SOLUTION OF THE CONGRUENCE PROBLEM FOR ARBITRARY HERMITIAN AND SKEW-HERMITIAN MATRICES OVER POLYNOMIAL RINGS

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Abstract. Let * be the involutorial automorphism of the complex polynomial algebra $\mathbb{C}[t]$ which sends $t$ to $-t$. Answering a question raised by V.G. Kac, we show that every hermitian or skew-hermitian matrix over this algebra is congruent to the direct sum of $1 \times 1$ matrices and $2 \times 2$ matrices with zero diagonal. Moreover we show that if two $n \times n$ hermitian or skew-hermitian matrices have the same invariant factors, then they are congruent. The complex field can be replaced by any algebraically closed field of characteristic $\neq 2$.

1. Introduction

Let $R$ be the polynomial algebra $F[t]$ in one variable $t$ over an algebraically closed field $F$ of characteristic $\neq 2$. Let * denote the involution of $R$ which is the identity on $F$ and sends $t$ to $-t$. (We remark that all nontrivial $F$-involutions of $F[t]$ are conjugate in $\text{Aut}_F(F[t])$.) It induces the $\mathbb{Z}_2$-gradation $R = R_0 \oplus R_1$ of $R$ with $R_0 = F[t^2]$ and $R_1 = tR_0$. We shall refer to the elements of $R_0$ (resp. $R_1$) as even (resp. odd).

Let $M_n(R)$ denote the algebra of $n$ by $n$ matrices over $R$. If $A = (a_{ij}) \in M_n(R)$, we define $A^*$ to be the matrix $B = (b_{ij}) \in M_n(R)$ where $b_{ij} = a_{ji}^*$. Thus * is now made into an involution of $M_n(R)$. We say that $A \in M_n(R)$ is hermitian (resp. skew-hermitian) if $A^* = A$ (resp. $A^* = -A$). Two hermitian (resp. skew-hermitian) matrices $A,B \in M_n(R)$ are said to be congruent if $B = S^*AS$ for some $S \in \text{GL}_n(R)$.

Not long ago V.G. Kac [3] posed the following question to the first author (see also [1]).

If $F$ is the complex field, is it true that every hermitian or skew-hermitian matrix $A \in M_n(R)$ is congruent to the direct sum of $1 \times 1$ matrices and $2 \times 2$ matrices with zero diagonal?

Note that no condition is imposed on the determinant of $A$. (The usual restriction is that $A$ be unimodular.) The two cases, hermitian and skew-hermitian, of this problem are tightly linked because if $A$ is hermitian then $tA$ is skew-hermitian, and vice versa.
The main objective of our paper is to give an affirmative answer to Kac’s question (Theorem 4.3), which we find quite surprising. The case \( n = 2 \) is dealt with in Section 3 and the general case in Section 4. In Section 4, we also prove that two hermitian (or skew-hermitian) matrices \( A, B \in M_n(R) \) are congruent if and only if they have the same invariant factors (Theorem 4.5). Then, in Section 5, we are able to characterize the sequence of invariant factors of a hermitian or skew-hermitian matrix, and to give the canonical form under congruence for such a matrix. In the last section we make comments on other fields and characterize those for which Kac’s question has positive answer.

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2. Preliminaries

The elements \( a \in R \) are polynomials and so they can be evaluated at any point \( \lambda \in F \). We denote by \( a(\lambda) \) the value of \( a \) at \( \lambda \). We say that a nonzero element \( a \in R \) is pure if \( \gcd(a, a^*) = 1 \). If \( a, b \in R \) with \( a \) pure and \( b \) even (resp. odd), then there exists \( x \in R \) such that \( ax + a^*x^* = b \) (resp. \( ax - a^*x^* = b \)).

Since \( F \) is algebraically closed, each \( a \in R_0 \) can be written as \( a = bb^* \), \( b \in R \).

If \( I = Ra \) is a homogeneous (i.e., \(*\)-invariant) ideal of \( R \), then its generator \( a \) is also homogeneous, i.e., it is either even or odd. If \( A = (a_{ij}) \in M_n(R) \) is hermitian or skew-hermitian, then the ideal generated by all entries \( a_{ij} \) is \(*\)-invariant and we denote its generator by \( \gcd(A) \). Hence \( \gcd(A) \) is the first invariant factor of \( A \).

