THE $F$-SIGNATURE AND STRONG $F$-REGULARITY

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Abstract. We show that the $F$-signature of a local ring of characteristic $p$, defined by Huneke and Leuschke, is positive if and only if the ring is strongly $F$-regular.

In [7], Huneke and Leuschke define the $F$-signature of an $F$-finite local ring of prime characteristic with perfect residue field. The $F$-signature, denoted $s(R)$, is an asymptotic measure of the proportion of $R$-free direct summands in a direct-sum decomposition of $R^{1/p^e}$, the ring of $p^e$th roots of $R$. This proportion seems to give subtle information on the nature of the singularity defining $R$. For example, the $F$-signature of any of the two-dimensional quotient singularities $(A_n)$, $(D_n)$, $(E_6)$, $(E_7)$, $(E_8)$ is the reciprocal of the order of the group $G$ defining the singularity [7, Example 18]. The main theorem of [7] on $F$-signatures is as follows.

Theorem 0.1. [7, Theorem 11] Let $(R, \mathfrak{m}, k)$ be a reduced complete $F$-finite Cohen–Macaulay local ring containing a field of prime characteristic $p$. Assume that $k$ is perfect. Then

1. If $s(R) > 0$, then $R$ is weakly $F$-regular.
2. If in addition $R$ is Gorenstein, then $s(R)$ exists, and is positive if and only if $R$ is weakly $F$-regular.

(See below for definitions of the $F$-signature and weak $F$-regularity.)

In this note, we extend this theorem in two directions: we remove the assumption in (2) that $R$ be Gorenstein, and we replace “weakly $F$-regular” by “strongly $F$-regular” throughout. Our main theorem is thus as follows.

Theorem 0.2. Let $(R, \mathfrak{m}, k)$ be a reduced excellent $F$-finite local ring containing a field of characteristic $p$, and let $d = \dim R$. Then the following are equivalent:

1. $\liminf \frac{a_q}{q^e + a(R)} > 0$.
2. $\limsup \frac{a_q}{q^e + a(R)} > 0$.
3. $R$ is strongly $F$-regular.

In particular, if the $F$-signature $s(R)$ is known to exist, then $s(R)$ is positive if and only if $R$ is strongly $F$-regular.

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We also extend the definition of the $F$-signature to the case of an imperfect residue field. This allows us to prove that $s(R)$ behaves well with respect to localization (Proposition 1.3).

Our results do not address the existence of the limit defining $s(R)$. Yao has shown that $s(R)$ exists whenever $R$ is Gorenstein on the punctured spectrum [10].

1. The Main Result

Throughout what follows, $(R, \mathfrak{m}, k)$ is a reduced Noetherian local ring of dimension $d$, containing a field of positive characteristic $p$. We use $q$ to denote a varying power of $p$. Set $d = \dim(R)$ and $\alpha(R) = \log \left( \frac{k}{k^p} \right)$. We assume throughout that $R$ is $F$-finite, that is, the Frobenius endomorphism $F : R \rightarrow R$ defined by $F(r) = r^p$ is a module-finite ring homomorphism. Equivalently, for each $q = p^e$, $R^{1/q} = \{ r^{1/q} \mid r \in R \}$ is a finitely generated $R$-module. In particular, this implies that $\alpha(R) < \infty$, and that $R$ is excellent [8, Propositions 1.1 and 2.5]. Also, when computing length over $R$, we have $\lambda(R/I) = \lambda(R^{1/q}/IR^{1/q})/q^{\alpha(R)}$.

We first define the $F$-signature of $R$.

**Definition 1.1.** Let $(R, \mathfrak{m}, k)$ be as above. For each $q = p^e$, decompose $R^{1/q}$ as a direct sum of finitely generated $R$-modules $R^a_q \oplus M_q$, where $M_q$ has no nonzero free direct summands. The $F$-signature of $R$ is

$$s(R) = \lim_{q \to \infty} \frac{a_q}{q^{d+\alpha(R)}},$$

provided the limit exists.

