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A NOTE ON ARTIN MOTIVES

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Abstract. We define a category of (pure) motives analogous to the category of absolute Hodge cycles and show that in this category every motive with trivial Hodge filtration is isomorphic to an Artin motive. We discuss relations with conjectures of Deligne, and of Fontaine-Mazur.

Introduction

In [FM] Fontaine and Mazur made the following conjecture.

Conjecture (Unramified Fontaine-Mazur Conjecture)**.** *Let K be a number field, p a rational prime, and S a finite set of primes of K not containing any primes above p. Let $G_{K,S}$ be the Galois group of the maximal extension of K unramified outside S. Then any continuous representation of $G_{K,S}$ on a finite dimensional $\mathbb{Q}_p$-vector space V factors through a finite quotient.*

In the same paper, Fontaine and Mazur also made other conjectures predicting when a p-adic representation of the absolute Galois group G_K of K should "come from geometry" in the sense of arising as a subquotient in the p-adic étale cohomology of some algebraic variety defined over K. They remark that these conjectures together with the Tate conjectures on algebraic cycles imply the conjecture above.

The main purpose of this note is to point out that if one assumes that the representation of $G_{K,S}$ on V in the above conjecture is "motivic", then the action of $G_{K,S}$ does indeed factor through a finite quotient. Here "motivic" means that as well as the representation V, we have a collection of auxiliary data, which looks like the set of realisations of a motive, with V being the p-adic étale realisation. This fits nicely with the following conjecture, which Fontaine has informed us was behind the discussion in [FM]

Conjecture. *A motive of Hodge type* $(0,0)$ *is an Artin motive.*

This conjecture may in turn be viewed as a refinement of the Mumford-Tate conjecture for motives of type $(0,0)$.

The proof of our finiteness result is not particularly difficult, but one nice feature is that it uses some of the more subtle properties of motives: compatible systems of Galois representations, comparison theorems between p-adic étale and crystalline cohomology, and the Weil conjecture.

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To formalise the situation, we begin in §1 by introducing for any collection of realisations, a suitable category of motives. The construction is entirely analogous to Deligne's construction of absolute Hodge cycle motives, which corresponds to taking the ℓ-adic, de Rham, and Betti realisations. On the other hand, in our situation we will be more interested in taking the ℓ-adic, de Rham, and crystalline realisations. In particular, built into the theory is the fact that such motives admit a comparison theorem between ℓ-adic and crystalline cohomology (in characteristic ℓ) for sufficiently large ℓ.

The above formalism has the advantage that it gives a supply of motives to which our basic finiteness result, proved in §2, applies. However it is also useful for making more refined statements about what can and cannot be proved, and in particular for describing how close one can come to proving the above conjecture on motives of type $(0,0)$ in various categories of motives.

In §3 we introduce motives with coefficients. A conjecture of Deligne says that a motive over $\mathbb{Q}$ of rank 1, with coefficients in a finite extension $E/\mathbb{Q}$ should be of the form $E(\varepsilon)(n)$, where ε is a finite E-valued character of $\mathbb{Q}$. This conjecture can be regarded as a special case of the conjecture on motives of type $(0,0)$. We briefly relate it to our previous results.

Finally we return to the remark of Fontaine and Mazur, that their conjectures together with the Tate conjectures imply the unramified Fontaine-Mazur conjecture. We give a proof of this remark assuming in addition that the action of $G_{K,S}$ on the p-adic étale cohomology of a smooth projective variety over K is semi-simple. This last assumption is known as the Grothendieck-Serre conjecture! Several people had expressed to us an interest in seeing such a proof.

One slightly subtle point here is that not any subquotient in the p-adic étale cohomology of a smooth projective variety is motivic. For example, if E is an elliptic curve with complex multiplication by a field K, and p is a prime which splits in K, then the p-adic étale cohomology of E is a sum of two characters, which are not motivic. The existence of non-motivic subquotients is the reason one has to assume a condition on the action of inertia at *every* prime over p in the unramified Fontaine-Mazur conjecture. For motivic representations, if there is one prime over p where the inertia acts through a finite quotient, then the existence of the Hodge filtration forces the same condition as the other primes over p.

