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TOTAL DEGREE BOUNDS FOR ARTIN L-FUNCTIONS AND PARTIAL ZETA FUNCTIONS

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0. Introduction

Let $\mathbf{F}_q$ be a finite field of characteristic p with q elements. Choose an algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. Throughout this paper, schemes, morphisms and sheaves defined over the base field $\mathbf{F}_q$ are denoted by letters with subscripts 0. We indicate the base extension from $\mathbf{F}_q$ to $\mathbf{F}$ by dropping the subscripts 0. Schemes and morphisms are separated and of finite type.

Let X_0 be a scheme over $\mathbf{F}_q$ and let $F : X \to X$ be the geometric Frobenius correspondence. For any endomorphism $f : X \to X$, denote by $\Lambda(f)$ the number of fixed points of f, that is,

$$\Lambda(f) = \#\{x \in X(\mathbf{F}) | f(x) = x\}.$$

Let G be a finite group acting on X_0 on the right and let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional $\overline{\mathbf{Q}}_l$-representation of G, where l is a prime number different from p. For each positive integer k, set

$$v_k = \frac{1}{\#G} \sum_{g\in G} \mathrm{Tr}(\rho(g)^{-1})\Lambda(gF^k).$$

The Artin L-function $L(X_0, \rho, t)$ is defined by

$$L(X_0, \rho, t) = \exp\left(\sum_{k=1}^{\infty} v_k \frac{t^k}{k}\right).$$

By a well known theorem of Grothendieck ([G]), this is a rational function. Define its total degree $\mathrm{totdeg}L(X_0, \rho, t)$ to be the sum of the number of its zeros and the number of its poles counted with multiplicities. In [BS], under the assumption that X_0 is a closed subscheme of the n-dimensional affine space $\mathbf{A}_0^n$ defined by the vanishing of polynomials of degrees at most d and G acts linearly on X_0 in the sense that $G \subset \mathrm{GL}(\mathbf{A}_0^n)$, Bombieri and Sperber prove that

$$\mathrm{totdeg}L(X_0, \rho, t) \le (\dim\rho)^2 (4d + 9)^{4n}.$$

If the action of G on X_0 is not linear, an explicit total degree bound can also be derived from the remark in [BS], but the resulting bound would depend on the order of the group G. As indicated in [BS], in some special cases, better

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bounds are known and these improved bounds are of considerable importance in estimating character sums, because they lead to estimates independent of the order of G.

In this paper, we give a bound for $\text{totdeg}L(X_0, \rho, t)$ without assuming G acts linearly on X_0. Our bound is independent of the order of G and hence is important for estimating character sums. Precisely, we have

Theorem 0.1. *Let X_0 be a closed subscheme of $\mathbf{A}_0^n$ defined by the vanishing of r polynomials of degrees at most d. Let G be a finite group acting on X_0 on the right and let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional $\overline{\mathbf{Q}}_l$-representation. Let $V = n_1V_1 \oplus \cdots \oplus n_kV_k$ be the decomposition of V into a direct sum of irreducible representations of G, where each n_j is the multiplicity of the irreducible representation V_j appeared in V. Then we have*

$$\text{totdeg}L(X_0, \rho, t) \leq 3 \cdot 2^{r+1} \cdot (3 + rd)^{n+1} \cdot (\sup_j \{\frac{n_j}{\dim V_j}\}).$$

Obviously we have $\sup_j\{\frac{n_j}{\dim V_j}\} \leq \dim\rho$. So we have

Corollary. *Notation as in Theorem 0.1. We have*

$$\text{totdeg}L(X_0, \rho, t) \leq 3 \cdot 2^{r+1} \cdot (3 + rd)^{n+1} \cdot (\dim\rho).$$

The main idea of our proof is to use Katz's bound on the sum of Betti numbers ([K1]) and a fixed point formula. Originally we only get the estimate in the Corollary. The estimate in Theorem 0.1 is suggested by the referee.

