FAMILIES OF K3 SURFACES OVER CURVES REACHING THE ARAKELOV-YAU TYPE UPPER BOUNDS AND MODULARITY

XIAOTAO SUN, SHENG-LI TAN, AND KANG ZUO

Let $f : X \to C$ be a family of semi-stable curves of genus $g$ over a smooth projective $C$ of genus $q$, and $S \subset C$ the degeneration locus of $f$. The so-called Arakelov inequality states that

$$\deg f_* \omega_{X/C} \leq \frac{g}{2} \deg \Omega^1_C (\log S) = \frac{g}{2} (2q - 2 + \# S).$$

When $g \geq 2$ and $\# S = 0$, the Miyaoka-Yau inequality for surfaces implies a much stronger inequality

$$\deg f_* \omega_{X/C} \leq \frac{g - 1}{6} \deg \Omega^1_C.$$

In general, Tan [28] proved that the Arakelov inequality for a family $f : X \to C$ of semi-stable curves of genus $\geq 2$ holds strictly.

If $g = 1$, then $\deg f_* \omega_{X/C}$ can reach the upper bound in the inequality. Beauville has classified such families over $C = \mathbb{P}^1$ with $\# S = 4$. More precisely, there are exactly 6 non-isotrivial families of semi-stable elliptic curves over $\mathbb{P}^1$ with 4 singular fibres. All of them are modular families of elliptic curves [2].

In this paper, we will consider the similar question for families of higher dimensional varieties. The Arakelov inequality is a special case of some more general inequalities for Hodge bundles. To state them, let $V$ denote a polarized real variation of Hodge structure on a smooth projective curve $C \setminus S$ such that the local monodromies around $S$ are all unipotent, let

$$(\oplus_{p+q=k} E^{p,q}, \theta)$$

denote the corresponding Hodge bundles. In [8] the following Arakelov-Yau type inequality was proven (also see [18] for a similar inequality):

If $k = 2l + 1$, then

$$\deg E^{k,0} \leq \left( \frac{1}{2} (h^{k-l,l} - h^{k-l,l}_0) + \sum_{j=0}^{l-1} (h^{k-j,j} - h^{k-j,j}_0) \right) \cdot \deg (\Omega^1_C (\log S)).$$

Received May 24, 2002.

This work is supported by a direct grant for Research from the Chinese University of Hong Kong (Project Code: 2060197). The second author is also supported by the 973 Foundation, the Foundation of EMC for Key Teachers and the PAD of Shanghai. The first author is supported by a grant of NSFC for outstanding young researcher (Project Code: 10025103).
If \( k = 2l \),
\[
\deg E^{k,0} \leq \sum_{j=0}^{l-1} (h^{j,k-j} - h^{j,k-j}_0) \cdot \deg(\Omega_C^1(\log S)).
\]

These inequalities generalize the original Arakelov inequality for a family \( f : A \to C \) of semi-stable abelian varieties due to Deligne. In general, Yau [31] proved the so-called Yau’s Schwarz type inequality, which can be formulated as follows. Let \( (M, ds) \) be a Hermitian manifold with holomorphic sectional curvature bounded above by a negative constant \( K \), and let \( (C \setminus S, ds_\mu) \) be a Poincare type metric. Then there exists a positive constant \( c \), such that for any holomorphic map \( \phi : C \setminus S \to M \), one has \( \phi^* ds \leq c ds_\mu \). It is the reason why we call such inequalities are of Arakelov-Yau type.

We now consider a family \( f : X \to C \) of semi-stable algebraic K3 surfaces. Let \( X^0 \) denote the largest subscheme where \( f \) is smooth and projective, and assume that \( R^2 f_* \mathcal{Z}_{X^0} \) extends to a local system \( \mathcal{V} \) on \( C \setminus S \). We call \( S \) the singularities of \( R^2 f_* \mathcal{Z}_X \) and write \( \Delta := f^*(S) \), which is a normal crossing divisor. Then the corresponding Hodge bundles read
\[
(f_* \omega_{X/C} \oplus R^1 f_* \Omega^1_{X/C}(\log \Delta) \oplus R^2 f_* (\mathcal{O}_X), \theta).
\]

If \( f \) is non-isotrivial, it is known that \( f_* \omega_{X/C} \) is ample on \( C \) by Fujita [6] (also see [9] [29] for higher dimensional base). Applying the above Arakelov-Yau type inequality for Hodge bundles of weight-2 one obtains
\[
\deg f_* \omega_{X/C} \leq \deg \Omega_C^1(\log S). \tag{0.0.1}
\]

If the iterated Kodaira-Spencer map of this family is zero, one shows a stronger inequality
\[
\deg f_* \omega_{X/C} \leq \frac{1}{2} \deg \Omega_C^1(\log S). \tag{0.0.2}
\]

In this note we shall study non-isotrivial algebraic families of semi-stable K3 surfaces over curves when the inequality (0.0.1), or (0.0.2) becomes an equality. One shows in Theorem 0.1 below that such a numerical equality has a strong consequence for the geometry of the generic fibre. The corresponding question has been considered in [30] for families of abelian varieties. The final presentation of this note has been influenced by [30]. It has also been motivated by Mok’s work on rigidity theorems of locally Hermitian symmetric spaces [13] and [14], where he use the Gaussian curvature of the induced metric on a holomorphic curves in a locally Hermitian symmetric space to characterize when this curve will be a totally geodesic embedding.

To state the main result, we recall some notation. Let \( a : A^0 \to C^0 \) be a family of abelian surfaces with a section, then the desingularization \( Z^0 \to C^0 \) of the quotient \( A^0/\{\pm 1\} \to C^0 \) is a family of Kummer surfaces (the so called Kummer construction). The rational map \( A^0 \to Z^0 \) is called a rational quotient of \( A^0 \). The family \( a : A^0 \to C^0 \) is called the associated family of abelian surfaces
of $Z^0 \to C^0$. In general, it is not true that every family of Kummer surfaces has
an associated family of abelian surfaces.

An involution $\iota$ on a $K3$ surface $X$ is called a Nikulin involution if $\iota^* \omega = \omega$
for every $\omega \in H^0(X, \Omega^2_X)$. It is known (Nikulin [17]) that every Nikulin
involution $\iota$ has eight isolated fixed points, and the rational quotient $X \to Z$ by $\iota$ is a $K3$
surface.

**Theorem 0.1.** Let $f : X \to C$ be a family of semi-stable $K3$ surfaces over $C$,
and $S \subset$ the singular locus of $\nabla := R^2 f_*(\mathbb{Z}_{X^0})$. If $S \neq \emptyset$ and if $\deg f_* \omega_{X/C}$
reaches the Arakelov bound in (0.0.1), then the following properties hold true:

a) The general fibre of $f : X \to C$ has Picard number 19.

b) There exist a finite étale cover $\sigma : C' \to C$, a Zariski open set $C'^0 \subset C'$
and a global Nikulin involution $\iota$ on $f : X^0 = f^{-1}(C'^0) \to C'^0$ such that
the rational quotient $X^0 \to Z^0$ by $\iota$ is a family of Kummer surfaces over $C'^0$,
which has an associated family of abelian surfaces that is isogenous to
the square product of a family of elliptic curves $g : E \to C'$.

c) The projective monodromy representation of the local system $R^1 g_*(\mathbb{Z}_{E^0})$
extends to

$$
\tau : \pi_1(C' \setminus \sigma^{-1}S, *) \to PSL_2(\mathbb{Z})
$$

such that

$$
C' \setminus \sigma^{-1}S \cong \mathcal{H}/\tau \pi_1(C' \setminus \sigma^{-1}S, *).
$$

A family of $K3$ surfaces satisfying Property b) will be called a family coming
from Nikulin-Kummer construction of the square product of a family of elliptic
curves.

**Theorem 0.2.** If the second iterated Kodaira-Spencer map of the family $f : X \to C$
is zero and if the family reaches the Arakelov bound in (0.0.2). Then
then the following properties hold true:

a) The general fibres of $f : X \to C$ have the Picard number at least 18.

b) After passing to a finite étale cover $\sigma : C' \to C$, the monodromy representation $\rho$ of $R^2 f_*(\mathbb{Z}_{X^0})$
is of the form

$$
\rho = \text{trivial rank-2 representation} \otimes (\tau : \pi_1(C' \setminus \sigma^{-1}S, *) \to SL_2(\mathbb{Z})),
$$

and $C' \setminus \sigma^{-1}S \cong \mathcal{H}/\tau \pi_1(C' \setminus \sigma^{-1}S, *)$.