1. The category of $\mathcal{C}$-motives

Let $K/\mathbb{Q}$ be an algebraic number field. We fix an algebraic closure $\bar{K}$ and denote by $G_K = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group. In the following we want to define a modification of the absolute Hodge cycles introduced by Deligne (see [De] and [DMOS, II,§ 6]).

1.1. Definition of $\mathcal{C}$-cycles. If X is a smooth projective variety over K one has the following cohomology theories at hand ($r \geq 0$).

COH-1. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$ the singular cohomology $H^r_\sigma(X) = H^r(\sigma X(\mathbb{C}), \mathbb{Q})$, which is a $\mathbb{Q}$-vector space with a $\mathbb{Q}$-Hodge structure of weight r.

COH-2. The hypercohomology $H^r_{DR}(X) = \mathbb{H}^r(X, \Omega^\bullet_X)$ of the differential complex $\Omega^\bullet_X$. This is a K-vector space together with a K-linear decreasing filtration $(F^i)_{i\in\mathbb{Z}}$.

COH-3. For each prime number ℓ the ℓ-adic cohomology $H^r_\ell(X) = H^r_{\acute{e}t}(X \times_K \bar{K}, \mathbb{Q}_\ell)$, which is a $\mathbb{Q}_\ell$-vector space with a continuous G_K-action. Outside a finite set S of finite primes this action is unramified and for $v \notin S$ let $\Phi_v \in D_v$ be a lifting of the geometric Frobenius $f_v \in D_v/I_v$. (As usual D_v denotes a decomposition subgroup of G_K and I_v its inertia subgroup.) Φ_v acts on $H^r_\ell(X)$ and the Weil conjectures tell us that the eigenvalues of the characteristic polynomial of this action are algebraic numbers whose absolute values (under any archimedean norm) are equal to $q_v^{r/2}$, where $q_v = p^m$ is the cardinality of the residue field $\kappa(v)$ of v.

COH-4. After enlarging S we can assume that it contains all places of bad reduction of X and all primes ramified in $K/\mathbb{Q}$. For any $v \notin S$ the crystalline cohomology $H^r_{\mathrm{cris},v}(X) = H^r_{\mathrm{cris}}(\mathcal{X} \otimes_{\mathcal{O}_v} \kappa(v)) \otimes_{\mathcal{O}_v} K_v$ is a K_v-vector space equipped with a σ_v-semi-linear automorphism φ_v, the crystalline Frobenius. (Here $\sigma_v = \mathrm{Frob}_{K_v}$.) Its mth power φ_v^m, the relative Frobenius, acts linearly.

These cohomology theories are related as follows.

COMP-1. Firstly, the de Rham isomorphism gives for every $\sigma : K \hookrightarrow \mathbb{C}$:

$$\mathrm{comp}_{DR,\sigma} : H^r_\sigma(X) \otimes \mathbb{C} \xrightarrow{\sim} H^r_{DR}(X) \otimes_{K,\sigma} \mathbb{C}.$$

This isomorphism is compatible with the Hodge-filtration on both sides.

COMP-2. Next we have for every ℓ and for every $\sigma : K \hookrightarrow \mathbb{C}$ and $\bar{\sigma} : \bar{K} \hookrightarrow \mathbb{C}$ with $\bar{\sigma}_{|K} = \sigma$ an isomorphism

$$\mathrm{comp}_{\ell,\bar{\sigma}} : H^r_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^r_{\acute{e}t}(\bar{\sigma}(X \times_K \bar{K}), \mathbb{Q}_\ell) \simeq H^r_\ell(X)$$

These isomorphisms satisfy $\tau \circ \mathrm{comp}_{\ell,\bar{\sigma}} = \mathrm{comp}_{\ell,\bar{\sigma}\tau}$, i.e. they are compatible with the G_K-action of H^r_ℓ.

COMP-3. For all $v \notin S$, we have the canonical isomorphism

$$\mathrm{comp}_{\mathrm{cris},v,DR} : H^r_{DR}(X) \otimes_K K_v \xrightarrow{\sim} H^r_{\mathrm{cris},v}(X).$$

We will consider the right hand side with the induced filtration.