Let us fix a hermitian or skew-hermitian matrix \( A \in M_n(R) \). Let \( R^n \) denote the free \( R \)-module of rank \( n \) consisting of column vectors. We shall denote by \( e_1, \ldots, e_n \) the standard basis vectors of \( R^n \). The matrix \( A \) defines a hermitian or skew-hermitian form \( f_A : R^n \times R^n \to R \) by

\[
 f_A(v, w) = v^*Aw.
\]

By [2, Lemma 1], \( A \) is congruent to the direct sum of a zero matrix and a hermitian or skew-hermitian matrix with nonzero determinant. (The proof given there in the hermitian case is also valid in the skew-hermitian case.) This argument shows that it suffices to consider only the hermitian or skew-hermitian matrices with nonzero determinant.

As \( F \) is algebraically closed, if \( n \geq 2 \) there exist nonzero isotropic vectors, i.e., nonzero vectors \( v \in R^n \) such that \( f_A(v, v) = 0 \). At the referee’s suggestion, we include a proof. Clearly, it suffices to consider the hermitian case (otherwise replace \( A \) by \( tA \)). We may also assume that \( n = 2 \). Then let

\[
 A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},
\]

where \( a, c \in R \) and \( b \in R_0 \). If \( b \) is odd, then \( A \) is skew-hermitian and \( \gcd(A) = b \). If \( b \) is even, then \( A \) is hermitian and \( \gcd(A) = a \). In either case, \( \gcd(A) \neq 0 \) and \( A \) is congruent to a matrix with nonzero determinant.
and we may assume that \( a \neq 0 \). As \( bb^* - ac \) is even, it can be written as \( dd^* \) for some \( d \in R \). Then \((d - b)e_1 + ac_2\) is a nonzero isotropic vector.

Assume that \( \det(A) \neq 0 \) and set \( d = \gcd(A) \). Then \( A = dB \) for some matrix \( B \in M_n(R) \) such that \( B^* = \pm B \) and \( \gcd(B) = 1 \). Therefore, without any loss of generality, we may assume that \( \det(A) \neq 0 \) and \( \gcd(A) = 1 \).

**3. The case \( n = 2 \)**

In this section we show that the answer to Kac’s question is affirmative if \( n = 2 \). We start with the hermitian case.

**Proposition 3.1.** If \( A = A^* \in M_2(R) \), \( \det(A) \neq 0 \), and \( \gcd(A) = 1 \), then \( A \) is congruent to \( \text{diag}(1, \det(A)) \).

**Proof.** Since there exist nonzero isotropic vectors, we may assume that

\[
A = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix}.
\]

The element \( a_0 = \gcd(a, a^*) \) is homogeneous, i.e., \( a_0^* = \pm a_0 \). We have a factorization \( a = a_0a_1 \) where \( a_1 \) is pure. By the hypothesis, \( \gcd(a_0, b) = 1 \). Consequently, there exist homogeneous elements \( x \) and \( y \), with \( y \) even, such that \( a_0x + by = 1 \). Clearly we may assume that \( y(0) \neq 0 \). Choose a factorization \( y = zz^* \) such that \( a_1z \) is pure. Then there exists \( w \in R \) such that

\[
a_1zw + a_1^*z^*w^* = 1.
\]

Since \( a_0x \) is even, we find that

\[
1 = a_0x + by
= a_0x(a_1zw + a_1^*z^*w^*) + bzz^*
= axwz + a^*x^*w^*z^* + bzz^*
= f_A(x^*w^*e_1 + ze_2, x^*w^*e_1 + ze_2).
\]

The assertion of the proposition is now obvious.

Next we consider the skew-hermitian case. We shall need the following simple lemma.

**Lemma 3.2.** Let \( a, b \in R \) satisfy \( \gcd(a, b) = \gcd(b, b^*) = 1 \). Then there exist \( x, y \in R \), with \( x \) even, such that \( ax + by = 1 \).