**Remark 0.3.**

i) Theorem 0.1 can be used to explain the observation of B. Lian and S.-T.
Yau ([10], [11]) that the weight-2 VHS attached to a certain one dimensional families of $K3$ surfaces coming from the Mirror of $K3$ surfaces of Picard number $\geq 1$ can be expressed as the square products of the weight 1 VHS attached to a certain one dimensional families of elliptic curves (also see [5]). Note that such a family must reach the Arakelov bound in (0.0.1). We thank A. Todorov for pointing that out to us. Note that, if $S = \emptyset$ then there is another type families of $K3$ surfaces reaching the Arakelov bound (0.0.1). Namely, let $a : A \to C$ be a modular family of false elliptic curves,
i.e. abelian surface whose endomorphism ring is isomorphic to an order of an indefinite quaternion algebra over \( \mathbb{Q} \) ([26]). Then the Kummer construction gives rise to a family \( f : X \to C \) of smooth \( K3 \) surfaces reaching the Arakelov bound (0.0.1), and \( C \) is a Shimura curve. One likes to know what is the mirror pair of this family.

ii) For a family \( f : X \to C \) as in Theorem 0.2 one can find a family \( f' : X' \to C \), which comes from the Nikulin-Kummer construction of a product of a modular family of elliptic curves \( g : E_1 \to C \) with an elliptic curve \( E_2 \) over \( \mathbb{C} \), and such that sub VHSs of transcendental lattices of \( f \) and \( f' \) are Hodge isometric to each other. Are there closer geometric relations among these families?

Let \( f : X \to \mathbb{P}^1 \) be a Calabi-Yau 3-fold fibred by non-constant families of semi-stable \( K3 \) surfaces. The triviality of \( \omega_X \) implies that \( \deg f_* \omega_{X/\mathbb{P}^1} = 2 \).

**Corollary 0.4.** Let \( f : X \to \mathbb{P}^1 \) be a Calabi-Yau 3-fold fibred by non-constant semi-stable \( K3 \) surfaces. Then the followings hold true:

i) If the iterated Kodaira-Spencer map of \( f \) is non-zero, then \( f \) has at least 4 singular fibres. If \( f \) has 4 singular fibres, then \( X \) is rigid and birational to the Nikulin-Kummer construction of a square product of a family of elliptic curves \( g : E \to \mathbb{P}^1 \). After passing to (if necessary) a double cover \( E' \to E \), the family \( g' : E' \to \mathbb{P}^1 \) is one of the 6 modular families of elliptic curves constructed by Beauville.

ii) If the iterated Kodaira-Spencer map of \( f \) is zero, then \( f \) has at least 6 singular fibres. If \( f \) has 6 singular fibres over \( S \subset \mathbb{P}^1 \), then \( X \) is non-rigid, the general fibres have Picard number 18, and \( \mathbb{P}^1 \setminus S \simeq \mathcal{H}/\Gamma \), where \( \Gamma \) is a subgroup of \( SL_2(\mathbb{Z}) \) of index 24.

**Remark 0.5.**

i) Any \( K3 \)-fibred Calabi-Yau 3-fold \( f : X \to \mathbb{P}^1 \) in 0.4, i) is rigid because of the modular construction for \( X \). Since all 6 examples of Beauville are defined over \( \mathbb{Z} \), we may assume that \( X \) has a suitable integral model. The \( L \)-series of \( X \) is defined to be the \( L \)-series of the Galois representation on \( H^3_{et}(\overline{X}, \mathbb{Q}) \). One should be able to verify the so-called modularity conjecture for \( X \). M.-H. Saito and N. Yui [20] checked for one example that up to a finite Euler factor, \( L(X, s) = L(f, s) \) for \( f \in S_4(\Gamma_0(N)) \).

ii) Does any rigid Calabi-Yau 3-fold fibred by semi-stable \( K3 \) surfaces come from the modular construction in 0.4, i)?

iii) One can construct an example for the case ii) of Corollary 0.4. Let

\[
g : E(4) \to X(4)
\]

be the modular family of elliptic curves corresponding to the congruence group \( \Gamma(4) \). Then \( X(4) \simeq \mathbb{P}^1 \) with six cusps, and \( \deg g_* \omega_{E(4)/X(4)} = 2 \). The Nikulin-Kummer construction applied to the product of \( g : E(4) \to X(4) \)
with a constant family of elliptic curves gives an $K3$ fibred Calabi-Yau
3-fold reaching the upper bound in $(0, 0.2)$, which is non-rigid.

1. Weight-2 VHS and $\mathbb{R}$-Splitting

Let $f : X \to C$ be a family of semi-stable $K3$ surfaces. Consider its weight-2
variation of Hodge structure (VHS for simplicity)

$$\mathcal{V}_0 = R^2f_*(\mathbb{Z}_X).$$

Let $S \subset C$ denote the subset, where the local monodromies of $\mathcal{V}_0$ are
non-trivial, hence of infinite order and $\Delta := f^*(S)$. We will write $\mathcal{V}$ for the extension
of $\mathcal{V}_0$ to $C \setminus S$. One has the canonical extension of Hodge bundles

$$E^{p,q} = R^d f_*(\Omega^p_X \otimes (\log \Delta)), \quad p + q = 2,$$

together with the cup product of Kodaira-Spencer map

$$\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_C(\log S).$$

$\theta = \theta^{2,0} + \theta^{1,1}$ is called the Higgs field of $\mathcal{V}$.

Lemma 1.1. We have $\deg E^{2,0} \leq \deg \Omega^1_C(\log S)$, and if the equality

$$\deg E^{2,0} = \deg \Omega^1_C(\log S)$$

holds, then there is a real splitting $\mathcal{V} \otimes \mathbb{R} = \mathcal{W}_\mathbb{R} \oplus \mathcal{U}_\mathbb{R}$, which is orthogonal w.r.t.
the polarization, and $\mathbb{U}$ is unitary. The corresponding Higgs bundle splitting is

$$(E^{2,0} \oplus E^{1,1}_1 \oplus E^{0,2} \otimes \theta) \oplus (E^{1,1}_2, 0)$$

where $E^{1,1} = E^{1,1}_1 \oplus E^{1,1}_2$ and $E^{1,1}_1$ is a line bundle of degree zero such that

$$\theta : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S), \quad \theta : E^{1,1}_1 \to E^{0,2} \otimes \Omega^1_C(\log S)$$

are isomorphisms.

Proof. We consider the map $\theta^{1,1} : E^{1,1}_1 \to E^{0,2} \otimes \Omega^1_C(\log S)$, and let $E^{2,1}_1 \subset E^{1,1}_1$ denote the kernel of $\theta^{1,1}$. Then $(E^{2,1}_1, 0)$ is a Higgs sub-bundle.

Claim. $\deg E^{2,1}_1 \leq 0$, and if the equality holds then the Higgs subbundle

$$(E^{2,1}_1, 0) \subset (E, \theta)$$

induces a splitting $(E, \theta) = (E^{2,0} \oplus E^{1,1}_1 \oplus E^{0,2} \otimes \theta) \oplus (E^{1,1}_2, 0)$, which corresponds to a $\mathbb{C}$-splitting of the local system $\mathcal{V} \otimes \mathbb{C} = \mathcal{W}_\mathbb{C} \oplus \mathcal{U}_\mathbb{C}$.

Proof of the claim. Let $h$ denote the Hodge metric on $E|_{C \setminus S}$, and let $\Theta(E|_{C \setminus S}, h)$ be its curvature form. Then we have ([7], Chapter II)

$$\Theta(E|_{C \setminus S}) + \theta \wedge \bar{\theta} + \bar{\theta} \wedge \theta = 0,$$

where $\bar{\theta}$ is the complex conjugation of $\theta$ with respect to $h$. Consider the $C^\infty$ $h$–orthogonal decomposition $E|_{C \setminus S} = E^{1,1}_2|_{C \setminus S} \oplus E^{1,1}_2|_{C \setminus S}^\perp$. One has

$$\Theta(E^{1,1}_2|_{C \setminus S}, h) = \Theta(E|_{C \setminus S}, h)|_{E^{1,1}_2} + \bar{A} \wedge A = -((\theta \wedge \bar{\theta})|_{E^{1,1}_2} - (\bar{\theta} \wedge \theta)|_{E^{1,1}_2} + \bar{A} \wedge A,$$
where $A \in A^{1,0}(\text{Hom}(E_{2}^{1,1}, E_{2}^{1,1}))$ is the second fundamental form of the sub-bundle $E_{2}^{1,1} \subset E$ and $\bar{A}$ is the complex conjugation with respect to $h$. Since $\theta(E_{2}^{1,1}) = 0$, we have $(\bar{\theta} \wedge \theta)|_{E_{2}^{1,1}} = 0$. Hence

$$\Theta(E_{2}^{1,1}|_{C \setminus S'}, h) = -(\theta \wedge \bar{\theta})E_{2}^{1,1} + \bar{A} \wedge A.$$ 

