Phi(\mathbb{C}) \setminus (\Phi(\Sigma) \cup r \overline{B})$ there is a path $\eta$: $[0, 1] \to \Phi(\mathbb{C}) \setminus (\Phi(\Sigma) \cup r \overline{B})$ such that $\eta(0) = q$ and $|\eta(t)| \to +\infty$ as $t \to 1$. The statement of the lemma now follows from [BF, Lemma 3.1]. This completes the proof.

Remark. It is easy to see that the proof of Lemma 3.1 in [BF] works for $\mathbb{C}^N$, $N \geq 2$, and so Lemma 5.1. holds for proper holomorphic embeddings $\Phi$: $\mathbb{C} \to \mathbb{C}^N$, $N \geq 2$.

Proof of the induction step, continued. We have already mentioned that for each $m$, $f_{m+1} = (\Psi_{m+1} \circ \Theta_{m+1} \circ A_m) \circ (G_{m+1} \circ g_m \circ p_{m+1}) = A_{m+1} \circ g_{m+1}$. Thus, $f_n = H_n \circ g_1 \circ (p_2 \circ \cdots \circ p_n)$ where $H_n = (\Psi_n \circ \Theta_n) \circ \cdots \circ (\Psi_2 \circ \Theta_2) \circ (G_n \circ \cdots \circ G_2)$ is a holomorphic automorphism of $\mathbb{C}^2$. It follows that $f_n(K_n)$ is a compact subset of $(H_n \circ g_1)(\mathbb{C})$, a closed submanifold of $\mathbb{C}^2$ biholomorphically equivalent to $\mathbb{C}$, whose boundary $f_n(bK_n)$ is a smooth Jordan curve which is, by (ii), contained in $\mathbb{C}^2 \setminus r_n \overline{B}$. By Lemma 5.1 the set $f_n(K_n) \cup r_n \overline{B}$ is polynomially convex. By (ii) $f_n(K_n) \cup r_n \overline{B}$ contains no point of $f_n((K_{n+1} \setminus K_n) \cap S)$. Since $f_n$ is one to one it follows that $f_n(\zeta) \neq f_n(\eta)$ if $\zeta, \eta \in (K_{n+1} \setminus K_n) \cap S$. By (i), $\varphi((K_{n+1} \setminus K_n) \cap S)$ does not meet $r_n \overline{B}$. However, some points of $\varphi((K_{n+1} \setminus K_n) \cap S)$ may lie in $f_n(K_n)$. Since $f_n(K_n)$ is contained in $(H_n \circ g_1)(\mathbb{C})$, a closed one dimensional complex submanifold of $\mathbb{C}^2$, one can change $\varphi$ slightly on $K_{n+1} \cap S$ to $\tilde{\varphi}$ so that

$$|\tilde{\varphi} - \varphi| < \theta_n \text{ on } K_{n+1} \cap S,$$

so that $\tilde{\varphi}$ is one to one on $K_{n+1} \cap S$ and that $f_n(K_n) \cup r_n \overline{B}$ contains no point of $\tilde{\varphi}((K_{n+1} \setminus K_n) \cap S)$. By [FGS] there is an automorphism $\Theta_{n+1}$ of $\mathbb{C}^2$ which fixes each point of $f_n(K_n) \cap S$, that moves each point $f_n(\zeta)$, $\zeta \in (K_{n+1} \setminus K_n) \cap S$ to $\tilde{\varphi}(\zeta)$, and that satisfies

$$|\Theta_{n+1} - \text{Id}| < \theta_n \text{ on } f_n(K_n) \cup r_n \overline{B}. \tag{5.2}$$

By (iv) we have $f_n|K_n \cap S = \varphi|K_n \cap S$. Almost the same equality holds for $\Theta_{n+1} \circ f_n$ in place of $f_n$ since $\Theta_{n+1} \circ f_n|K_{n+1} \cap S = \tilde{\varphi}|K_{n+1} \cap S$. Applying on both sides on the left an automorphism $\Psi$ provided by Lemma 3.1 which satisfies $\Psi \circ \tilde{\varphi} = \varphi$ on $K_{n+1} \cap S$, would produce a map from $\Delta$ to $\mathbb{C}^2$ that would satisfy (iv) with $n$ replaced by $n + 1$. However, such a map does not necessarily satisfy (ii) with $n + 1$ in place of $n$ or (iii) since we have no control over what $\Theta_{n+1}$ does with $f_n(\Delta \setminus K_n)$.
6. The induction step, Part 3