**Proof.** Choose \( u, v \in R \) such that \( au + bv = 1 \) and \( z \in R \) such that \( bz - b^*z^* = u^* - u \). Then \( x = u + bz \in R_0 \) and \( y = v - az \) satisfy \( ax + by = 1 \).

**Proposition 3.3.** Let \( A \in M_2(R) \), \( A^* = -A \), \( \det(A) \neq 0 \), and \( \gcd(A) = 1 \). Then \( A \) is congruent to a matrix with zero diagonal.
Proof. We may assume that 
\[ A = \begin{pmatrix} 0 & a \\ -a^* & b \end{pmatrix}. \]
As \( a(0) \neq 0 \), we can write \( a = a_1cc^* \) with \( a_1c \) pure. By Lemma 3.2, there exist \( x, d \in R \), with \( x \) even, such that 
\[ bx + cd = 1. \]
By replacing \((x,d)\) with \((x + \lambda cc^*, d - \lambda bc^*)\), where \( \lambda \in F \) is suitably chosen, we may assume that \( \gcd(a_1, x) = 1 \). Since \( b \) is odd, we can choose \( w \in R \) such that 
\[ c^*w - cw = b. \]
Since \( \gcd(a_1, cx) = 1 \), there exist \( v, p \in R \) such that 
\[ a_1^*v - cxp = xw^*-d^*. \]
Choose \( q \in R \) such that 
\[ a_1^*q - a_1q^* = p - p^* \]
and set 
\[ y = v + cxq, \quad z = w + c^*(p + a_1q^*). \]
Then 
\[ c^*z^* - cz = c^*w^* + cc^*(p^* + a_1q) - cw - cc^*(p + a_1q^*) \]
\[ = c^*w^* - cw = b, \]
\[ xz^* - a_1^*y = xw^* + cx(p^* + a_1q) - a_1^*v - a_1^*cxq \]
\[ = cxp^* + xw^* - a_1^*v = d^*. \]
Hence 
\[ a_1^*c^*y = c^*xz^* - c^*d^* = c^*xz^* - (1 + bx) \]
\[ = x(c^*z^* - b) - 1 = cxz - 1, \]
and so 
\[ S = \begin{pmatrix} cx \\ a_1^*c^* \\ y \\ z \end{pmatrix} \in SL_2(R). \]
We have 
\[ S^* \begin{pmatrix} 0 & a_1c^2 \\ -a_1^*(c^*)^2 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & r \\ -r^* & s \end{pmatrix}, \]
where 
\[ r = a_1cc^*(cxz - a_1^*c^*y) = a_1cc^* = a, \]
\[ s = a_1c^2y^*z - a_1^*(c^*)^2yz^* \]
\[ = a_1cy^* \cdot cz - a_1^*c^*y \cdot c^*z^* \]
\[ = (c^*xz^* - 1)cz - (cxz - 1)c^*z^* \]
\[ = c^*z^* - cz = b. \]
Hence 
\[ \begin{pmatrix} 0 & r \\ -r^* & s \end{pmatrix} = A. \]
4. Equivalence implies congruence

The first main theorem is a simple consequence of the following two propositions. The first one deals with hermitian matrices.

**Proposition 4.1.** If \( A = (a_{ij}) \in M_n(R), \quad A^* = A \) and \( \gcd(A) = 1 \), then there exists \( v \in R^n \) such that \( f_A(v, v) = 1 \).

**Proof.** We may assume that \( \det(A) \neq 0 \). The proof will be by induction on \( n \). The case \( n = 1 \) is obvious. For the case \( n = 2 \) see Proposition 3.1. Thus let \( n > 2 \).

Since there exist nonzero isotropic vectors, we may assume that \( a_{11} = 0 \). Moreover, we may assume that \( a_{1j} = 0 \) for \( j < n \). Denote by \( E_{ij} \) the matrix of order \( n \) whose \((i,j)\)-th entry is 1 and all other entries are 0, and by \( I_n \) the identity matrix. For any \( \lambda \in F \), the matrix \( A_\lambda = (I_n + \lambda E_{21})A(I_n + \lambda E_{12}) \) is congruent to \( A \). Let \( A'_\lambda \) denote the submatrix of \( A_\lambda \) obtained by deleting the first row and column. Set \( A'_0 = A'_\lambda \). Note that for \( \lambda, \mu \in F \) we have

\[
A'_\lambda - A'_\mu = (\lambda - \mu)(a_{1n}E_{1,n-1} + a_{n1}E_{n-1,1}),
\]

where now \( E_{ij} \)'s have order \( n - 1 \).