$\Theta(E_{2}^{1,1}|_{C \setminus S'}, h)$ is negative semidefinite since $\theta \wedge \bar{\theta}E_{2}^{1,1}$ is positive semidefinite and $\bar{A} \wedge A$ is negative semidefinite. Since the local monodromies around points in $S$ are unipotent, $\text{Tr} \Theta(E_{2}^{1,1}|_{C \setminus S'}, h)$ represents (by [22]) the Chern class $c_{1}(E_{2}^{1,1})$ as a current. Thus

$$\deg E_{2}^{1,1} = \int_{C \setminus S} \text{Tr} \Theta(E_{2}^{1,1}|_{C \setminus S}, h) \leq 0,$$

and $\Theta(E_{2}^{1,1}|_{C \setminus S}, h) = 0$ if $\deg E_{2}^{1,1} = 0$. This implies that $\bar{\theta}(E_{2}^{1,1}) = 0$ and $A = 0$. Altogether this shows that the sub-Higgs bundle $(E_{2}^{1,1}, 0)$ of $(E, \theta)$ induces a splitting of the Higgs bundle

$$(E, \theta) = (E^{2,0} \oplus E_{1}^{1,1} \oplus E^{0,2}, \theta) \oplus (E_{2}^{1,1}, 0)$$

and the corresponding splitting $V \otimes \mathbb{C} = W_{C} \oplus U_{C}$ of the complex local system. Thus the claim is proved. \hfill \Box

Let $I \subset E^{0,2} \otimes \Omega_{C}^{1}(\log S)$ denote the image of $\theta^{1,1}$. Then the exact sequence

$$0 \to E_{2}^{1,1} \to E^{1,1} \to I \to 0,$$

together with $\deg E_{2}^{1,1} = 0$ implies that

$$-\deg E_{2}^{1,1} = \deg I.$$

Hence,

$$-\deg E_{2}^{2,0} + \deg \Omega_{C}^{1}(\log S) = \deg(E^{0,2} \otimes \Omega_{C}^{1}(\log S)) \geq \deg I = -\deg E_{2}^{1,1} \geq 0.$$ 

Thus the inequality $\deg E_{2}^{2,0} \leq \deg \Omega_{C}^{1}(\log S)$ becomes an equality if and only if $\deg E_{2}^{1,1} = 0$ and $I = E^{0,2} \otimes \Omega_{C}^{1}(\log S)$, which is our $E_{1}^{1,1}$. It is easy to see that the Higgs field of $W_{C}$ is an isomorphism, thus $W_{C}$ is irreducible over $\mathbb{C}$. Now we only need to show that the decomposition $V \otimes \mathbb{C} = W_{C} \oplus U_{C}$ can be, in fact, defined over $\mathbb{R}$. Taking the complex conjugation on $W_{C}$ one has

$$\overline{W_{C}} \subset V \otimes \overline{\mathbb{C}} = V \otimes \mathbb{C}.$$ 

$\overline{W_{C}}$ is again of the Hodge type $(2, 0) + (1, 1) + (0, 2)$, irreducible and with non-zero Higgs field. The projection $p : \overline{W_{C}} \subset V \otimes \mathbb{C} \to U_{C}$ can not be injective since $U_{C}$ is unitary. Moreover, since $\overline{W_{C}}$ cannot have a proper sub local system, this projection must be zero. Thus $W_{C} = \overline{W_{C}}$ and we obtain a real sub local system $W_{R} \subset V \otimes \mathbb{R}$. The intersection form restricted to $W_{R}$ is non-degenerated. Thus the orthogonal complement of $W_{R}$ with respect to the intersection form gives the desired real decomposition $V \otimes \mathbb{R} = W_{R} \oplus \overline{U_{R}}$. \hfill \Box
Lemma 1.2. If the iterated Kodaira-Spencer map \( \theta^{1,1}\theta^{2,0} = 0 \), then

\[
\deg E^{2,0} \leq \frac{1}{2} \deg \Omega^1_C(\log S).
\]

When the equality \( \deg E^{2,0} = \frac{1}{2} \deg \Omega^1_C(\log S) \) holds, then there is a real splitting

\[
\mathbb{V} \otimes \mathbb{R} = \mathbb{W} \oplus \mathbb{U},
\]

which is orthogonal w.r.t. the polarization, and \( \mathbb{U} \) is unitary. The corresponding Higgs bundle splitting is

\[
(E^{2,0} \oplus (E_1^{1,1} \oplus E_1^{1,1*} \oplus E^{0,2}, \theta) \oplus (E_2^{1,1}, 0)
\]

where \( E_1^{1,1} \) and \( E_1^{1,1*} \) are sub line bundles of \( E^{1,1} \) with

\[
\deg E_1^{1,1} = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S),
\]

and \( E^{1,1} = E_1^{1,1} \oplus E_1^{1,1*} \oplus E_2^{1,1} \). The Higgs field

\[
\theta : (E^{2,0} \oplus (E_1^{1,1} \oplus E_1^{1,1*}) \oplus E^{0,2}) \to (E^{2,0} \oplus (E_1^{1,1} \oplus E_1^{1,1*}) \oplus E^{0,2}) \otimes \Omega^1_C(\log S)
\]

is defined by \( \theta = \tau \oplus -\tau^* \), where \( \tau : E^{2,0} \simeq E_1^{1,1} \otimes \Omega^1_C(\log S), \quad E_1^{1,1} \to 0 \).

Proof. Since \( \theta^{1,1}\theta^{2,0} = 0 \), the map \( \theta^{2,0} \) factors through

\[
\theta^{2,0} : E^{2,0} \to E_1^{1,1} \otimes \Omega^1_C(\log S),
\]

where \( E_1^{1,1} \subset E^{1,1} \) is a sub-line bundle such that \( \theta^{1,1}(E_1^{1,1}) = 0 \). Thus

\[
(E^{2,0} \oplus E_1^{1,1}, \theta^{2,0}) \subset (E, \theta)
\]

is a rank-2 Higgs sub bundle. By the same arguments as in the proof of Lemma 1.1, one has \( \deg E^{2,0} \oplus E_1^{1,1} \leq 0 \), thus

\[
\deg E^{2,0} \leq \frac{1}{2} \Omega^1_C(\log S).
\]

If the equality holds, then \( \theta^{2,0} =: \tau : E^{2,0} \to E_1^{1,1} \otimes \Omega^1_C(\log S) \) is an isomorphism with \( \deg E_1^{1,1} = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S) \), and the Higgs sub bundle \( (E^{2,0} \oplus E_1^{1,1}, \theta^{2,0}) \subset (E, \theta) \) gives rise to a complex sub local system \( \mathbb{W}_1 \subset \mathbb{V} \otimes \mathbb{C} \). The dual \( \mathbb{W}_1 \subset \mathbb{V} \otimes \mathbb{C} \) corresponds to Higgs subbundle

\[
(E^{2,0} \oplus E_1^{1,1*}) = E_1^{1,1*} \oplus E^{0,2}
\]

together with the Higgs field \( -\tau^* : E_1^{1,1*} \to E^{0,2} \otimes \Omega^1_C(\log S) \). The sub-local system \( \mathbb{W} := \mathbb{W}_1 \oplus \mathbb{W}_1 \) is real, and the intersection form restricted to \( \mathbb{W} \) is non-degenerated. Hence, the orthogonal complement defines the desired decomposition. \( \square \)
2. Splitting over $\overline{\mathbb{Q}}$

We start with a very simple observation. Suppose that $V$ is a local system defined over $\overline{\mathbb{Q}}$. Fixing a positive integer $r$, let $G(r, V)$ denote the set of all rank-$r$ sub-local systems of $V$. Then $G(r, V)$ is a projective variety defined over $\overline{\mathbb{Q}}$. The following property is well known.

**Lemma 2.1.** If $[W] \in G(r, V)$ is an isolated point, then $W$ is defined over $\overline{\mathbb{Q}}$.

**Lemma 2.2.** The $\mathbb{R}$-splittings $V \otimes \mathbb{R} = W \oplus U$ in Lemma 1.1 and Lemma 1.2 can be defined over $\overline{\mathbb{Q}}$.