We now describe our second result. Let X_0 be a closed subscheme of $\mathbf{A}_0^n$ defined by the vanishing of some polynomials. Let $d_1, \ldots, d_n$ be n positive integers and let $N_{d_1\ldots d_n}(k, X_0)$ be the number of points $(x_1, \ldots, x_n)$ in $X(\mathbf{F})$ such that $x_1 \in \mathbf{F}_{q^{kd_1}}, \ldots, x_n \in \mathbf{F}_{q^{kd_n}}$. In [W1], Wan defines the partial zeta function $Z_{d_1\ldots d_n}(X_0, t)$ to be

$$Z_{d_1\ldots d_n}(X_0, t) = \exp\bigg(\sum_{k=1}^{\infty} N_{d_1,\ldots d_n}(k, X_0)\frac{t^k}{k}\bigg).$$

He proves that $Z_{d_1\ldots d_n}(X_0, t)$ is rational if the integers d_i's can be arranged such that $d_1|d_2|\cdots|d_n$, generalizing Dwork's classical rationality theorem. In general, it is not clear if the partial zeta function is always rational, but Faltings [W1] noted that the partial zeta function is always nearly rational in the sense that there are some roots of unity λ_k and some algebraic numbers μ_k $(k = 1, \ldots, m)$ such that

$$Z_{d_1\ldots d_n}(X_0, t) = \prod_{k=1}^{m} (1 - \mu_k t)^{\lambda_k}.$$

In the above expression, by collecting similar terms, we may also assume that the μ_k $(k = 1, \ldots, m)$ are distinct, but then the λ_k may not be roots of unity anymore. In this case, we define the total degree of $Z_{d_1\ldots d_n}(X_0, t)$ to be $\sum_{k=1}^{m} |\lambda_k|$. Here we fix an archimedean absolute value on the field of algebraic numbers.

Theorem 0.2. *Let X_0 be a closed subscheme of $\mathbf{A}_0^n$ defined by the vanishing of r polynomials of degrees at most d. Let $d_1, \ldots, d_n$ be n positive integers and let m be their least common multiple. Then we have*

$$\mathrm{totdeg} Z_{d_1 \ldots d_n}(X_0, t) \leq 3 \cdot 2^{mr+1} \cdot \left(3 + mrd\right)^{d_1 + \cdots + d_n + 1}.$$

The main idea of the proof of Theorem 0.2 is again to use Katz's bound on the sum of Betti numbers, an observation of Faltings, and a fixed point formula. As an immediate application of Theorem 0.2, one obtains an explicit total degree bound for the geometric moment zeta functions [W5] (which are rational functions) attached to a family of algebraic varieties over finite fields. In particular, one obtains explicit estimates for the constants in the higher degree excess theorems in Katz [K2] in his statistical study of the universal family of smooth projective hypersurfaces over finite fields. In some special cases, Theorem 0.2 can be greatly improved, especially about the asymptotic behavior of the bound as m goes to infinity while some of the d_i's are fixed. The universal family of elliptic curve and its relations to dimensions of modular forms of high weights provide such an example. Another interesting case to consider is the higher moment zeta function arising from the universal family of smooth projective hypersurfaces.

As an archimedean analogue of Dwork's p-adic unit root zeta functions, certain pure weight L-functions arising from a family of algebraic varieties were introduced in [W2]. The rationality of these pure weight L-functions depends on the full strength of Deligne's main theorem [D] which says that the l-adic higher direct image sheaves with compact support are mixed. We do not yet know how to give an explicit total degree bound for these deeper pure weight L-functions. The "degree bound" problem, that is, the lower bound problem for the p-adic Newton polygon, for the pure (p-adic) slope L-functions in Dwork's original conjecture is even more difficult, as these pure slope L-functions [W3][W4] are p-adic meromorphic functions and in general not rational any more. In the so-called rank one case, an explicit lower bound for the p-adic Newton polygon has been worked out in [W4]. No non-trivial bound is known in the higher rank case.