Our formulation differs slightly from the original definition in [7], where it is assumed that $k$ is perfect, or equivalently that $\alpha(R) = 0$. This reformulation allows us to show that $s(R)$ cannot decrease upon localization. We use a lemma due to Kunz ([8]).

**Lemma 1.2.** Let $R$ be an $F$-finite Noetherian ring of characteristic $p$. Then for any prime ideals $P \subseteq Q$ of $R$, $[k(P) : k(P)^p] = [k(Q) : k(Q)^p]p^{\dim R_Q/PR_Q}$. In other words, $\alpha(R_P) = \alpha(R_Q) + \st Q/P$.

**Proposition 1.3.** Let $(R, \mathfrak{m})$ be an $F$-finite local ring and $P$ a prime ideal. For $q = p^e$, let $a_q$ be the number of nonzero $R$-free direct summands in $R^{1/q}$, and let $b_q$ be the corresponding quantity for $R_P$. Then

$$\frac{b_q}{q^{\dim(R_P)+\alpha(R_P)}} \geq \frac{a_q}{q^{\dim(R)+\alpha(R)}}.$$ 

In particular, if both $s(R)$ and $s(R_P)$ exist, then $s(R_P) \geq s(R)$.

**Proof.** We have $(R_P)^{1/q} \cong (R^{1/q})_P$, so the number of $R_P$-free direct summands in $(R_P)^{1/q}$ is at least the number of $R$-free summands in $R^{1/q}$. A straightforward computation using Lemma 1.2 now gives the result. \qed
We now begin to work toward showing that \( s(R) \) is positive if and only if \( R \) is strongly \( F \)-regular. We refer the reader to [6] for basic notions concerning the theory of tight closure, including finitistic tight closure, but review briefly the ideas used in the proof.

A Noetherian ring \( R \) of characteristic \( p \) is said to be weakly \( F \)-regular provided every ideal of \( R \) is tightly closed. Equivalently, the zero module is finitistically tightly closed in \( E = E_R(k) \), the injective hull of the residue field of \( R \). In other symbols, \( 0^*_{E^f} = 0 \). We say that \( R \) is strongly \( F \)-regular if for every \( c \in R \) not in any minimal prime of \( R \), the inclusion \( Rc^{1/q} \subseteq R^{1/q} \) splits for \( q \gg 0 \). Equivalently, the zero module is tightly closed in \( E \), that is, \( 0^*_{E} = 0 \). Weak and strong \( F \)-regularity are conjecturally equivalent, but this is known only in low dimension and in some special cases.

A test element for \( R \) is an element \( c \), not in any minimal prime of \( R \), such that \( cI^* \subseteq I \) for every ideal \( I \) of \( R \), and the test ideal, denoted \( \tau(R) \), is the ideal generated by all test elements. For a reduced local ring \( R \), \( \tau(R) = \text{Ann}_R 0^*_{E^f} \) by [5, Theorem 8.23]. Thus \( R \) is weakly \( F \)-regular if and only if \( \tau(R) = R \). On the other hand, the CS test ideal, cf. [9] and [2], is the ideal \( \tilde{\tau}(R) = \text{Ann}_R 0^*_{E} \). By work of [9] and [2], the CS test ideal behaves well under localization, so defines the non-strongly \( F \)-regular locus of \( \text{Spec}(R) \). In particular, \( R \) is strongly \( F \)-regular if and only if \( \tilde{\tau}(R) = R \).

It is known that a weakly \( F \)-regular ring is \( F \)-pure, that is, the Frobenius morphism is a pure homomorphism, and that for an \( F \)-pure ring both \( \tau(R) \) and \( \tilde{\tau}(R) \) are radical ideals.