As \( \det(A) \neq 0 \), we have \( a_{rs} \neq 0 \) for some \( r, s \in \{2, 3, \ldots, n-1\} \). Since \( a_{rs} \) has only finitely many monic divisors, there exist \( \lambda, \mu \in F \), with \( \lambda \neq \mu \), such that \( \gcd(A'_\lambda) = \gcd(A'_\mu) \). Denote this common gcd by \( d \). The displayed formula for \( A'_\lambda - A'_\mu \) shows that \( d \) divides \( a_{1n} \) (and \( a_{n1} \)). It also divides all entries of \( A' \). Since \( a_{1j} = 0 \) for \( j < n \), it follows that \( d \) divides all entries of \( A \). As \( \gcd(A) = 1 \), we conclude that \( d = 1 \).

We have shown that \( \gcd(A'_\lambda) = 1 \) for some \( \lambda \in F \). By the induction hypothesis there exists \( w \in R^{n-1} \) such that \( f_{A'_\lambda}(w, w) = 1 \). As \( A \) and \( A_\lambda \) are congruent, there exists \( v \in R^n \) such that \( f_A(v, v) = 1 \).

The second proposition is a skew-hermitian analog of the first one. We shall need the following definition. Let \( \nu_A \) denote the minimum degree of nonzero polynomials \( f_A(v, w) \) over all \( v, w \in R^n \) with \( f_A(v, v) = 0 \).

**Proposition 4.2.** If \( A = (a_{ij}) \in M_n(R), \quad A^* = -A \) and \( \gcd(A) = 1 \), then \( A \) is congruent to the direct sum \( B \oplus D \), where

\[
B = \begin{pmatrix}
0 & f \\
-f^* & 0
\end{pmatrix},
\]

with \( f \) pure of degree \( \nu_A \). Furthermore, \( ff^* \) divides all entries of \( D \), i.e., \( \det(B) \) is the second invariant factor of \( A \).

**Proof.** For the case \( n = 2 \) see Proposition 3.3. Thus let \( n > 2 \).

After a suitable change of basis, we may assume that \( f_A(e_1, e_1) = 0 \) and that there exists \( w \in R^n \) such that \( f_A(e_1, w) \) is nonzero and has degree \( \nu_A \). Thus
$a_{11} = 0$. By performing some additional congruence transformations, we may also assume that $a_{12} \neq 0$ has degree $\nu_A$ and that $a_{1j} = 0$ for $j > 2$.

Denote by $A$ the set of all skew-hermitian matrices $X = (x_{ij}) \in M_n(R)$ which are congruent to $A$ and such that $x_{1j} = 0$ for $j \neq 2$ while $x_{12}$ has degree $\nu_A$. For $X \in A$ let $d_X = \gcd(x_{12}, x_{21}, x_{22})$ where we require $d_X$ to be monic. Let $A_0$ denote the set of all $X \in A$ such that $d_X$ has the minimum degree. Without any loss of generality, we assume that $A \in A_0$.

Our first objective is to show that $d_A$ is 1 or $t$. Let $2 \leq r < s \leq n$ and for $x \in R$ define $A_x \in A$ by

$$A_x = (I_n + x^* E_{rs}) A (I_n + x E_{sr})$$

and set $d_x = d_{A_x}$. For $\lambda \in F$, the $(r, r)$-th entry of $A_{\lambda x}$ is

$$a_{rr} + \lambda (a_{rs} x^* - a_{sr}^* x) + \lambda^2 a_{ss} xx^*.$$

We take first $r = 2$. As $a_{12}$ has only finitely many monic divisors, we can choose distinct $\alpha, \beta, \gamma \in F$ such that $d_{\alpha x} = d_{\beta x} = d_{\gamma x}$. Denote this common gcd by $d$. As the Vandermonde determinant of $\alpha, \beta, \gamma$ is not 0, $d$ must divide $a_{22}$, $a_{2s} x^* - a_{2s}^* x$ and $a_{ss} xx^*$. It follows that $d$ divides $d_A$, and consequently we must have $d = d_A$. By taking $x = 1$, we infer that $d_A$ divides the diagonal entries of $A$. As $d_A$ divides $a_{2s} x^* - a_{2s}^* x$ for all $x \in R$, we deduce that $d_A$ divides $a_{2s}$.

