**L^p BOUNDS FOR THE FUNCTION OF MARCINKIEWICZ**

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1. Introduction

Let \( \Omega \) denote a homogeneous function of degree 0 on \( \mathbb{R}^n \) which is locally integrable and satisfies

\[
\int_{S^{n-1}} \Omega(y) d\sigma(y) = 0,
\]

where \( d\sigma \) represents the normalized Lebesgue measure on the unit sphere \( S^{n-1} \).

For \( n \geq 2 \) and \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), the Marcinkiewicz function of \( f \) is given by

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty \left| \int_{|y| \leq t} \frac{\Omega(y)}{|y|^{n-1}} f(x-y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.
\]

The above operator was introduced by E.M. Stein in [7] as an extension of the notion of Marcinkiewicz function from one dimension to higher dimensions. By using the \( L^p \) boundedness of the 1-dimensional Marcinkiewicz function, Stein showed that \( \mu_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) whenever \( \Omega \) is odd.

For a general kernel function \( \Omega \), the \( L^p \) boundedness of \( \mu_\Omega \) has been established under various conditions on \( \Omega \). For example, Stein proved that \( \mu_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p \leq 2 \) if \( \Omega \in \text{Lip}(S^{n-1}) \). Benedek, Calderón and Panzone proved in [2] that the \( L^p \) boundedness of \( \mu_\Omega \) holds for \( 1 < p < \infty \) under the condition that \( \Omega \in C^1(S^{n-1}) \).

In 1972 T. Walsh showed that the \( L^p \) boundedness of \( \mu_\Omega \) can still hold even if \( \Omega \) is quite rough.

**Theorem 1** (Walsh [11]). Suppose that \( p \in (1, \infty) \), \( r = \min\{p, p'\} \), and \( \Omega \in L((\log L)^{1/r}((\log \log L)^{2(1-2/r')}(S^{n-1})) \). Then \( \mu_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \).

When \( p = 2 \), the condition in Theorem 1 is simply \( \Omega \in L((\log L)^{1/2}(S^{n-1})) \), which was shown by Walsh to be optimal in the sense that the exponent 1/2 in \( L((\log L)^{1/2}) \) cannot be replaced by any smaller numbers.

On the other hand, Walsh did not consider his condition to be in any sense optimal when \( p \neq 2 \). Indeed, by comparing with the result of Calderón and Zygmund on singular integrals, one is naturally led to the question whether the condition \( \Omega \in L((\log L)^{1/2}(S^{n-1})) \) is also sufficient for the \( L^p \) boundedness of \( \mu_\Omega \) even when \( p \neq 2 \). This problem, which was formally proposed by Y. Ding in [4], is resolved by our next theorem.

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Theorem 2. If $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $p \in (1, \infty)$, then $\mu_{\Omega}$ is bounded on $L^p(\mathbb{R}^n)$.

The method employed in this paper is based in part on ideas from [1], [3], [5] and [10], among others. A great deal more can be obtained by applying variations of this scheme to more general integral operators of Marcinkiewicz type. An extensive discussion of these results will appear in a forthcoming paper.

Throughout the rest of the paper the letter $C$ will stand for a constant but not necessarily the same one in each occurrence.

2. The Main Lemma and Proof of Theorem 2

For a suitable family of measures $\tau = \{\tau_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$, we define the operators $\Delta_\tau$ and $\tau^*$ by

$$\Delta_\tau(f)(x) = \left(\int_{\mathbb{R}} |\tau_t * f(x)|^2 dt \right)^{1/2}$$

and

$$\tau^*(f)(x) = \sup_{t \in \mathbb{R}} |\tau_t |^* |f(x)|.$$

The following is our main lemma:

Lemma 3. Let $a \geq 2$, $A > 0$, $\gamma > 0$, $q > 1$ and $C_q > 0$. Suppose that the family of measures $\{\tau_t : t \in \mathbb{R}\}$ satisfies the following:

(i) $\|\tau_t\| \leq A$ for $t \in \mathbb{R}$;
(ii) $|\hat{\tau}_t(\xi)| \leq A\min\{a^t|\xi|, (a^t|\xi|)^{-1}\}\gamma / \ln a$ for $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$;
(iii) $\|\tau^*(f)\|_q \leq C_q A\|f\|_q$ for $f \in L^q(\mathbb{R}^n)$.