We perform our induction process in such a way that

\[(6.1') \quad A_n(\{(z, w): |w| > T_n/2\}) \text{ misses } 2r_{n-1}B \quad \text{if } n \text{ is odd.}\]

and

\[(6.1'') \quad A_n(\{(z, w): |z| > S_n/2\}) \text{ misses } 2r_{n-1}B \quad \text{if } n \text{ is even.}\]

Recall that \((6.1')\) holds for \(n = 1\). We are describing the induction step for odd \(n\) so assume that \((6.1')\) holds. To handle the problem described at the end of the previous section we replace \(g_n\) in \(\Theta_{n+1} \circ A_n \circ g_n = \Theta_{n+1} \circ f_n\) by \(G_{n+1} \circ g_n\) where \(G_{n+1}\) is an automorphism of \(\mathbb{C}^2\) of the form \((2.1')\). Passing to a slightly smaller \(U_n\) if necessary we may assume that \(g_n(U_n)\) is bounded. We want that \(G_{n+1}\) changes \(g_n\) only slightly on \(K_n\) and on \(K_{n+1} \cap S\) while it maps \(g_n(U_n \setminus \text{Int}Q_{n+1})\) so far from the origin that

\[(6.2) \quad (\Theta_{n+1} \circ A_{n+1}) \circ (G_{n+1} \circ g_n)(U_n \setminus \text{Int}Q_{n+1}) \subset \mathbb{C}^2 \setminus 2r_{n+1}B\]

which, since \(g_n(U_n)\) is bounded, and since \(\Theta_{n+1} \circ A_{n+1}\) is an automorphism of \(\mathbb{C}^2\), holds if

\[(6.3) \quad \left| S_{n+1} \left( \frac{w}{T_n} \right)^{M_{n+1}} \right| \geq \rho_n \quad (|w| \geq T_n)\]

provided that \(\rho_n\) is sufficiently large. Choose \(\tau_n > 0\) so small that

\[(6.4) \quad |(\Theta_{n+1} \circ A_n)(p) - (\Theta_{n+1} \circ A_n)(q)| < \theta_n \quad (q \in g_n(P_{n+1}), \ |p - q| < 2\tau_n).\]

We want that

\[(6.5) \quad \left| S_{n+1} \left( \frac{w}{T_n} \right)^{M_{n+1}} \right| \leq \tau_n \quad (|w| \leq T_{n+1})\]

which will imply that \(G_{n+1}\) changes \(g_n\) on \(P_{n+1}\) for at most \(\tau_n\). Let

\[S_{n+1} = \rho_n \left( \frac{T_n}{T_{n+1}} \right)^{M_{n+1}}.\]

Notice that \(S_{n+1}\) is arbitrarily large provided that \(M_{n+1}\) is large enough. The choice of \(S_{n+1}\) implies \((6.3)\) while \((6.5)\) becomes equivalent to

\[(6.7) \quad \rho_n \left( \frac{T_{n+1}}{T_{n+2}} \right)^{M_{n+1}} < \tau_n\]

which will hold provided that \(M_{n+1}\) is large enough. Choose \(M_{n+1}\) so large that \(S_{n+1}\) becomes so large that

\[(6.8) \quad (\Theta_{n+1} \circ A_n)(\{|z| > S_{n+1}/2\}) \text{ misses } (2r_n + \varepsilon_n)B.\]

Notice that if an automorphism \(G: \mathbb{C}^2 \to \mathbb{C}^2\) satisfies \(|G(z) - z| < \tau (z \in R\mathbb{B})\) where \(0 < \tau < R\) then \((R-\tau)\mathbb{B} \subset G(R\mathbb{B})\). Choose a compact set
$K'_n \subset \text{Int} K_n$ such that $f_n(\Delta \setminus K'_n) \subset f_n(\Delta \setminus K_n) + \theta_n \mathbb{B}$. Now, (ii) implies that $A_n(g_n(\Delta \setminus K'_n)) = f_n(\Delta \setminus K'_n)$ misses $(r_n - \theta_n)\mathbb{B}$ and (5.2) implies that

$$ (6.9) \quad (\Theta_{n+1} \circ A_n \circ g_n)(\Delta \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_n - 2\theta_n)\mathbb{B}. $$

By (6.5), $|G_{n+1} \circ g_n - g_n| \leq \tau_n$ on $P_{n+1}$ so by (6.4)

$$ |(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) - (\Theta_{n+1} \circ A_n \circ g_n)(\zeta)| \leq \theta_n \quad (\zeta \in P_{n+1}) $$

which, by (6.9) gives

$$ (6.10) \quad (\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(P_{n+1} \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_n - 3\theta_n)\mathbb{B}. $$