COMP-4. Finally let $v \notin S$ with residue characteristic p. There is a B_{cris}-linear, D_v-equivariant isomorphism, respecting filtrations and Frobenius actions.

$$\mathrm{comp}_{\mathrm{cris},v,p} : H^r_p(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^r_{\mathrm{cris},v}(X) \otimes_{K_v} B_{\mathrm{cris}}.$$

($*_{\text{char}}$) If ℓ is a prime not dividing v one has the action of Φ_v on $H^r_\ell(X)$ and the action of φ_v^m on $H^r_{\text{cris},v}(X)$. The characteristic polynomials of these actions have $\mathbb{Q}$-coefficients and we have [KM, Thm 1]

$$P^r_{\ell,v}(T) = \det(1 - T\Phi_v; H^r_\ell(X)) = \det(1 - T\varphi_v^m; H^r_{\text{cris},v}(X)) = P^r_{\text{cris},v}(T).$$

We can summarise this in the following diagram

$$\begin{tikzcd}
(\sigma) \arrow[rr, leftrightarrow, dashed, "\text{comp}_{DR,(\sigma)}"] \arrow[d, leftrightarrow, dashed, "\text{comp}_{(\ell),(\bar\sigma)}"'] & & DR \arrow[d, leftrightarrow, dashed, "\text{comp}_{(\text{cris},v),DR}"] \\
(\ell) \arrow[rr, leftrightarrow, dashed, "\text{comp}_{(\text{cris},v),(p)}"'] & & (\text{cris}, v).
\end{tikzcd}$$

We will use the abbreviations $H^r_\alpha(M)$ and $\text{comp}_{\alpha,\alpha'}$ for the objects appearing in this diagram.

Remark. The above diagram leads to a conjectural relation between complex and p-adic periods. We learnt from Fontaine that it is a special case of a more elaborate and precise conjecture he has made (unpublished).

We fix embeddings $\overline{K} \hookrightarrow \mathbb{C}$ and $\overline{K} \hookrightarrow \mathbb{C}_p$ and omit them for ease of notation, in particular we write $H_B(X)$ for the singular cohomology defined via these embeddings, and we denote by v the induced valuation on K. We consider in the following only varieties X with good reduction at v.

For a K-algebra A we denote by $H_B \otimes A$ and $H_{DR} \otimes A$ the functors $X \mapsto H_B(X) \otimes_{\mathbb{Q}} A$ and $X \mapsto H_{DR}(X) \otimes_K A$. The torsor $\underline{\text{Isom}}(H_B, H_{DR})$ assigns to an affine K-scheme $\operatorname{Spec} A$ the set of isomorphisms $\text{Isom}(H_B \otimes A, H_{DR} \otimes A)$. This torsor is representable by a scheme $\operatorname{Spec} R$, with R an inductive limit of finite type K-algebras.

It has a $\mathbb{C}$-valued point coming from COMP-1. It seems reasonable to conjecture that the resulting map $R \to \mathbb{C}$ is injective. In fact such a claim is related to the Hodge conjecture. On the other hand, we have a B_{cris}-valued point given by

$$H_B(X) \otimes B_{\text{cris}} \simeq H_p(X) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \simeq H_{\text{cris},v}(X) \otimes_{K_v} B_{\text{cris}} \simeq H_{DR}(X) \otimes_K B_{\text{cris}}$$

and the resulting map $R \to B_{\text{cris}}$.

Thus, conjecturally, we have a correspondence between complex and p-adic periods extending the well known correspondence $2\pi i \mapsto t$

Definition 1.1. Let $\mathcal{C} \subset \{(\sigma), (\ell), DR, (\text{cris}, v)\}$. The group $C^r_{\mathcal{C}}$ of $\mathcal{C}$-cycles of codimension r is defined by

$$C^r_{\mathcal{C}}(X) = \{x = (x_\alpha)_{\alpha\in\mathcal{C}} \in \prod_{\alpha\in\mathcal{C}} H^{2r}_\alpha(X)(r) \mid \text{comp}_{\alpha,\alpha'}(x_{\alpha'}) = x_\alpha\}/\sim_S,$$

satisfying the following properties:

(i) x_σ is a Hodge cycle, i.e. is of type $(0,0)$ if $(\sigma) \in \mathcal{C}$;

(ii) $x_{DR} \in F^0(H^{2r}_{DR}(X)(r))$ if $DR \in \mathcal{C}$;

(iii) $g(x_\ell) = x_\ell$ if $(\ell) \in \mathcal{C}$ for all $g \in G_K$.