Then, for every $p$ satisfying $|1/p - 1/2| < 1/(2q)$, there exists a positive constant $C_p$ which is independent of $a$ and $A$ such that

$$\|\Delta_\tau(f)\|_p \leq C_p A\|f\|_p$$

for $f \in L^p(\mathbb{R}^n)$.

This lemma can be viewed as a continuous analogue of Theorem B in [5]. The novel feature, which keys its application to the current problem, is the uniformness of the bound on the operator norm with respect to the parameter $a$.

Proof of Theorem 2. Let $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and satisfy (1.1). For $k \in \mathbb{N}$ let $E_k = \{y \in S^{n-1} : 2^{k-1} \leq |\Omega(y)| < 2^k\}$ and

$$\Omega_k(y) = \Omega(y) \chi_{E_k}(y) - \int_{E_k} \Omega d\sigma.$$

Thus

$$\int_{S^{n-1}} \Omega_k(y)d\sigma(y) = 0$$
for $k \in \mathbb{N}$. Let $\Lambda = \{ k \in \mathbb{N} : \sigma(E_k) > 2^{-4k} \}$ and

$$\Omega_0 = \Omega - \sum_{k \in \Lambda} \Omega_k.$$ 

It then follows that $\Omega_0 \in L^2(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega_0(y) d\sigma(y) = 0.$$ 

For every $k \in \Lambda$ we define the family of measures $\tau^{(k)} = \{ \tau_{k,t} : t \in \mathbb{R} \}$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f d\tau_{k,t} = 2^{-kt} \int_{|y| \leq 2^kt} \frac{\Omega_k(y)}{|y|^{n-1}} f(y) dy.$$ 

If we set $a_k = 2^k, A_k = 2 \int_{E_k} |\Omega(y)| d\sigma(y)$ and $\gamma = \frac{\ln 2}{6}$, then the following holds for $t \in \mathbb{R}, \xi \in \mathbb{R}^n$, and $p > 1$:

$$\begin{align*}
(\text{i}) & \quad \| \tau_{k,t} \| \leq A_k, \\
(\text{ii}) & \quad |\hat{\tau}_{k,t}(\xi)| \leq A_k(a_k^t|\xi|)^{\gamma/\ln a_k}, \\
(\text{iii}) & \quad |\hat{\tau}_{k,t}(\xi)| \leq CA_k(a_k^t|\xi|)^{-\gamma/\ln a_k}, \\
(\text{iv}) & \quad \| (\tau^{(k)})^* \|_{p,p} \leq C_p A_k,
\end{align*}$$

where $C$ and $C_p$ are independent of $k$.

While (2.5.i) is obvious, (2.5.ii) follows immediately from (2.4) and (2.5.i). In addition, (2.5.iv) can be obtained in a straightforward manner (see, for example, Page 823 in [6]).

On the other hand, by the proof of Corollary 4.1 on P. 551 of [5],

$$|\hat{\tau}_{k,t}(\xi)| \leq C \| \Omega_k \|_2 (a_k^t|\xi|)^{-1/6}. \quad (2.6)$$

Thus, by (2.5.i), (2.6) and the inequality $\| \Omega_k \|_2 \leq 2^{2k+2} A_k$,

$$|\hat{\tau}_{k,t}(\xi)| \leq A_k^{(k-1)/k} [C 2^{2k+2} A_k(a_k^t|\xi|)^{-1/6}]^{1/k} \leq CA_k(a_k^t|\xi|)^{-\gamma/\ln a_k},$$

which proves (2.5.iii).

By Minkowski’s inequality,

$$\mu_\Omega(f) \leq \mu_{\Omega_0}(f) + \sum_{k \in \Lambda} (k \ln 2)^{1/2} \Delta_{\tau^{(k)}}(f). \quad (2.7)$$

Finally, by (2.5), (2.7), Theorem 1 and Lemma 3, we obtain

$$\| \mu_\Omega(f) \|_p \leq C_p \left( 1 + \sum_{k \in \Lambda} \sqrt{k} A_k \right) \| f \|_p \leq C_p (1 + \| \Omega \|_{L(\log L)^{1/2}}) \| f \|_p$$

for $1 < p < \infty$. The proof of Theorem 2 is now complete.
References


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