Although our principal motivation of this paper comes from theoretical considerations, we have also been influenced by practical algorithmic computations of zeta functions and L-functions. Our explicit bounds are useful in explicitly computing the related zeta functions and L-functions. This is an emerging new subject, which we shall not discuss it here, but see [W6] for an expository introduction.

1. Proof of the Theorems

We need the following result of Katz ([K1], Corollary of Theorem 1):

Proposition 1.1 (Katz)**.** *Let X be a closed subscheme of $\mathbf{A}^n$ over $\mathbf{F}$ defined by the vanishing of r polynomials of degrees at most d. We have*

$$\sum_{i=0}^{2\mathrm{dim}X} \mathrm{dim}H_c^i(X, \overline{\mathbf{Q}}_l) \leq 3 \cdot 2^{r+1} \cdot (3+rd)^{n+1}.$$

We also need the following fixed point formula:

Proposition 1.2. *Let X_0 be a quasi-projective scheme over $\mathbf{F}_q$. For any $\mathbf{F}_q$-automorphism $\sigma_0 : X_0 \to X_0$ of finite order and for any positive integer k, we have*

$$\Lambda(\sigma F^k) = \sum_{i=0}^{2\mathrm{dim}X} (-1)^i \mathrm{Tr}(\sigma F^k, H_c^i(X, \overline{\mathbf{Q}}_l)),$$

where $F : X \to X$ is the geometric Frobenius correspondence and $\Lambda(\sigma F^k)$ is the number of fixed points of $\sigma F^k : X \to X$.

To prove this fixed point formula, we use Galois decent theory to show that there exists another scheme X_0' over $\mathbf{F}_q$ such that X' is isomorphic to X over $\mathbf{F}$ and the geometric Frobenius correspondence on X' coincides with σF on X. Proposition 1.2 for the case $k = 1$ then follows from the fixed point formula for the geometric Frobenius correspondence on X'. The general case can be proved by working over the base $\mathbf{F}_{q^k}$.

Another result we need is the following well-known fact:

Lemma 1.3. *Let G be a finite group, $\rho : G \to GL(V)$ a finite dimensional $\overline{\mathbf{Q}}_l$-representation, and $F : V \to V$ a G-equivariant endomorphism of V. Then we have*

$$\mathrm{Tr}(F, V^G) = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(gF, V).$$

Proof. Note that since F is G-equivariant, the subspace V^G is invariant under the action of F. So we may talk about $\mathrm{Tr}(F, V^G)$. Consider the homomorphism

$$\pi : V \to V, x \mapsto \frac{1}{\#G} \sum_{g \in G} gx.$$

Then we have $\mathrm{im}(\pi) = V^G$ and $\pi|_{V^G} = \mathrm{id}$. So we have

$$\mathrm{Tr}(F, V^G) = \mathrm{Tr}(\pi F, V^G) = \mathrm{Tr}(\pi F, V) = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(gF, V).$$

Let's prove Theorem 0.1. Let G be a finite group acting on the right of a quasi-projective scheme X_0 over $\mathbf{F}_q$, let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional $\overline{\mathbf{Q}}_l$-representation of G and let

$$L(X_0, \rho, t) = \exp\left(\sum_{k=1}^{\infty} v_k \frac{t^k}{k}\right)$$

be the Artin L-function, where

$$v_k = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(g^{-1}, V)\Lambda(gF^k).$$

By Proposition 1.2, we have

$$\begin{aligned} v_k &= \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(g^{-1}, V)\Lambda(gF^k) \\ &= \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(g^{-1}, V) \sum_{i=0}^{2\mathrm{dim}X} (-1)^i \mathrm{Tr}(gF^k, H_c^i(X, \overline{\mathbf{Q}}_l)) \\ &= \sum_{i=0}^{2\mathrm{dim}X} (-1)^i \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}(gF^k, H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*), \end{aligned}$$

where V^* denotes the representation of G dual to V and F acts trivially on V^*. Combining with Lemma 1.3 applied to the representation $H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*$ of G and the equivariant endomorphism F^k, we get

$$v_k = \sum_{i=0}^{2\mathrm{dim}X} (-1)^i \mathrm{Tr}(F^k, (H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*)^G).$$