Let $\zeta \in Q_{n+1} \setminus P_{n+1}$. Since $g_n(Q_{n+1} \setminus P_{n+1}) \subset \{|w| > T_n/2\}$ and since $G_{n+1}$ does not change the $w$ coordinate we have $(G_{n+1} \circ g_n)(\zeta) \in \{|w| > T_n/2\}$ and so $(G_{n+1} \circ g_n)(\zeta) \in A_n(\{|w| > T_n/2\})$. By (6.1') $A_n(\{|w| > T_n/2\})$ misses $2r_{n-1}\mathbb{B}$ which implies that $(G_{n+1} \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus 2r_{n-1}\mathbb{B}$. By (5.2) it follows that $(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus s\mathbb{B}$ where $s = \min\{r_n - \theta_n, 2r_{n-1} - \theta_n\}$, by (3.4), satisfies $s > r_{n-1} + \theta_n + \varepsilon_n$. By (6.10), (6.2) and (3.4) it follows that

$$ (6.11) \quad (\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(U_n \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n)\mathbb{B}. $$

### 7. The induction step, Part 4

Note first that $\Theta_{n+1} \circ A_n \circ g_n|K_{n+1} \cap S = \varphi|K_{n+1} \cap S$. This does not necessarily hold if we replace $g_n$ by $G_{n+1} \circ g_n$. However, since all points of $K_{n+1} \cap S$ lie in $P_{n+1}$, since $|G_{n+1} \circ g_n - g_n| < \tau_n$ on $P_{n+1}$ and since $|\varphi - \varphi| < \theta_n$ on $K_{n+1} \cap S$ it follows by (6.4) that

$$ (7.1) \quad |\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n - \varphi| < 2\theta_n \quad \text{on} \quad K_{n+1} \cap S. $$

The problem now is that $(G_{n+1} \circ g_n)^{-1}(\{(z, w): |z| = S_{n+1}\})$ is not necessarily equal to $b\Delta$ so we cannot use $\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n$ as $f_{n+1}$ even after composing with a correction automorphism provided by Lemma 3.1. However, $(G_{n+1} \circ g_n)^{-1}(\{|z| = S_{n+1}\})$ is a real analytic curve that is arbitrarily small $C^1$ perturbation of $b\Delta$ independently of $M_{n+1}$ if only $S_{n+1}$ is large enough [G, Sec. 5]; in our case this means if only $M_{n+1}$ is large enough.

Thus, provided that $M_{n+1}$ is large enough the conformal map $p_{n+1}$ mapping $\Delta$ to the domain $(G_{n+1} \circ g_n)^{-1}(\{|z| < S_{n+1}\})$ and satisfying $p_{n+1}(0) = 0$, $p'_{n+1}(0) > 0$, is arbitrarily close to the identity on $\Delta$ provided that $M_{n+1}$ is sufficiently large [P, p. 286]. Once we have chosen $M_{n+1}$ the map $p_{n+1}$ extends holomorphically to a neighbourhood $U_{n+1} \subset U_n$ of $\overline{\Delta}$ so that the extended map $p_{n+1}$ maps $U_{n+1}$ biholomorphically onto $p_{n+1}(U_{n+1})$ and so that the map $g_{n+1} = G_{n+1} \circ g_n \circ p_{n+1}: U_{n+1} \to \mathbb{C}^2$ is transverse to $\{(z, w): |z| = S_{n+1}\}$ and satisfies $g_{n+1}^{-1}(\{|z| = S_{n+1}\}) = b\Delta |G|$.

Passing to a larger $M_{n+1}$ if necessary we may assume that $p_{n+1}$ is so close to the identity on $\overline{\Delta}$ that

$$ (7.2) \quad |g_{n+1} \circ p_{n+1} - g_n| < \tau_n \quad \text{on} \quad \overline{\Delta} $$
and that

\[ \begin{aligned}
&\left\{ \begin{array}{ll}
K_n \subset p_{n+1}^{-1}(P_{n+1}), \\
K_{n+1} \cap S \subset p_{n+1}^{-1}(P_{n+1}) \\
K_n \cap S \subset p_{n+1}^{-1}(Q_{n+1}) \subset \text{Int}K_{n+1}, \\
p_{n+1}^{-1}(K'_n) \subset K_n.
\end{array} \right.
\end{aligned} \tag{7.3} \]