Next we take $r > 2$. Since $d_A$ must divide (4.2) for all $\lambda \in F$ and $x \in R$, we infer that $d_A$ divides $a_{rs} x^* - a_{rs}^* x$ for all $x \in R$. Consequently, $d_A$ divides $a_{rs}$. As $\gcd(d) = 1$, it follows that $d_A$ is either 1 or $t$.

We shall now rule out the possibility $d_A = t$. Suppose that $d_A = t$. Assume that all entries $a_{2j}$ are divisible by $t$. In the above construction, we take once again $r = 2$ and choose $s > 2$ such that $a_{sk}$ is not divisible by $t$ for some $k > 1$ and $k \neq s$. As above, we can choose a nonzero $\lambda \in F$ such that $d_{\lambda x} = d_A$. Then the $(2, k)$-th entry of $A_{\lambda x}$ is not divisible by $t$. Hence we can assume that one of the entries in the second row, say $a_{23}$, is not divisible by $t$.

The $3 \times 3$ submatrix in the upper left hand corner of $A$ has the form

$$
\begin{pmatrix}
0 & at^k & 0 \\
-a^*(-t)^k & bt & c \\
0 & -c^* & d
\end{pmatrix},
$$

where $a, b, c$ are not divisible by $t$, $k \geq 1$, $\gcd(a, a^* b) = 1$, and the degree of $at^k$ is equal to $\nu_A$. Moreover, by using Proposition 3.1, we may also assume that $a$ is pure. Hence we can choose $x \in R$ such that $ax^* + a^* x = (-1)^k b$. Then the vector $v = xe_1 + t^{k-1} e_2$ is isotropic and

$$f_A(v, e_1) = a^* t^{2k-1}, \quad f_A(v, e_3) = c(-t)^{k-1}.$$

Hence there exists $w \in R^a$ such that

$$f_A(v, w) = t^{k-1} \gcd(a^*, c),$$

contradicting the fact that $at^k$ has degree $\nu_A$. We conclude that $d_A = 1$. 
It is now easy to finish the proof. By Proposition 3.3, we may assume that the $2 \times 2$ submatrix $B$ in the upper left hand corner of $A$ has the form (4.1) with $f$ pure. From the definition of $\nu_A$ it follows that the entries $a_{1j}$, $j > 2$, are divisible by $f$. By performing suitable elementary congruence transformations, we may assume that all these entries are 0. A similar argument can be used to make the entries $a_{2j} = 0$ for $j > 2$.

Thus we have $A = B \oplus D$ where $D = (d_{ij})$. Replace the zero in the $(2,2)$ position of $A$ by $-d_{ii}$. It follows from Proposition 3.3 that this change can be achieved by a congruence transformation on the block $B$. Now add the $(i+2)$-nd row of $A$ to the second row and then the $(i+2)$-nd column to the second column. The entry in the $(2,2)$ position will become 0 again. From the definition of $\nu_A$ it follows that the $(2,j+2)$-nd entry of this new matrix must be divisible by $f^*$. As this entry is equal to $d_{ij}$, we conclude that all entries of $D$ are divisible by $f^*$. As $D^* = -D$, they are also divisible by $f$. As $f$ is pure, all entries of $D$ are divisible by $ff^*$.

We are now able to answer Kac’s question.

**Theorem 4.3.** If $A \in M_n(R)$ is hermitian or skew-hermitian, then $A$ is congruent to the direct sum of $1 \times 1$ matrices and $2 \times 2$ matrices with zero diagonal.

**Proof.** As observed in Section 2, we may assume that $\det(A) \neq 0$ and $\gcd(A) = 1$. We already know that the theorem is true if $n \leq 2$. It remains to use induction and apply the Propositions 4.1 and 4.2. □

To prove our second main result, we need the following simple lemma.