So we have

$$L(X_0, \rho, t) = \exp\left(\sum_{k=1}^{\infty} v_k \frac{t^k}{k}\right) = \prod_{i=0}^{2\mathrm{dim}X} \det(I - Ft, (H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*)^G)^{(-1)^{i+1}}.$$

Therefore

$$\mathrm{totdeg} L(X_0, \rho, t) \leq \sum_{i=0}^{2\mathrm{dim}X} \dim(H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*)^G.$$

Let $V = n_1 V_1 \oplus \cdots \oplus n_k V_k$ be the decomposition of V into a direct sum of irreducible representations of G, where each n_j is the multiplicity of the irreducible representation V_j appeared in V. Then we have

$$\dim(H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*)^G = \sum_{j=1}^{k} n_j \dim(H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V_j^*)^G = \sum_{j=1}^{k} n_j m_j,$$

where $m_j = \dim(H_c^i(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V_j^*)^G$. Note that m_j equals the multiplicity of V_j appeared in the representation $H_c^i(X, \overline{\mathbf{Q}}_l)$ of G. So we have $\sum_{j=1}^{k} m_j \mathrm{dim} V_j \leq$

$\dim H^i_c(X, \overline{\mathbf{Q}}_l)$. Therefore

$$\begin{aligned}\dim(H^i_c(X, \overline{\mathbf{Q}}_l) \otimes_{\overline{\mathbf{Q}}_l} V^*)^G &= \sum_{j=1}^{k} \frac{n_j}{\dim V_j} m_j \dim V_j \\ &\leq (\sup_j\{\frac{n_j}{\dim V_j}\})(\sum_{j=1}^{k} m_j \dim V_j) \\ &\leq (\sup_j\{\frac{n_j}{\dim V_j}\})(\dim H^i_c(X, \overline{\mathbf{Q}}_l)),\end{aligned}$$

and hence

$$\mathrm{totdeg} L(X_0, \rho, t) \leq (\sup_j\{\frac{n_j}{\dim V_j}\})(\sum_{i=0}^{2\dim X} \dim H^i_c(X, \overline{\mathbf{Q}}_l)).$$

Theorem 0.1 then follows from Katz's bound in Proposition 1.1. □

Now we turn to the proof of Theorem 0.2. Let X_0 be a closed subscheme of $\mathbf{A}_0^n$ over $\mathbf{F}_q$ defined by the vanishing of r polynomials of degrees at most d. Let $d_1, \ldots, d_n$ be n positive integers and let m be their least common multiple. Let Y_0 be the subvariety of the m-fold product X_0^m defined as follows: For any point $(x_1, \ldots, x_m)$ in $X^m(\mathbf{F})$, where each x_j is a point in $X(\mathbf{F}) \subset \mathbf{A}^n(\mathbf{F})$, write $x_j = (x_{1j}, \ldots, x_{nj})$. We define Y_0 to be the closed subvariety of $X_0^m \subset \mathbf{A}_0^{mn}$ defined by the equations

$$x_{ij} = x_{ij'} \text{ whenever } j \equiv j' \bmod d_i \ (i \in \{1, \ldots, n\},\ j, j' \in \{1, \ldots, m\}).$$