**Proof.** By Lemma 2.1, one only needs to show that $W$ is a rigid sub-local system of $V \otimes \mathbb{C}$. Suppose that there is a family of sub-local systems $\{W_t\}$, $W_0 = W$.

By semi-continuity, the Higgs fields $\theta^{p,q}$ of $W_t$ are again isomorphisms for $t$ being sufficiently close to 0. Then the projection $W_t \to V \otimes \mathbb{C} \to U$ must be zero, otherwise, $W_t$ would contain a non-trivial unitary component, which contradicts that $\theta^{p,q}$ are isomorphisms. Hence $W_t = W$.

Similarly, we show that the sub-local system $W = W_1 \oplus \overline{W}_1 \subset V = W \oplus U$ is rigid. Suppose that there is a family of sub local systems $\{W_t\}$ with $W_0 = W$, we decompose $W_t$ into the direct sum of irreducible components over $\mathbb{C}$, which has only following possible types up to isomorphism

$$W_1 \oplus \overline{W}_1; \quad W_1 \oplus U'; \quad \overline{W}_1 \oplus U''; \quad U''',$$

where $U', U'', U'''$ are unitary. By semicontinuity, the last three cases are impossible if $t$ is sufficiently close to 0 (otherwise $\theta^{1,1}$ would be zero). Thus

$$W_t \simeq W_1 \oplus \overline{W}_1,$$

which implies that the projection $W_t \to V \otimes \mathbb{C} \to U$ must be zero. Otherwise, $W_t$ would contain a non-trivial unitary component, which contradicts that the Higgs fields of $W$ are isomorphisms. $\square$

3. Splitting over $\mathbb{Q}$ and $\mathbb{Z}$-structures

We call the splitting in Lemma 1.1 of type (0.0.1) and the splitting in Lemma 1.2 of type (0.0.2).

**Lemma 3.1.** If $S \neq \emptyset$, the splittings in Lemma 2.2 can be defined over $\mathbb{Q}$.

**Proof.** Let $V \otimes K = W \oplus U$ be the splitting of type (0.0.1) in Lemma 2.2, where $K$ is a Galois extension of $\mathbb{Q}$. For any $\sigma \in \text{Gal}(K/\mathbb{Q})$, we claim that $\sigma W = W$. Otherwise, the projection $p : \sigma W \to V \otimes K \to U$ must be non-zero and $\sigma W$ is isomorphic to a unitary sub local system $U' \subset U$ under $p$ since $W$ is irreducible (thus $\sigma W$ is also irreducible). Let $\gamma$ be a short loop around $s \in S$. Then the monodromy matrix $\rho_{\sigma W}(\gamma)$ has infinite order, hence $\rho_{\sigma W}(\gamma)$ has also infinite order, which contradicts that $\rho_{LV}(\gamma)$ is identity. We proved that $W$
FAMILIES OF K3 SURFACES 331

is invariant under Gal(K/Q). Hence W is defined over Q and the orthogonal complement of \( W \subset V \otimes Q \) w.r.t. the intersection form defines an \( \mathbb{Q} \)-splitting

\[
V \otimes \mathbb{Q} = W \oplus U.
\]

By the same argument, we show that the splitting of type (0.0.2) in Lemma 2.2 is also defined over Q. \( \square \)

**Lemma 3.2.** After passing to a finite etale cover of \( C \) the splittings in Lemma 3.1 induce \( \mathbb{Z} \)-sub lattices

\[
V \supset W_Z \oplus \mathbb{Z}^\nu,
\]

where \( \nu = 19 \) under the assumptions of 1.1 and \( \nu = 18 \) under those in 1.2 such that

\[
V \otimes \mathbb{Q} = (W_Z \oplus \mathbb{Z}^\nu) \otimes \mathbb{Q},
\]

where \( \mathbb{Z}^\nu \) is respectively a rank-\( \nu \) constant \( \mathbb{Z} \)-lattice of type-(1,1).

**Proof.** Let \( W_Z = V \cap W, \quad U_Z = V \cap U \). It is easy to check that

\[
W_Z \otimes \mathbb{Q} = W, \quad U_Z \otimes \mathbb{Q} = U,
\]

thus \( W_Z \) and \( U_Z \) are lattices in \( W \) and \( U \). Since \( U \) is unitary and carries an \( \mathbb{Z} \)-structure, the monodromy group of \( U \) is finite. Since the local monodromies of \( U \) around \( S \) are trivial, \( U \) extends to a local system on \( C \). Therefore, after passing to the cover corresponding to this monodromy group, \( U \) becomes a constant local system \( Z^{19}, Z^{18} \) respectively. \( \square \)

**Corollary 3.3.** Let \( f : X \to C \) be a family of semi-stable K3 surfaces over a curve \( C \). When it reaches the upper bound \( \deg f_* \omega_{X/C} = \deg \Omega^1_C(\log S) \), then the Picard number of the general fibres is at least 19. If \( \theta^{1-1} \theta^{2,0} = 0 \) and \( f \) reaches the upper bound \( \deg f_* \omega_{X/C} = \frac{1}{2} \deg \Omega^1_C(\log S) \), then the Picard number of the general fibres is at least 18.

### 4. Nikulin and Kummer construction

Let \( f : X \to C \) be a family of semi-stable K3 surfaces, which reaches the upper bound \( \deg f_* \omega_{X/C} = \deg \Omega^1_C(\log S) \). By Lemma 3.2, after passing to a finite étale cover of \( C \), one has

\[
V \otimes \mathbb{Q} = W \oplus \mathbb{Q}^{19},
\]

where \( W \) is an \( \mathbb{C} \)-irreducible representation of \( \pi_1(C \setminus S, \ast) \) and \( \mathbb{Q}^{19} \) is a constant local system of rank 19 such that \( \mathbb{Q}^{19}_t \subset NS(X_t) \otimes \mathbb{Q} \) for any \( t \in C \setminus S \). We obtain therefore,

**Lemma 4.1.** For any \( t \in C \setminus S \), the Picard number \( \rho(X_t) \geq 19 \) and for any class \( s_t \in \mathbb{Q}^{19}_t \subset \text{Pic}(X_t) \otimes \mathbb{Q} \) there is a \( \mathbb{Q} \)-divisor \( D \in \text{Div}(X) \otimes \mathbb{Q} \) such that \( D|_{X_t} = s_t \).
Let $Y$ be an algebraic K3 surface and $H^2(Y, Z) = T_Y \oplus NS(Y)$ be the orthogonal decomposition. $T_Y$ is the so-called transcendental lattice of $Y$, which is even and has signature $(2, 20 - \rho(Y))$. It is well-known that as lattices
\[ H^2(Y, Z) \cong U^3 \oplus E_8(-1)^2. \]
We recall some results about embeddings of lattices (see [15] and references given there).

**Lemma 4.2** (Theorem 2.4 of [12], or Corollary 2.6 of [15]). Let $T$ be a non-degenerate even lattice of rank $r$. Then there is a primitive embedding
\[ T \hookrightarrow U^r. \]
In particular, if $\rho(X) \geq 19$, then there is a primitive embedding
\[ T_X \hookrightarrow U^3. \]

**Lemma 4.3.** If $12 < \rho \leq 20$, then every even lattice $T$ of signature $(2, 20 - \rho)$ occurs as the transcendental lattice of some algebraic K3 surface and the primitive embedding $T \hookrightarrow U^3 \oplus E_8(-1)^2$ is unique.

**Theorem 4.4** ([15]). If $\rho(Y) \geq 19$, then there exists a primitive embedding
\[ \varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y, Z) \]
and a Nikulin involution $\tau : Y \to Y$ such that $\tau^* : H^2(Y, Z) \to H^2(Y, Z)$ is identity on $(\varphi(E_8(-1)^2))^\perp$.