(iv) $\varphi_v(x_v) = x_v$ if $(\text{cris}, v) \in \mathcal{C}$;

and the equivalence relation $\sim_S$ is trivial if $(\text{cris}, v) \notin \mathcal{C}$ and is defined by $(x_\alpha) \sim_S (y_\alpha)$ if and only if $x_\alpha = y_\alpha$ for $\alpha \in \{(\sigma), (\ell), DR\} \cap \mathcal{C}$, and there exists a finite set of finite primes S such that $x_{\text{cris},v}$ and $y_{\text{cris},v}$ are defined and coincide for all $v \notin S$.

Remark. The conditions (i)-(iv) are necessarily fulfilled if $\mathcal{C}$ is big enough. For example if $\{(\ell), (\sigma)\} \subset \mathcal{C}$ then property (iii) is forced by COMP-2.

Example. For each of the cohomology theories above one has cycle class maps

$$cl^r_\alpha : \mathcal{Z}^r(X)/\sim_{rat} \to H^{2r}_\alpha(X)(r)$$

(see [DMOS, pp20-21] for the definition if $\alpha \in \{(\sigma), (\ell), DR\}$, and [GM, I] for $\alpha = (\text{cris}, v)$ where v is a place of K where X has good reduction).

The equivalence class of $(cl^r_\alpha(Z))_{\alpha\in\mathcal{C}}$ defines a $\mathcal{C}$-cycle. To see this we have to check the condition $\text{comp}_{\alpha,\alpha'}(cl^r_{\alpha'}(Z)) = cl^r_\alpha(Z)$ for Z a codimension r cycle on X. For $\{\alpha, \alpha'\}$ of the form $\{DR, \sigma\}$ or $\{\ell, \sigma\}$ this follows from [DMOS, I, 2.1(a)]. For $\{\alpha, \alpha'\}$ of the form $\{(\text{cris}, v), DR\}$, with v a place of good reduction of X, this is proved in [GM, Appendix B.3.1]. Finally for $\{\alpha, \alpha'\}$ of the form $\{(\text{cris}, v), p\}$ with v a place of good reduction of X of characteristic p, this follows from the construction of $cl^r_{(\text{cris},v)}$ in [GM] using Chern classes, and the compatibility of $\text{comp}_{(\text{cris},v),p}$ with Chern classes [Fa, Lemma 5.1].

1.2. The category of $\mathcal{C}$-motives. The construction is done as in [DMOS, II,§ 6]. The major difference is that we may not have a $\mathbb{Q}$-linear fibre functor, if we do not have a Betti-realisation. We briefly repeat the steps. Let $\mathcal{V}_K$ be the category of smooth projective varieties over K. The category $\mathcal{CV}^{\mathcal{C}}_K$ is the additive category with objects $h(X)$ for each object X of $\mathcal{V}_K$ and morphisms given via $\mathcal{C}$-correspondences: $\text{Hom}_{\mathcal{CV}^{\mathcal{C}}_K}(h(X), h(Y)) = C^n_{\mathcal{C}}(X \times Y)$, if X is connected of dimension n and by additivity, via $h(X \amalg Y) = h(X) \oplus h(Y)$, in general. A tensor product $\mathcal{CV}^{\mathcal{C}}_K$ is defined via $h(X)\otimes h(Y) = h(X\times Y)$ together with the usual commutativity and associativity constraints. The identity is $h(\text{Spec}(K))$. Adjoining the images of projectors one obtains the pseudo-abelian envelope $\mathcal{M}^{\mathcal{C},+}_K$ of $\mathcal{CV}^{\mathcal{C}}_K$. In particular the gradings of the cohomology theories $H^*_\alpha(X) = \oplus H^r_\alpha(X)$ induce gradings $h(X) = \oplus h^r(X)$. Formal inversion of the Lefschetz-motive $\mathbb{L} = h^2(\mathbb{P}^1)$ gives $\mathcal{M}^{\mathcal{C}}_K$ and after a suitable modification of the sign in the definition of the

$\otimes$-product, we end up with the desired category $\mathcal{M}_K^{\mathcal{C}}$. If $\mathcal{C} = \{(\sigma), (\ell), DR\}$ $\mathcal{C}$-motives are just motives for absolute Hodge cycles.

Remarks.

(a) It is clear from the construction that $\mathcal{C}$-motives satisfy the comparison isomorphisms COMP-1 - COMP-4 whenever they are defined, but one has to impose the condition $(*_{\text{char}})$ on the characteristic polynomials if it is required.