Since \(|G_{n+1} \circ g_n \circ p_{n+1} - g_n \circ p_{n+1}| \leq \tau_n\) on \(P_{n+1}\) it follows that \(|G_{n+1} \circ g_n \circ p_{n+1} - g_n \circ p_{n+1}| \leq \tau_n\) on \(p_{n+1}^{-1}(P_{n+1})\) which, by (7.2) and (7.3) implies that

\[ |G_{n+1} \circ g_n \circ p_{n+1} - g_n| < 2\tau_n \text{ on } K_n \cup (K_{n+1} \cap S). \]

Since \(K_n \cup (K_{n+1} \cap S) \subset P_{n+1}\), (6.4) implies that

\[ |(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n \circ p_{n+1})(\zeta) - (\Theta_{n+1} \circ A_n \circ g_n)(\zeta)| < \theta_n \quad (\zeta \in K_n \cup (K_{n+1} \cap S)). \]

By (5.2), \(|\Theta_{n+1} f_n(\zeta) - f_n(\zeta)| < \theta_n \quad (\zeta \in K_n)\) so it follows that

\[ |v(\zeta)| < 2\theta_n \quad (\zeta \in K_n). \tag{7.4} \]

Further, since \((\Theta_{n+1} \circ A_n \circ g_n)|K_{n+1} \cap S = \tilde{\varphi}\) and since \(|\tilde{\psi} - \psi| < \theta_n\) on \(K_{n+1} \cap S\) it follows also that

\[ |v(\zeta)| < 3\theta_n \quad (\zeta \in K_{n+1} \cap S). \tag{7.5} \]

The choice of \(R\) and (3.3) imply that there is a holomorphic automorphism \(\Psi_{n+1}\) of \(\mathbb{C}^2\) such that

\[ |\Psi_{n+1} - \text{Id}| < \varepsilon_n/2^{n+1} \text{ on } R \mathbb{B} \tag{7.6} \]

and such that

\[ (\Psi_{n+1} \circ \Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n \circ p_{n+1})(\zeta) = \varphi(\zeta) \quad (\zeta \in K_{n+1} \cap S). \tag{7.7} \]

Put \(f_{n+1} = A_{n+1} \circ g_{n+1}\), where \(A_{n+1} = \Psi_{n+1} \circ \Theta_{n+1} \circ A_n\), and \(g_{n+1} = G_{n+1} \circ g_n \circ p_{n+1}\). By (7.7), (iv) is satisfied with \(n+1\) in place of \(n\). Since \(\theta_n < \varepsilon_n/2^{n+2}\) and since \(f_n(K_n) + \mathbb{B} \subset R \mathbb{B}\), (7.4) and (7.6) imply that \(|f_{n+1}(\zeta) - f_n(\zeta)| < 2\theta_n + \varepsilon_n/2^{n+1} < \varepsilon_n/2^n\) \((\zeta \in K_n)\) so that (v) is satisfied.

By (7.3), \(\zeta \in \Delta \setminus \text{Int}K_{n+1}\) implies that \(p_{n+1}(\zeta) \in U_n \setminus Q_{n+1}\) which, by (6.2) implies that \((\Theta_{n+1} \circ A_n \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus 2r_{n+1} \mathbb{B}\). By (7.6), by the fact that \(R > 2r_{n+1}\) and by (3.2) it follows that \(f_{n+1}(\zeta) \in \mathbb{C}^2 \setminus (2r_{n+1} - \varepsilon_n/2^{n+1}) \mathbb{B} \subset \mathbb{C}^2 \setminus (2r_{n+1} - r_1) \mathbb{B} \subset \mathbb{C}^2 \setminus (r_{n+1} \mathbb{B})\). Thus (ii) holds with \(n\) replaced by \(n+1\).

By (6.11)

\[ (\Theta_{n+1} \circ A_n \circ g_n)(U_n \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n) \mathbb{B}. \]

If \(\zeta \in \Delta \setminus K_n\) then, by (7.3), \(p_{n+1}(\zeta) \in p_{n+1}(\Delta) \setminus K'_n \subset U_n \setminus K'_n\) so

\[ (\Theta_{n+1} \circ A_n \circ g_n)(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n) \mathbb{B}, \]

and since \(R_{n-1} + \theta_n + \varepsilon_n < R\) it follows by (7.6) that \(f_{n+1}(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_{n-1} \mathbb{B}\), that is, (iii) is satisfied.

Finally, (6.8) implies that

\[ (\Psi_{n+1} \circ \Theta_{n+1} \circ A_n)(\{|z| > S_{n+1}/2\}) \text{ misses } \Psi_{n+1}(2r_n + \varepsilon_n) \mathbb{B}. \]

Since \(2r_n + \varepsilon_n < R\), (7.6) implies that \(2r_n \mathbb{B} \subset \Psi_{n+1}(2r_n + \varepsilon_n) \mathbb{B}\) so \(A_{n+1}(\{|z| > S_{n+1}/2\}) \text{ misses } 2r_n \mathbb{B}\), that is, \(6.1''\) holds with \(n\) replaced by \(n+1\).
This completes the proof of the induction step.