**Proof.** By Lemma 4.2, there is a primitive embedding $\phi : T_Y \hookrightarrow U^3$, thus a primitive embedding $\phi \oplus 0 : T_Y \hookrightarrow U^3 \oplus E_8(-1)^2$. By Lemma 4.3 (uniqueness), the above embedding is isomorphic to
\[ T_Y = NS(X)^\perp \subset H^2(Y, Z) \cong U^3 \oplus E_8(-1)^2. \]
Thus, there is a primitive embedding
\[ \psi : E_8(-1)^2 \hookrightarrow T_Y^\perp = NS(Y) \subset H^2(Y, Z). \]
Let $\{c_j\}_{1 \leq j \leq 8}$ and $\{c_j\}_{1 \leq j \leq 8}$ be the bases of $E_8(-1) \oplus 0$ and $0 \oplus E_8(-1)$ and
\[ g : H^2(Y, Z) \to H^2(Y, Z) \]
be defined as: $g(\psi(c_j^1)) = \psi(c_j^2)$, $g(\psi(c_j^2)) = \psi(c_j^1)$ and $g(e) = e$ for any $e \in (\psi(E_8(-1)^2))^\perp$. Then, by theorems of Nikulin (see Theorem 5.6 of [Mo]), there is a Nikulin involution $\tau : Y \to Y$ and $w \in W(Y)$ (the group of Picard-Lefschetz reflections) such that $\tau^* = w \cdot g \cdot w^{-1}$. Let
\[ \varphi : E_8(-1)^2 \to H^2(Y, Z) \to H^2(Y, Z), \]
then $\varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y, Z)$ is another primitive embedding, and
\[ \tau^*(\varphi(c_j^1)) = \varphi(c_j^2), \quad \tau^*(\varphi(c_j^2)) = \varphi(c_j^1), \quad \tau^*(e) = e, \quad \forall e \in (\varphi(E_8(-1)^2))^\perp. \]
\[ \square \]
Let $t_0 \in C \setminus S$ be a point such that the fibre $X_{t_0}$ satisfying $\rho(X_{t_0}) = 19$. Thus,

$\mathbb{Q}_{t_0}^{19} = NS(X_{t_0}) \otimes \mathbb{Q}$.

Since the monodromy action of $\pi_1(C \setminus S, t_0)$ on $\mathbb{Q}_{t_0}^{19}$ is trivial, $\varphi(c_j^1)$ and $\varphi(c_j^2)$, $1 \leq j \leq 8$ can be lifted to divisors $D_j^1$ and $D_j^2$, $1 \leq j \leq 8$ on $X$. Then we have

**Lemma 4.5.** For any $t \in C \setminus S$, let $d_{j,t}^i = D_j^i|_{X_t} \in H^2(X_t, \mathbb{Z})$. Then $\{d_{j,t}^i\}_{1 \leq j \leq 8}$ ($i = 1, 2$) generate a sublattice of $H^2(X_t, \mathbb{Z})$, which is isomorphic to $E_8(-1)^2$ such that $E_8(-1)^2 \xrightarrow{\sim} H^2(X_t, \mathbb{Z})$ is a primitive embedding, $E_8(-1) \oplus 0$ and $0 \oplus E_8(-1)$ are isomorphic to $\mathbb{Z}\{d_{j,t}^i, j = 1, \ldots, 8\}$ and $\mathbb{Z}\{d_{j,t}^2, j = 1, \ldots, 8\}$

**Proof.** The proof is straightforward. For example, to prove that $\{d_{j,t}^i\}_{1 \leq j \leq 8}$ are $\mathbb{Z}$-linearly independent: if $\sum n_j d_{j,t}^i = 0$ in $H^2(X_t, \mathbb{Z})$, we claim that $\sum n_j \varphi(c_j^i) = 0$, which will imply the $\mathbb{Z}$-linear independence of $\{d_{j,t}^i\}_{1 \leq j \leq 8}$. The claim is clear.

To see that the embedding $E_8(-1)^2 \xrightarrow{\sim} H^2(X_t, \mathbb{Z})$ is primitive, let $B \in H^2(X_t, \mathbb{Z})$ be a class with $mB \in \mathbb{Z}\{d_{j,t}^i, i = 1, 2, j = 1, \ldots, 8\}$. Since $B$ is invariant under the monodromy, one finds a lifting $\tilde{B}$ of $B$. Since $\varphi : E_8(-1)^2 \xrightarrow{\sim} H^2(X_t, \mathbb{Z})$ is primitive and $m\tilde{B}|_{X_{t_0}} = \varphi(E_8(-1)^2)$, $\tilde{B}|_{X_{t_0}} = \sum n_j \varphi(c_j^i)$. Then

$$
\left( m(\tilde{B} - \sum n_j D_j^i) \right)|_{X_t}, m\left( \tilde{B} - \sum n_j D_j^i \right)|_{X_t} = 0
$$

and $(m(\tilde{B} - \sum n_j D_j^i)|_{X_t}, H|_{X_t}) = 0$. By Hodge index theorem one obtains $m(\tilde{B} - \sum n_j D_j^i)|_{X_t} = 0$, hence, $(\tilde{B} - \sum n_j D_j^i)|_{X_t} = 0$. \hfill $\Box$

Let $E = \bigoplus_{p+q=2} E^{p,q}$ denote the canonical extension of the Hodge bundle associated to the local system $R^2 f_* (\mathbb{Z})$, and $End(E) \to C$ denote the sheaf of endomorphisms of the vector bundle $m(\tilde{B} - \sum n_j D_j^i)|_{X_t} = 0$.

$$
End(E)^2 : \{ \text{ schemes over } C \} \to \{ \text{ sets} \}
$$

where $End(E)^2(T) = \{ \text{ bundle morphism } E_T \to E_T \text{ over } T \}$. For $t \in C \setminus S$, by Lemma 4.5, we can define an isometric involution

$g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$

by $g_t(d_{j,t}^i) = d_{j,t}^2$, $g_t(d_{j,t}^2) = d_{j,t}^1$, $g_t(e) = e$ for all $e \in \mathbb{Z}\{d_{j,t}^i\}$ and $1 \leq j \leq 8$. It is easy to see that $g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ is a morphism of $\pi_1(C \setminus S)$-modules. Thus, they give rise an involution

$$
g : R^2 f_* (\mathbb{Z}) \to R^2 f_* (\mathbb{Z})
$$

of local system, which corresponds to a section $g \in H^0(C \setminus S, End(E))$.
Lemma 4.6. The section \( g \in H^0(C \setminus S, \text{End}(E)) \) defined above can be extended to a section in \( H^0(C, \text{End}(E)) \), and thus \( g \) is an algebraic section.

Proof. Recall that \( R^2f_*({\mathcal{Z}_X}) \otimes \mathbb{Q} = \mathbb{W} \oplus \mathbb{Q}^{19} \) and the canonical extension of the Hodge bundle corresponding to \( R^2f_*({\mathcal{Z}_X}) \) can be written into

\[
(E, \theta) = (E_{W}, \theta) \oplus (\mathcal{O}_C^{19}, 0),
\]

where \((E_W, \theta)\) and \((\mathcal{O}_C^{19}, 0)\) are the canonical extension of the Hodge bundles corresponding to \( \mathbb{W} \) and \( \mathbb{Q}^{19} \) respectively. By the construction of \( g \), it is identity on \( \mathbb{W} \) (thus extended to \( E_W \)), and is well-defined on the constant lattice \( \mathbb{Z}^{19} \). Thus it is clear that \( g \) can be extended on \( C \).

\[ \square \]

Lemma 4.7. Let \( H \) be an ample divisor on \( X \) and \( g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z}) \) be the Hodge isometry involutions defined above. Then there exists a non-empty Zariski open set \( C^0 \subset C \setminus S \) such that \( g_t|_{X_t} \) is an ample divisor for any \( t \in C^0 \). In particular, \( g_t \) is an effective Hodge isometry for any \( t \in C^0 \).

Proof. We may write \( H|_{X_{t_0}} = \sum n_j^1 \varphi(c_j^1) + \sum n_j^2 \varphi(c_j^2) + e \), where \( e \in \varphi(E_8(-1)^2) \).

Let \( E \) be a lifting of \( e \) and

\[
D = \sum_{j=1}^{8} n_j^1 D_j^1 + \sum_{j=1}^{8} n_j^2 D_j^2 + E, \quad \tilde{D} = \sum_{j=1}^{8} n_j^1 D_j^2 + \sum_{j=1}^{8} n_j^2 D_j^1 + E.
\]

Then, for any \( t \in C \setminus S, H|_{X_t} = D|_{X_t} \) and \( g_t(D|_{X_t}) = \tilde{D}|_{X_t} \). Thus \( D \) is a relative ample divisor on \( f^{-1}(C \setminus S) \) and \( \tilde{D}|_{X_{t_0}} \) is ample (here we have chosen \( t_0 \) such that \( g_{t_0} \) is effective). Thus there exists a Zariski open set \( C^0 \subset C \setminus S \) such that \( \tilde{D} \) is relative ample on \( f^{-1}(C^0) \).

\[ \square \]

Lemma 4.8. The \( g \) induces an involution \( \tau : f^{-1}(C^0) \to f^{-1}(C^0) \) over \( C^0 \) such that \( \tau_t : X_t \to X_t \) (for \( t \in C^0 \)) are Nikulin involutions with \( \tau_t^* = g_t \).