A local ring \((R, m, k)\) is said to be approximately Gorenstein provided there is a sequence \( \{I_t\} \) of \( m \)-primary irreducible ideals cofinal with the powers of \( m \). When \( R \) is Cohen–Macaulay and has a canonical ideal \( J \) (so is Gorenstein at all associated primes), such a family can be obtained as follows: Let \( x_1, \ldots, x_d \) be a system of parameters such that \( x_1 \in J \) and \( x_2, \ldots, x_d \) form a system of parameters for \( R/J \). Then \( I_t := (x_1^{t-1}J, x_2^t, \ldots, x_d^t)R \), for \( t \geq 1 \), gives the required family. Furthermore, the direct limit \( \lim_{\longrightarrow t} R/I_t \), where the maps in the direct system are \( R/I_t \xrightarrow{x^{\frac{1}{t}}-x^d} R/I_{t+1} \), is isomorphic to \( E_R(k) \). If \( u_1 \in R \) is a representative for the socle generator of \( R/I_1 \), then \( u_t := (x_1 \cdots x_d)^{t-1}u_1 \) generates the socle of \( R/I_t \), and each \( u_t \) maps in the limit to \( u \), the socle element of \( E_R(k) \).

More generally ([4, Thm. 1.7]), if \( R \) is any locally excellent Noetherian ring that is locally Gorenstein at associated primes, then \( R \) is approximately Gorenstein.

The following result of Hochster, together with its corollary below, explains our interest in approximately Gorenstein rings. It can be thought of as a generalization of [7, Lemma 12].

**Proposition 1.4.** [4, Theorem 2.6] Let \( (R, m) \) be an approximately Gorenstein local ring and let \( \{I_t\} \) be a sequence of irreducible ideals cofinal with the powers
of $m$. Let $f : R \rightarrow M$ be a homomorphism of finitely generated $R$-modules. Then $f$ is a split injection if and only if $f \otimes_R R/I_t$ is injective for every $t$.

**Proposition 1.5.** Let $(R, m)$ be an approximately Gorenstein local ring with a family of irreducible ideals $\{I_t\}$ as above, and let $u_t \in R$ represent a socle generator for $R/I_t$. Let $f : R \rightarrow M$ be a homomorphism of finitely generated $R$-modules. If $M$ has no free summands, then there exists $t_0 > 0$ such that $u_t M \subseteq I_t M$ for all $t \geq t_0$.

**Proof.** By Proposition 1.4, $f \otimes_R R/I_t$ fails to be injective for some $t$. Since $u_t$ is the unique socle element of $R/I_t$, we have $f(u_t) \in I_t M$, that is, $u_t M \subseteq I_t M$. This continues to hold for all $t' \geq t$, since there is an injection $R/I_t \rightarrow R/I_{t'}$ with $u_t \mapsto u_{t'}$.

We also use a result of Aberbach, which says that, in some sense, elements not in tight closures are very far from being in Frobenius powers.

**Theorem 1.6.** [1, Prop. 2.4] Let $(R, m)$ be an excellent local domain such that the completion is also a domain. Let $N = \lim_{\rightarrow t} R/J_t$ be a direct limit system of cyclic modules. Fix $u \not\in 0^*_N$. Then there exists $q_0$ such that

$$\bigcup_t (j_t^{[q]} : u_t^q) \subseteq m^{[q/q_0]}$$

for all $q \gg 0$ (where the sequence $\{u_t\}$ represents $u \in N$ and $u_t \mapsto u_{t+1}$).

**Proof of Theorem 0.2.** The Cohen-Macaulayness of $R$ is forced by the assumptions ([7, Theorem 11] and [5]), so we may assume throughout that $R$ is Cohen-Macaulay.

That (1) implies (2) is trivial. So assume that (2) holds. We proceed by induction on the dimension $d$, the case $d = 0$ being trivial. If $d > 0$, then Proposition 1.3 shows that we may assume by induction on $d$ that $R$ is strongly $F$-regular on the punctured spectrum. We will show that $0^*_E = 0$, where as above $E = E_R(k)$ is the injective hull of the residue field of $R$.

Since $\overline{\tau}(R) = \text{Ann}_R 0^*_E$ is a radical ideal and is known to define the non-strongly $F$-regular locus of $R$ (see [2]), and $R$ is strongly $F$-regular on the punctured spectrum, $\text{Ann}_R 0^*_E$ contains the maximal ideal $m$. If $\overline{\tau}(R) = R$, then we are done, so we assume $\overline{\tau}(R) = m$. Then $0^*_E = \text{soc}(E)$.