Since the map \( \varphi \) is proper, (vii) and the fact that (3.1) holds for every \( n \) imply that \( r_n \to +\infty \) as \( n \to \infty \). The proof of Theorem 1.1 is complete. \( \square \)

8. Remarks

Theorem 1.1 holds with \( \mathbb{C}^2 \) replaced by \( \mathbb{C}^N, \ N \geq 2 \).

**Theorem 8.1** Let \( N \geq 2 \). Given a discrete set \( S \subset \Delta \) and a proper injection \( \varphi : S \to \mathbb{C}^N \) there is a proper holomorphic embedding \( f : \Delta \to \mathbb{C}^N \) that extends \( \varphi \).

If \( N \geq 3 \) then one proves Theorem 8.1. as in the case \( N = 2 \) with a slight modification: Let \( \iota : \mathbb{C}^2 \to \mathbb{C}^N \) be the standard embedding \( \iota(\zeta_1, \zeta_2) = (\zeta_1, \zeta_2, 0, \ldots, 0) \). In the proof we replace \( f_n = A_n \circ g_n \) by \( f_n = A_n \circ \iota \circ g_n \) where \( A_n \) is a holomorphic automorphism of \( \mathbb{C}^N \) and \( g_n \), as in the proof in the case \( N = 2 \), is a one to one and regular holomorphic map from an open neighbourhood \( U_n \) of \( \Delta \) to \( \mathbb{C}^2 \) which, for even \( n \) is transverse to \( \{(z, w) : |z| = S_n \} \) and satisfies \( g_n^{-1}(\{|z| = S_n\}) = b\Delta \), and for odd \( n \), is transverse to \( \{(z, w) : |w| = T_n \} \) and satisfies \( g_n^{-1}(\{|w| = T_n\}) = b\Delta \). Also, in the induction step, the maps \( \Theta_{n+1} \) and \( \Psi_{n+1} \) are automorphisms of \( \mathbb{C}^N \) and \( G_{n+1} \) is an automorphism of \( \mathbb{C}^2 \). In (6.1') and (6.1'') we replace \( A_n \) by \( A_n \circ \iota \).

We say that two proper holomorphic embeddings \( f_1, f_2 : \Delta \to \mathbb{C}^N \) are \( \text{Aut}(\mathbb{C}^N) \)-equivalent if there is an automorphism \( \Psi : \mathbb{C}^N \to \mathbb{C}^N \) such that \( f_2 = \Psi \circ f_1 \).

**Corollary 8.2** Let \( N \geq 2 \). The set of \( \text{Aut}(\mathbb{C}^N) \)-equivalence classes of proper holomorphic embeddedings of \( \Delta \) into \( \mathbb{C}^N \) is uncountable.

**Proof.** [BFo] It is known [RR, Remark 5.2] that there is an uncountable family \( E \) of discrete injective sequences in \( \mathbb{C}^N \) such that if \( \{z_n, \ n \in \mathbb{N}\} \) and \( \{w_n, \ n \in \mathbb{N}\} \) are distinct elements of \( E \) then there is no automorphism \( \Psi \) of \( \mathbb{C}^N \) such that \( \Psi(z_n) = w_n \ (n \in \mathbb{N}) \). Let \( \{\zeta_n\} \subset \Delta \) be an injective sequence, \( \lim_{n \to \infty} |\zeta_n| = 1 \), and let \( \{z_n, \ n \in \mathbb{N}\}, \{w_n, \ n \in \mathbb{N}\} \) be distinct elements of \( E \). By Theorem 8.1 there are proper holomorphic embeddings \( f_1, f_2 : \Delta \to \mathbb{C}^N \) such that \( f_1(\zeta_j) = z_j, f_2(\zeta_j) = w_j (j \in \mathbb{N}) \). Every automorphism \( \Psi \) of \( \mathbb{C}^N \) such that \( f_2 = \Psi \circ f_1 \) would have to satisfy \( \Psi(z_n) = w_n \ (n \in \mathbb{N}) \) and there is no such \( \Psi \). Thus, in this way, each element of \( E \) produces a proper holomorphic embedding of \( \Delta \) into \( \mathbb{C}^N \) and the embeddings associated with distinct elements of \( E \) are not \( \text{Aut}(\mathbb{C}^N) \)-equivalent. This completes the proof. \( \square \)

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