Proof. Let \( \mathcal{L} = D + \tilde{D} \), where \( D \) and \( \tilde{D} \) are the divisors defined in the proof of Lemma 4.7. Then we know that \( \mathcal{L} \) is relative ample on \( f^{-1}(C^0) \) and \( \mathcal{L}|_{X_t} = \mathcal{L}|_{X_{t_0}} \) is invariant under the involution \( g_t \). Let \( \pi : \text{Aut}(f^{-1}(C^0)/C^0) \to C^0 \) denote the automorphism group scheme, which represents the functor

\[
\text{Aut}^\mathcal{L}(f^{-1}(C^0)/C^0, T) = \left\{ \text{Isomorphisms } h : f^{-1}(C^0) \times_{C^0} T \to f^{-1}(C^0) \times_{C^0} T \text{ over } T \text{ such that } h^*(p_T^*\mathcal{L}) = p_T^*(\mathcal{L}) \right\}.
\]

Thus there exists a universal automorphism

\[
f^{-1}(C^0) \times_{C^0} \text{Aut}^\mathcal{L}(f^{-1}(C^0)/C^0) \xrightarrow{h} f^{-1}(C^0) \times_{C^0} \text{Aut}^\mathcal{L}(f^{-1}(C^0)/C^0)
\]

and \( h^* \) induces an endomorphism \( \pi^*E \to \pi^*E \), which gives a homomorphism

\[
\text{Aut}^\mathcal{L}(f^{-1}(C^0)/C^0) \xrightarrow{\alpha} \text{End}(E)
\]

Thus

\[
\text{End}(E) = C^0.
\]
By Torelli theorem of K3 surfaces, \( \alpha \) is injective. On the other hand, the fibres of \( \alpha \) are isomorphic to group schemes, which are smooth. Thus \( \alpha \) is an embedding. By Lemma 4.6 and Lemma 4.7, \( g(C^0) \) is algebraic and contained in the image of \( \alpha \), which gives a section of \( \pi : \text{Aut}^\ell(f^{-1}(C^0)/C^0) \to C^0 \). That is an automorphism
\[
\begin{align*}
    f^{-1}(C^0) & \xrightarrow{\tau} f^{-1}(C^0) \\
    f \downarrow C^0 & \quad = \quad f \downarrow C^0
\end{align*}
\]
such that \( \tau_t^* = g_t \) for any \( t \in C^0 \). Thus \( \tau_t \) are Nikulin involutions, i.e. \( \tau_t^* \omega = \omega \) for any \( \omega \in H^{2,0}(X_t) \).

Since all fibres \( X_t \) are algebraic K3 surfaces, the \( \tau_t \) gives rise a Shioda-Inose structure on \( X_t \) by theorems of Morrison (see Theorem 6.3 of [15]). Let \( g : Z^0 \to C^0 \) be the desingularization of \( f^{-1}(C^0)/\tau \to C^0 \). Then \( g : Z^0 \to C^0 \) is a family of Kummer surfaces and there exist divisors \( N_1, ..., N_8 \) on \( Z^0 \) such that their restrictions \( (N_1)_t, ..., (N_8)_t \) on \( Z^t \) are the exceptional \((-2)\)-curves of the double points of \( X_t/\tau_t \) (produced by the eight isolated fixed points of \( \tau_t \)). By Lemma 3.2, we write \( R^2 g_* (Z_{f^{-1}(C^0)}) = \mathbb{W} \oplus \mathbb{Z}^{19} \). Then we have (see Lemma 3.1 of [15])
\[
R^2 g_* (Z_{Z^0}) \simeq (\mathbb{W} \oplus \mathbb{Z}^{19'})(2) \oplus \mathbb{Z}[N_1, ..., N_8],
\]
where \( \mathbb{Z}^{19'} \) is the invariant sub local system of \( \mathbb{Z}^{19} \) under \( \tau \), \( (\mathbb{W} \oplus \mathbb{Z}^{19'})(2) \) has the same underlying local system as \( (\mathbb{W} \oplus \mathbb{Z}^{19'}) \), and with the intersection form defined by multiplication by 2 of the the intersection form on \( (\mathbb{W} \oplus \mathbb{Z}^{19'}) \).

**Lemma 4.9.** By making \( C^0 \) smaller, there exists a family of abelian surfaces
\[
a : A^0 \to C^0
\]
with \( \rho(A^0_t) \geq 3 \) such that \( g : Z^0 \to C^0 \) is its Kummer construction.

**Proof.** It is easy to see that, for any \( t \in C^0 \), \( NS(Z^0_t) \) contains a sub-lattice, which is isomorphic to \( \mathbb{Z}^{19'}(2) \oplus \mathbb{Z}[N_1, ..., N_8] \) as a trivial \( \tau_1(C \setminus S) \)-modules. Thus \( g : Z^0 \to C^0 \) is a family of Kummer surfaces with \( \rho(Z_t) \geq 19 \). Let \( t_0 \in C^0 \) with \( \rho(Z^0_{t_0}) = 19 \). Then \( NS(Z^0_{t_0}) \supset \mathbb{Z}^{19'} \oplus \mathbb{Z}[N_1, ..., N_8] \) and
\[
NS(Z^0_{t_0}) \otimes \mathbb{Q} = (\mathbb{Z}^{19'} \oplus \mathbb{Z}[N_1, ..., N_8]) \otimes \mathbb{Q}.
\]

Let \( F_1, ..., F_{16} \) be the liftings of the sixteen pairwise-disjoint \((-2)\)-curves on \( Z^0_{t_0} \) to \( Z^0 \). It is not difficult to see that we can choose \( F_i \) \( (i = 1, ..., 16) \) to be effective divisors on \( Z^0 \). In fact, since \( g_* \mathcal{O}_{Z^0}(F_i) \neq 0 \) (because \( H^0(F_i | Z^0_{t_0}) \neq 0 \) for any \( t \in C^0 \) by Riemann-Roch theorem), we have, for \( m \) large enough and a point \( p \in Z^0, H^0(\mathcal{O}_{Z^0}(F_i + mg^{-1}(p))) = H^0(\mathcal{O}_{Z^0}(mp) \otimes g_* \mathcal{O}_{Z^0}(F_i)) \neq 0 \). Thus there is an effective divisor \( D \) on \( Z^0 \) such that \( D|_{Z^0_{t_0}} \) is numerical equivalent to \( F_i|_{Z^0_{t_0}} \), which implies that \( D|_{Z^0_{t_0}} = F_i|_{Z^0_{t_0}} \) since a nodal class is represented by only one effective divisor. We can choose \( F_i \) \( (i = 1, ..., 16) \) to be irreducible further. In fact, we will show that \( F_i|_{Z^0_{t_0}} \) is irreducible if \( \rho(Z^0_{t_0}) = 19 \). Otherwise, let \( F_i|_{Z^0_{t_0}} = D_1 + D_2 \), where \( D_1 \) is irreducible with \( D_1^2 = -2 \) and \( D_2 \) is effective.
Note that for any lifting of an irreducible curve, whose restriction to any other fibre is equivalent to an effective divisor. Thus if $\tilde{D}_1$ and $\tilde{D}_2$ are the liftings of $D_1$ and $D_2$ (with lifting the irreducible components of $D_2$), we see that $\tilde{D}_1|_{Z_{t_0}}$ and $\tilde{D}_2|_{Z_{t_0}}$ are equivalent to effective divisors. On the other hand, $F_i|_{Z_{t_0}^0} - \tilde{D}_1|_{Z_{t_0}^0}$ is numerically equivalent to $\tilde{D}_2|_{Z_{t_0}^0}$ since it is so on $Z_{t_0}^0$. But this is impossible since $F_i|_{Z_{t_0}^0}$ is a nodal class. Let $g : Z \to C$ be a compactification of $g : Z^0 \to C^0$ with $Z$ smooth and $F_1, ..., F_{16}$ be extended to $Z$. It is known that $F_1|_{Z_{t_0}} + \cdots + F_{16}|_{Z_{t_0}} \equiv 2\delta$. Let $\Delta$ be a divisor on $Z$ such that $\Delta|_{Z_{t_0}} = \delta$. Then $F_1 + \cdots + F_{16} - 2\Delta$ is numerically equivalent to zero on the general fibres, thus

$$F_1 + \cdots + F_{16} - 2\Delta \equiv g^*D_a, \quad D_a \in \text{Div}(C).$$

Choose $C^0$ smaller so that $F_i|_{Z_t}$ ($i = 1, ..., 16$) are irreducible for $t \in C^0$ and $F_1 + \cdots + F_{16} \equiv 2\Delta$ on $Z^0$.