As in the discussion above, $E = E_R(k) \cong \lim_{\rightarrow t} R/I_t$ for a family of irreducible ideals $I_t$. Let $u$ be a socle generator for $E$ and $\{u_t\} \subseteq R$ a sequence of representatives for the socle generators of $R/I_t$, converging to $u$. 
Fix a power \( q \) of the characteristic, and decompose \( R^{1/q} \cong R^{a_q} \oplus M_q \), where \( M_q \) has no nonzero free summands. Then for each \( t \), we have

\[
\lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t, u_t)^{[q]}\right) = \frac{\lambda\left(R^{1/q}/I_t R^{1/q}\right)}{q^{\alpha(R)}} - \frac{\lambda\left(R^{1/q}/(I_t, u_t) R^{1/q}\right)}{q^{\alpha(R)}}
\]

\[
= \frac{a_q \lambda(R/I_t) + \lambda(M_q/I_t M_q)}{q^{\alpha(R)}} - \frac{a_q \lambda(R/(I_t, u_t)) - \lambda(M_q/(I_t, u_t) M_q)}{q^{\alpha(R)}}
\]

\[
= \frac{a_q \lambda(R/I_t) - a_q \lambda(R/(I_t, u_t)) + \lambda(M_q/I_t M_q) - \lambda(M_q/(I_t, u_t) M_q)}{q^{\alpha(R)}}
\]

\[
= \frac{a_q + c_{t,q}}{q^{\alpha(R)}},
\]

for some \( c_{t,q} \geq 0 \). By Proposition 1.5, there exists \( t_0 > 0 \) such that \( u_t M_q \subseteq I_t M_q \) for \( t \geq t_0 \), that is, \( c_{t,q} = 0 \) for \( t \geq t_0 \). On the other hand, \( \lambda(R/I_t^{[q]}) - \lambda(R/(I_t, u_t)^{[q]}) = \lambda(R/(I_t^{[q]} : u_t^{[q]})) \) is equal to 1 for large \( t \) since \( (I_t^{[q]} : u_t^{[q]}) = m \) for large \( t \). Thus, for large \( t \),

\[
\lim_{q \to \infty} \frac{a_q + c_{t,q}}{q^{d+\alpha(R)}} = \lim_{q \to \infty} \frac{1}{q^{d+\alpha(R)}} = 0,
\]

a contradiction.

Lastly, assume that \( R \) is strongly \( F \)-regular and keep the same notation. We then have \( 0_E^q = 0 \), so \( u \not\in 0_E^q \). By Theorem 1.6, then, there exists \( q_0 \) such that

\[(I_t^{[q]} : u_t^{[q]}) \subseteq m^{[q/q_0]}
\]

for all \( q \geq q_0 \). Fix \( q \geq q_0 \). Then there exists \( t_0 \) such that for all \( t \geq t_0 \) we have

\[
\frac{a_q}{q^{\alpha(R)}} = \lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t^{[q]}, u_t^{[q]})\right)
\]

\[
= \lambda\left(R/(I_t^{[q]} : u_t^{[q]})\right)
\]

\[
\geq \lambda\left(R/m^{[q/q_0]}\right).
\]

Divide by \( q^d \) and pass to the limit; we see that \( \liminf \frac{a_q}{q^{d+\alpha(R)}} \geq e_{HK}(m, R)/q_0^d > 0 \). Thus (1) holds.

The last statement is immediate if there is a limit. \( \square \)

The \( F \)-signature suggests a form of dimension that we may attach to an \( F \)-finite reduced local ring. Let \( s_j = \lim_{q \to \infty} \frac{a_q}{q^{d+\alpha(R)}} \) for \( 0 \leq j \leq d = \dim(R) \) and set \( s_{-1} = 1 \). Then we can define the \( s \)-dimension of \( R \) as \( \text{sdim}(R) = \max\{j \geq -1 | s_j > 0\} \). A ring which is \( F \)-pure then has non-negative \( s \)-dimension,
and Theorem 0.2 says that $R$ is strongly $F$-regular if and only if $s\dim(R) = \dim(R)$.

References


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