**Lemma 4.4.** Let $A, B \in M_2(R)$ be skew-hermitian, $\gcd(A) = \gcd(B) = 1$, and $\det(A) = \det(B) \neq 0$. Then $A$ and $B$ are congruent.

**Proof.** By Proposition 3.3, we may assume that

$$A = \begin{pmatrix} 0 & ab \\ -a^*b^* & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & ab^* \\ -a^*b & 0 \end{pmatrix},$$

with $ab$ and $ab^*$ pure. There exist $x, y \in R_0$ such that $bb^*x - aa^*y = 1$. If

$$S = \begin{pmatrix} b^*x & ay \\ a^* & b \end{pmatrix},$$

then $S \in SL_2(R)$ and $S^*BS = A$. □

Recall that two matrices $A, A' \in M_n(R)$ are said to be equivalent if there exist $S, T \in GL_n(R)$ such that $A' = SAT$. A necessary and sufficient condition for $A$ and $A'$ to be equivalent is that they have the same invariant factors.

**Theorem 4.5.** Let $A, A' \in M_n(R)$ be both hermitian or both skew-hermitian. If $A$ and $A'$ are equivalent, then they are congruent.
Proof. We use induction on $n$. The case $n = 1$ is trivial. Let $n > 1$. Denote the invariant factors of $A$ (and $A'$) by $f_1, \ldots, f_n$. If $f_n = 0$ then we can use the induction hypothesis. Assume that $f_n \neq 0$. By dividing $A$ and $A'$ by $f_1$, we may assume that $f_1 = 1$.

Now if $A$ and $A'$ are hermitian (resp. skew-hermitian) then Proposition 4.1 (resp. Proposition 4.2 and Lemma 4.4) allows us to finish the proof by using the induction hypothesis.

We shall give more details in the skew-hermitian case. By Proposition 4.2 we may assume that $A = B \oplus D$, where $B$ and $D$ are as stated there. Similarly, we may assume that $A' = B' \oplus D'$. Since $\det(B) = f_2 = \det(B')$, Lemma 4.4 implies that $B$ and $B'$ are congruent. Since $D$ and $D'$ have the same invariant factors, they are congruent by the induction hypothesis. Hence $A$ and $A'$ are congruent.

5. Canonical form under congruence

In the next theorem we characterize the invariant factors of hermitian and skew-hermitian matrices. Clearly these factors have to be homogeneous.

**Theorem 5.1.** Let $0 \leq r \leq n$. Let $f_1, \ldots, f_n$ be a sequence of homogeneous elements in $R$ such that $f_1, \ldots, f_r$ are monic, each dividing the next one, and $f_{r+1}, \ldots, f_n$ are zero. Then this sequence is the list of invariant factors of a hermitian (resp. skew-hermitian) matrix $A \in M_n(R)$ of rank $r$ if and only if the following two conditions hold:

(i) Any maximal subsequence $f_i, f_{i+1}, \ldots, f_j$ consisting of consecutive nonzero odd (resp. even) elements has even length. We shall write such subsequence as $g_i, h_i, g_{i+2}, h_{i+2}, \ldots, g_{j-1}, h_{j-1}$.

(ii) For each $(g_k, h_k)$ as above, $h_k = g_k p_k p_k^*$ with $p_k$ pure.

**Proof.** We prove necessity by induction on $n$. The cases $r = 0$ and $n = 1$ are trivial. Let $r \geq 1$ and $n \geq 2$. By replacing $A$ with $f_1^{-1} A$, we may assume that $f_1 = 1$.

If $A$ is hermitian, then Proposition 4.1 shows that $A$ is congruent to $(1) \oplus B$ and we can apply the induction hypothesis to $B$ to finish the proof.

If $A$ is skew-hermitian, then $A$ is congruent to the matrix $B \oplus D$ as stated in Proposition 4.2. In particular $f_2 = \det(B)$ is even and not divisible by $t$. We can now finish the proof by applying the induction hypothesis to $D$.

Sufficiency can be read off from the next theorem. \qed

It is now easy to obtain the canonical forms for hermitian and skew-hermitian matrices under congruence.