Let $A^0 \to Z^0$ be the double covering with branch locus $F_1 + \cdots + F_{16}$, and let $\pi : A^0 \to A^0$ be the uniform blowing down of the sixteen $(-1)$-curves on the fibres $A_{0,t}$. Then $a : A^0 \to C^0$ is the family of abelian surfaces with $\rho(A^0_t) \geq 3$.

5. Splitting on families of abelian surfaces

Let $a : A^0 \to C^0$ be the family of abelian surfaces constructed in Lemma 4.9. We take a compactification $a : A \to C$, (which may not be semi-stable). We consider the decomposition

$$R^2a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q} = \mathbb{Q}^\rho \oplus T_a,$$

where $\mathbb{Q}^\rho$ is the maximal constant sub local system and its complement $T_a$ is the so-called the sub VHS of the transcendental part of $R^2a_*(\mathbb{Z}_{A^0})$. It is known that $T_a$ is Hodge isometric to $T_g(2)$, where $T_g$ is the sub VHS of the transcendental part of the weight-$2$ VHS $R^2g_*\mathbb{Z}_{Z^0}$ attached to the family of Kummer surfaces $g : Z^0 \to C^0$ arisen from $a : A \to C$. Furthermore, $T_g$ is Hodge isometric to $T_f(2)$, where $T_f = \mathcal{W}$ is the sub VHS of the transcendental part of the weight-$2$ VHS $R^2f_*\mathbb{Z}_{f^{-1}(C^0)}$ attached to one original family $f : f^{-1}(C^0) \to C^0$. Since $\mathcal{W}$ is, in fact, defined on $C \setminus S$, $T_a$ can be extended to $C \setminus S$ as an VHS.

**Lemma 5.1.** The $\mathbb{Q}$–vector space of endomorphisms of

$$R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q}$$

has dimension 4, and is of $(0,0)$-type.

**Proof.** By the construction of $a : A^0 \to C^0$, we see $R^2a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q}$ contains a constant local system of dimension 3 of $(1,1)$-type (this corresponds to a sub-lattice of Picard lattice of $A^0$). Hence, it corresponds to a 3-dimensional subspace of
Lemma 5.2. The family \(a : A^0 \to C^0\) is isogenous to the square product of a family of elliptic curves \(e : E^0 \to C^0\).

Proof. Case 1). Suppose that there is a subset \(T \subset C^0\) of non-countable many points such that \(A_t\) is isogenous to \(E_t \times E_t, t \in T\). Since there are only countable many isomorphic classes of elliptic curves having complex multiplication, we find an \(t_0 \in T\) such that \(\text{End}(E_{t_0}) \otimes \mathbb{Q} = \mathbb{Q}\). Hence, the endomorphism algebra
\[
\text{End}(A_{t_0}) \otimes \mathbb{Q} \cong M_2(\mathbb{Q}).
\]
In the other words, we have \(\text{End}(R^1a_* (\mathbb{Z}_{A^0}) \otimes \mathbb{Q})|_{t_0} \cong M_2(\mathbb{Q})\). Since \(\text{End}(R^1a_* (\mathbb{Z}_{A^0}) \otimes \mathbb{Q})\) is constant local system, we have \(\text{End}(R^1a_* (\mathbb{Z}_{A^0}) \otimes \mathbb{Q}) \cong M_2(\mathbb{Q})\). The element
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\in \text{End}(R^1a_* (\mathbb{Z}_{A^0}) \otimes \mathbb{Q})
\]
gives a \(\mathbb{Q}\)-splitting \(R^1a_* (\mathbb{Z}_{A^0}) \otimes \mathbb{Q} = \mathbb{W}_Q \oplus \mathbb{W}_Q\), thus isogeny splitting of \(f : A^0 \to C^0\) into the square product of a family of elliptic curves \(e : E^0 \to C^0\).

Case 2). Suppose that there are non-countable many points \(\{t\} \subset C^0\) such that \(A_t\) is simple. Since the Picard number \(\rho(A_t) \geq 3\), one checks easily that \(\rho(A_t) = 3\) and \(\text{End}(A_t) \otimes \mathbb{Q}\) is the totally indefinite quaternion algebra over \(\mathbb{Q}\). An abelian surface with this type endomorphism algebra is called a false elliptic curve. There are countable many projective curves \(\{C_i\}_{i \in \mathbb{N}}\) in the moduli space of polarized abelian surfaces, which are Shimura curves of certain type and parameterize all false elliptic curves. So, the family \(a : A^0 \to C^0\) induces a morphism \(\phi : C^0 \to C_i\) for some \(i \in \mathbb{N}\), which extends to a morphism \(\phi : C \to C_i\). This implies that the local monodromies of \(R^2a_* (\mathbb{Z}_{A^0})\) around the singularity has finite order. It contradicts to \(S \neq \emptyset\). 

\(\square\)
6. Proof of Theorems 0.1, 0.2 and Corollary 0.4

Proof of Theorem 0.1. Only the modularity of $C' \setminus \sigma^{-1}S$ needs to be checked. The isogeny $a : A^0 \to C^0 \sim e^2 : E^0 \times_{C^0} E^0 \to C^0$ induces an isomorphism $S^2(R^1\epsilon_*(\mathbb{Z}_{E^0})) \simeq \mathbb{W}|_{C^0}$. There are natural group homomorphisms
\[ 1 \to \{ \pm 1 \} \to SL_2(\mathbb{R}) \to SO(1,2), \]
which induce an isomorphism between $\mathcal{H}$ and a connected component of the symmetric space $SO(1,2)/SO(2) \times O(1)$, say
\[ i : \mathcal{H} \simeq SO^+(1,2)/SO(2) \times O(1). \]

Since $\mathbb{W}|_{C^0}$ is the restriction of $\mathbb{W}$ on $C \setminus S$ to $C^0$, the local monodromies of $R^1\epsilon_*(\mathbb{Z}_{E^0})$ around $(C \setminus S) \setminus C^0$ are either $+1$, or $-1$. Thus the projective monodromy representation of $R^1\epsilon_*(\mathbb{Z}_{E^0})$ is actually defined on $C \setminus S$, say
\[ \rho_{R^1\epsilon_*(\mathbb{Z}_{E^0})} : \pi_1(C \setminus S, \ast) \to PSL_2(\mathbb{Z}). \]

Let $\tilde{\phi}_{R^1\epsilon_*(\mathbb{Z}_{E^0})} : \tilde{C} \setminus \tilde{S} \to \tilde{\mathcal{H}}$ be the period map corresponding to $R^1\epsilon_*\mathbb{Z}_{E^0}$ and
\[ \tilde{\phi}_{\mathbb{W}} : \tilde{C} \setminus \tilde{S} \to SO^+(1,2)/SO(2) \times O(1) \]
denote the period map corresponding to $\mathbb{W}$. Then $\tilde{\phi}_{\mathbb{W}} = i \cdot \tilde{\phi}_{R^1\epsilon_*(\mathbb{Z}_{E^0})}$ is an isomorphism. In fact, the tangent map of $\tilde{\phi}_{\mathbb{W}}$ is precisely the Kodaira-Spencer map of $\mathbb{W}$: $\theta^{2,0} : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S)$, which is isomorphic at each point by Lemma 1.1. Thus $\tilde{\phi}_{\mathbb{W}}$ is a local diffeomorphism. Since the Hodge metric on the Higgs bundle corresponding to $\mathbb{W}$ has logarithmic growth at $S$ and bounded curvature by Schmid [22], together with the remarks after Proposition 9.1 and Proposition 9.8 in [25], $\tilde{\phi}_{\mathbb{W}}$ is a covering map, hence an isomorphism. This implies that $\tilde{\phi}_{R^1\epsilon_*(\mathbb{Z}_{E^0})}$ is an isomorphism. Thus
\[ \phi_{R^1\epsilon_*(\mathbb{Z}_{E^0})} : C \setminus S \simeq \mathcal{H}/\rho_{R^1\epsilon_*(\mathbb{Z}_{E^0})} \]
is an isomorphism. \hfill $\square$

In order to prove Theorem 0.2, we need the following lemma.

Lemma 6.1. Let $f : X \to C$ be a family of semi-stable K3 surfaces, which has zero iterated Kodaira-Spencer map and reaches the Arakelov bound (II)
\[ \deg f_*\omega_{X/C} = \frac{1}{2} \deg \Omega^1_C(\log S). \]

Then, after passing to a finite étale covering $C' \to C$, the VHS $\mathbb{W}$ is non-rigid.