Keeping only the variables x_{ij} with $1 \leq j \leq d_i$ $(1 \leq i \leq n)$, one obtains a closed subvariety $Y_0' \subset \mathbf{A}_0^{d_1+\cdots+d_n}$ defined by mr equations of degrees at most d, and Y_0' is isomorphic to Y_0. Let $\sigma_0 : X_0^d \to X_0^d$ be the automorphism defined by

$$(x_1, \ldots, x_d) \to (x_d, x_1, \ldots, x_{d-1}).$$

Then Y_0 is invariant under the action of σ_0. For each positive integer k, as Faltings observed, one can show that

$$N_{d_1 \ldots d_n}(k, X_0) = \Lambda(\sigma F^k),$$

where the left-hand side is the number of points $(x_1, \ldots, x_n)$ in $X(\mathbf{F})$ such that $x_1 \in \mathbf{F}_{q^{kd_1}}, \ldots, x_n \in \mathbf{F}_{q^{kd_n}}$, and the right-hand side is the number of fixed points of the endomorphism $\sigma F^k : Y \to Y$. (Confer §3 in [W1]). For each i, let $\lambda_{i1}, \ldots, \lambda_{ik_i}$ be all the distinct eigenvalues of σ acting on $H^i_c(Y, \overline{\mathbf{Q}}_l)$ and let $H_{i1}, \ldots, H_{ik_i}$ be the corresponding eigenvector spaces. Since σ has finite order, each λ_{ij} is a root of unity and

$$H^i_c(Y, \overline{\mathbf{Q}}_l) = H_{i1} \oplus \cdots \oplus H_{ik_i}.$$

Since F commutes with σ, each H_{ij} is invariant under the action of F. Let $\mu_{ij1}, \ldots, \mu_{ijk_{ij}}$ be all the eigenvalues of F acting on H_{ij}, where $k_{ij} = \dim H_{ij}$.

Each μ_{ijl} is an algebraic number. We have

$$\mathrm{Tr}(\sigma F^k, H_c^i(Y, \overline{\mathbf{Q}}_l)) = \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} \lambda_{ij} \mu_{ijl}^k.$$

By Proposition 1.2, we have

$$\begin{aligned} N_{d_1 \dots d_n}(k, X) = \Lambda(\sigma F^k) &= \sum_{i=0}^{2\dim Y} (-1)^i \mathrm{Tr}(\sigma F^k, H_c^i(Y, \overline{\mathbf{Q}}_l)) \\ &= \sum_{i=0}^{2\dim Y} \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} (-1)^i \lambda_{ij} \mu_{ijl}^k. \end{aligned}$$

So

$$\sum_{k=1}^{\infty} N_{d_1 \dots d_n}(k, X) t^k = \sum_{i=0}^{2\dim Y} \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} (-1)^i \lambda_{ij} t \frac{d}{dt} \ln(1 - \mu_{ijl} t)^{-1},$$

that is,

$$t \frac{d}{dt} \ln Z_{d_1 \dots d_n}(X_0, t) = \sum_{i=0}^{2\dim Y} \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} (-1)^i \lambda_{ij} t \frac{d}{dt} \ln(1 - \mu_{ijl} t)^{-1},$$

where $Z_{d_1 \dots d_n}(X_0, t)$ is the partial zeta function. This implies that

$$Z_{d_1 \dots d_n}(X_0, t) = \prod_{i=0}^{2\dim Y} \prod_{j=1}^{k_i} \prod_{l=1}^{k_{ij}} (1 - \mu_{ijl} t)^{(-1)^{i+1} \lambda_{ij}},$$

and

$$\begin{aligned} \mathrm{totdeg} Z_{d_1 \dots d_n}(X_0, t) &\leq \sum_{i=0}^{2\dim Y} \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} |(-1)^{i+1} \lambda_{ij}| \\ &= \sum_{i=0}^{2\dim Y} \sum_{j=1}^{k_i} \sum_{l=1}^{k_{ij}} 1 \\ &= \sum_{i=0}^{2\dim Y} \dim H_c^i(Y, \overline{\mathbf{Q}}_l) \\ &= \sum_{i=0}^{2\dim Y} \dim H_c^i(Y', \overline{\mathbf{Q}}_l). \end{aligned}$$

Theorem 0.2 follows from this last inequality, Proposition 1.1, and the fact that Y_0' is a closed subscheme of $\mathbf{A}^{d_1 + \cdots + d_n}$ defined by the vanishing of mr polynomials of degrees at most d. □

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