**Theorem 5.2.** Let $A \in M_n(R)$ and $A^* = \varepsilon A$, where $\varepsilon = \pm$, let $r$ be the rank of $A$, and let $f_1, \ldots, f_n$ be the invariant factors of $A$. Form the direct sum, $B$, of the following blocks:
(i) The $1 \times 1$ matrices $(f_i)$ for each $f_i$ such that $f_i^* = \varepsilon f_i$.
(ii) The $2 \times 2$ matrices
\[
g_k \begin{pmatrix} 0 & p_k \\ -p_k^* & 0 \end{pmatrix},
\]
for each pair $(g_k, h_k = g_k p_k p_k^*)$ constructed in the previous theorem.

Then $A$ is congruent to $B$. Moreover such $B$ is unique up to the ordering of the diagonal blocks and the factorizations $h_k = f_k p_k p_k^*$.

Proof. The matrices $A$ and $B$ have the same invariant factors.

6. Comments on other fields

We introduce four conditions on a field $F$ assuming only that the characteristic is not 2.

(K) Kac’s question has affirmative answer for the field $F$.
(N) The norm map $R \to R_0$ sending $x \to xx^*$ is onto.
(U) The quadratic form $x^2 - ty^2$ over $R$ is universal.
(I) No element of $R_0$ is irreducible in $R$.

Proposition 6.1. For a field $F$ of characteristic $\neq 2$, the above four conditions are equivalent to each other.

Proof. (K) $\Rightarrow$ (N). Let $\alpha \in F$, $\alpha \neq 0$. As
\[
A = \begin{pmatrix} -\alpha t & \alpha \\ -\alpha & t \end{pmatrix}
\]
is skew-hermitian but not diagonalizable, it must be congruent to
\[
\begin{pmatrix} 0 & x \\ -x^* & 0 \end{pmatrix}
\]
for some $x \in R$. Hence $\det(A) = -\alpha(t^2 - \alpha)$ splits over $F$. We deduce that $F$ is quadratically closed, i.e., it has no quadratic extensions.

It remains to show that if $a = 1 + t^2 b$, with $b \in R_0$, then $a = xx^*$ for some $x \in R$. This follows by applying the above argument to
\[
\begin{pmatrix} tb & 1 \\ -1 & t \end{pmatrix}.
\]

(N) $\Rightarrow$ (K). Our proofs are valid under this weaker hypothesis.

(N) $\Rightarrow$ (U). Let $\sigma : R \to R_0$ be the isomorphism of $F$-algebras sending $t$ to $t^2$. For $b \in R$ we have $\sigma(b) = zz^*$ for some $z \in R$. By writing $z = \sigma(x) + t \sigma(y)$, $(x, y \in R)$, we obtain $\sigma(b) = \sigma(x)^2 - t^2 \sigma(y)^2$, i.e., $b = x^2 - ty^2$.

(U) $\Rightarrow$ (N). For $a \in R_0$ we have $a = \sigma(b)$ with $b \in R$. As $b = x^2 - ty^2$ for some $x, y \in R$, we have $a = zz^*$ with $z = \sigma(x) + t \sigma(y)$.

It is obvious that (N) implies (I) and the converse is not hard to prove. \qed
One can construct examples of fields $F$ satisfying the above conditions without being algebraically closed. Start with a finite Galois extension $K/E$ whose Galois group is not a 2-group. Let $\sigma$ be an $E$-automorphism of an algebraic closure $\overline{K}$ of $K$ whose restriction to $K$ is nontrivial and has odd order. Then one can take $F$ to be the quadratic closure of $(\overline{K})^\sigma$. In particular the quadratic closure of the prime field $\mathbb{F}_p$ ($p$ odd) is an example. On the other hand, it is easy to see that the quadratic closure of the rationals does not satisfy the condition (U).

In general, a hermitian or skew-hermitian matrix $A \in M_n(R)$ need not be congruent to the direct sum of any $1 \times 1$ or $2 \times 2$ matrices. For instance, this is the case when $F$ is the real field and

$$A = \begin{pmatrix}
t^2 & 1 & 0 \\
1 & t^2 & t \\
0 & -t & t^2
\end{pmatrix}.$$ 

References


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