Proof. One needs to show that, after passing through a finite étale covering of $C$, the local system $R^2f_*\mathbb{Z}_{X^0} \otimes \mathbb{C}$ admits a non-zero endomorphism of type $(-1,1)$. By Lemma 3.2, one has splitting
\[ R^2f_*\mathbb{Z}_{X^0} \supset \mathbb{W}_Z \oplus \mathbb{Z}^{18}, \quad R^2f_*\mathbb{Z}_{X^0} \otimes \mathbb{Q} = (\mathbb{W}_Z \oplus \mathbb{Z}^{18}) \otimes \mathbb{Q}. \]

By Lemma 1.2, the Higgs bundle corresponds to $\mathbb{W}$ has the form
\[ (E^{2,0} \oplus E^{1,1}_1) \oplus (E^{1,1}_1 \oplus E^{0,2}) \]
such that the Higgs fields
\[ \tau : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S), \quad \tau^* : E^{1,1}_1 \otimes \Omega^1_C(\log S) \]
are isomorphisms. These two Higgs subbundles correspond to two sub-local systems \( W_1 \) and \( W_2 \). We claim that, after passing to a finite étale covering of \( C \), one has \( W_1 \simeq \tilde{W}_1 \). To prove the claim, consider the sub-local system
\[ W_1 \to W. \]
If \( W_1 \) is not rigid, then there is a small deformation \( W_{1,t} \subset W \otimes \mathbb{C} \) such that both projections \( W_{1,t} \subset W \otimes \mathbb{C} \to W_1 \) and \( W_{1,t} \subset W \otimes \mathbb{C} \to \tilde{W}_1 \) are non-zero. Since \( W_1 \) is irreducible, one obtains
\[ W_1 \simeq W_{1,t} \simeq \tilde{W}_1. \]
If \( W_1 \) is rigid, then by Lemma 2.1 \( W_1 \) is defined over a number field \( K \). Let \( \mathcal{O}_K \) denote the ring of algebraic integers in \( K \), and let
\[ W_1 \mathcal{O}_K = W \otimes \mathcal{O}_K \cap W_1. \]
Then \( W_1 \mathcal{O}_K \otimes K = W_1 \), which means that the corresponding monodromy representation of \( W_1 \) can be defined over \( \mathcal{O}_K \). The determinant \( \det W_1 = E^{2,0}_1 \otimes E^{1,1}_1 \) is a rank-1 unitary local system \( \eta \in \text{Pic}^0(C) \) and takes values in \( \mathcal{O}_K \). By a theorem of Kronecker, \( \eta \) is a torsion. So, after passing to the finite étale covering corresponding to \( \eta \), one obtains \( E^{2,0}_1 \simeq E^{1,1}_1 \) and
\[ (E^{2,0}_1 \oplus E^{1,1}_1, \tau) \simeq (E^{1,1}_1 \oplus E^{0,2}, \tau^*). \]
Thus, in any case, we obtain a non-zero endomorphism
\[ (E^{2,0}_1 \oplus E^{1,1}_1) \oplus (E^{1,1}_1 \oplus E^{0,2}) \to (E^{2,0}_1 \oplus E^{1,1}_1) \oplus (E^{1,1}_1 \oplus E^{0,2}) \]
of type \((-1,1)\), which corresponds to an endomorphism of \( R^2 f_*(\mathcal{O}_X) \otimes \mathbb{C} \) of type \((-1,1)\).

**Proof of Theorem 0.2.** By Lemma 6.1, after passing to a finite étale covering \( C' \to C \), the VHS \( W \) is non-rigid. By Corollary 5.6.3 of [21], one has
\[ \text{End}(W) \otimes \mathbb{Q} \simeq M_2(\mathbb{Q}). \]
Taking an element in \( M_2(\mathbb{Q}) \) with two distinct rational eigenvalues, we get a \( \mathbb{Q} \)-splitting \( W \otimes \mathbb{Q} = W_1 \oplus W_2 \) such that \( W_1 \) is isomorphic to \( \tilde{W}_2 \) and the Higgs bundle corresponding to \( W_1 \) has the form
\[ (L \oplus L^{-1}, \theta), \quad \theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S). \]
\( W_1 \) has an \( \mathbb{Z} \)-structure defined by \( W_{1Z} = W_2 \cap W_1 \). Again by Proposition 9.1 of [25], the Higgs bundle \( \theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S) \) gives rise to the uniformization
\[ C \setminus S \simeq \mathcal{H}/\rho_{\tilde{W}_1} \pi_1(C \setminus S,*), \]
where \( \rho_{\tilde{W}_1} \pi_1(C \setminus S,* \subset SL_2(\mathbb{Z}) \) of finite index.
Proof of Corollary 0.4. i) By Theorem 0.1 there exists a family of elliptic curves
\( g : E^0 \to \mathbb{P}^{10} \subset \mathbb{P}^1 \setminus S \) such that the projective representation
\[
\rho R_{g, Z_{E^0}} : \pi_1(\mathbb{P}^1 \setminus S, \ast) \to \Gamma' \subset PSL_2(\mathbb{Z})
\]
extends to \( \mathbb{P}^1 \setminus S \) and \( \mathbb{P}^1 \setminus S \cong \mathbb{H}/\Gamma' \). By \([2]\), \( \Gamma' \subset PSL_2(\mathbb{Z}) \) is of index 12 and conjugates to one of the following 6 subgroups of \( PSL_2(\mathbb{Z}) \), which are images of \( \Gamma(3), \Gamma_0^0(4) \cap \Gamma(2), \Gamma_0^0(5), \Gamma_0^0(6), \Gamma_0^0(8) \cap \Gamma_0^0(4) \) and \( \Gamma_0^0(9) \cap \Gamma_0^0(3) \) in \( SL_2(\mathbb{Z}) \) of index 24, where
\[
\Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv c \equiv 0, a \equiv 1(\text{mod.} n) \right\}, \\
\Gamma_0^0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0, a \equiv 1(\text{mod.} n) \right\}, \\
\Gamma_0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0(\text{mod.} n) \right\}.
\]

In the proof of Theorem 0.1, we have seen already that the monodromy of \( R_{g, Z_{E^0}} \) of a short loop around a point of \( (\mathbb{P}^1 \setminus S) \setminus \mathbb{P}^{10} \) is either +1, or -1. If all of them equal to +1, then the representation \( \rho R_{g, Z_{E^0}} \) extends to \( \mathbb{P}^1 \setminus S \), and the image of \( \pi_1(\mathbb{P}^1 \setminus S, \ast) \) under this representation conjugates to one of the above 6 subgroups. Hence \( g : E^0 \to \mathbb{P}^{10} \) extends to a modular family of elliptic curves \( g : E \to \mathbb{P}^1 \setminus S \) from one of 6 examples in \([2]\). Suppose that the monodromies of \( R_{g, Z_{E^0}} \) of short loops around some points of \( (\mathbb{P}^1 \setminus S) \setminus \mathbb{P}^{10} \) equal to -1. Then the image of \( \pi_1(\mathbb{P}^1 \setminus S, \ast) \) conjugates to the preimage \( p^{-1}p\Gamma \), where \( \Gamma \) is one of \( \Gamma(3), \Gamma_0^0(4) \cap \Gamma(2), \Gamma_0^0(5), \Gamma_0^0(6), \Gamma_0(8) \cap \Gamma_0^0(4) \) and \( \Gamma_0(9) \cap \Gamma_0^0(3) \). The inclusion \( \Gamma \subset p^{-1}p\Gamma \) of index 2 defines an étale covering \( E^{0'} \to E^0 \), which is étale along the fibres and the family \( g' : E^{0'} \to \mathbb{P}^{10} \) extends to the modular family of elliptic curves \( g' : E' \to \mathbb{P}^1 \setminus S \) corresponding to \( \Gamma \).

ii) is straightforward. \( \square \)

Acknowledgments

This paper owes a lot to discussions with H. Esnault, B. Hassett, N. Mok, A. Todorov, E. Viehweg, and S.-T. Yau. Also, conversations with S.-W. Zhang on Shimura curves were very helpful for us. Viehweg read through the preliminary version, and made many valuable suggestions on how to improve the paper. We would like to thank all of them. Finally, we would like to express our appreciation to the referee for many valuable suggestions of corrections to the original version.

References

FAMILIES OF K3 SURFACES


Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, and Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, P. R. of China.

E-mail address: xsun@math08.math.ac.cn

Department of Mathematics, East China Normal University, Shanghai 200062, P. R. of China.

E-mail address: sltan@euler.math.ecnu.edu.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: kzuo@math.cuhk.edu.hk