(b) For any choice of $\mathcal{C}$ every object M of $\mathcal{M}_K^{\mathcal{C}}$ is a direct sum of direct factors of $h^n(X)(r)$ for some $X \in \mathcal{V}_K$ and $r, n \in \mathbb{Z}, n \geq 0$, i.e. the summands of M may be represented as triple $(h(X), p, r)$, where X, r are as above and $p \in C_{\mathcal{C}}(X \times X)$ is a projector. In particular, the $\ell-$adic and crystalline realisations satisfy the Weil conjecture on eigenvalues of Frobenii, i.e. they have complex absolute value $q_v^{-wt(M)/2}$, where $q_v = \#\kappa(v)$ and $wt(M) = n - 2r$.

(c) Though one expects the category to be a neutral semi-simple Tannakian category this can only be shown in special cases. Suppose that $\mathcal{C}$ contains (σ) and the $\mathbb{Q}$-linear fibre functor H_σ is faithful (this last condition can only fail if $\mathcal{C} \subset \{\sigma, (\text{cris}, v)\}$). By [Ja, Lemma 2] the semi-simplicity of the category follows from the semi-simplicity of the finite-dimensional $\mathbb{Q}$-algebra $\text{End}(h(X))$ for every object X of $\mathcal{V}_K$. As in the case of absolute Hodge cycles this follows by [Kl, Cor. 3.12] because the analogues of the standard conjectures hold for $\mathcal{C}$-cycles.

(d) In all cases there are defined on $\mathcal{M}_K^{\mathcal{C}}$ additive K_α-linear $\otimes$-functors taking values in Vec_{K_α}, where $K_\alpha = \mathbb{Q}, K, \mathbb{Q}_\ell$, or K_v respectively. If one can pass between any two realisations in $\mathcal{C}$ by a sequence of comparison isomorphisms, then these are faithful, and hence are fibre functors.

Definition 1.2. The full subcategory of $\mathcal{M}_K^{\{(\ell),(\text{cris},v),DR\}}$ whose objects satisfy $(*_{\text{char}})$ is denoted with $\mathcal{M}_K$.

For later use we denote the forgetful functor $\Psi : \mathcal{M}_K \to \mathcal{M}_K^{\{(\ell),(\text{cris},v)\}}$.

Example. Because cycle class maps are compatible with push forward, pull back and intersections (see [GM] for the case of crystalline cohomology) one has a canonical functor from the category of Grothendieck motives to the category $\mathcal{M}_K^{\mathcal{C}}$.

The objects in the essential image of this functor satisfy $(*_{\text{char}})$. To see this note that a Grothendieck motive is essentially a triple $(h^i(X), p, r)$ where X is a smooth algebraic variety, p is a projector, and $r \in \mathbb{Z}$ is a twist. If v is a place of K we have to show that the characteristic polynomial of the correspondence $\Phi_v \circ p$ on the ℓ-adic realisations $H^i_\ell(X)$ is independent of the choice of ℓ not divisible by v, and similarly for the crystalline realisations. Now p is a $\mathbb{Z}$-linear sum of algebraic correspondences c, so it is enough to show the independence for $\Phi_v \circ c$. The fixed point formula (which is a formal consequence of the properties of the cycle class map mentioned above) proves that the alternating product of the characteristic polynomials on the $H^i_\ell(X)$ and $H^i_{(\mathrm{cris},v)}(X)$ is independent of the cohomology theory, and the independence of the characteristic polynomial in each degree i then follows by looking at weights (for the $H^i_{(\mathrm{cris},v)}(X)$ these exist by [KM]).

Thus, this functor has image in $\mathcal{M}_K$. Conjecturally all motives in $\mathcal{M}_K$ should arise in this way.

1.3. The subcategory $\mathcal{M}_K^\circ$ of Artin motives. The definition of this subcategory follows that of [DMOS, II,6.17]. Let $\mathcal{CV}_K^{\mathcal{C}\circ}$ be the subcategory of $\mathcal{CV}_K^{\mathcal{C}}$ formed by the $h(X)$ with X a zero-dimensional variety over K. We define the category of Artin motives $\mathcal{M}_K^{\mathcal{C},\circ}$ to be the smallest pseudo-abelian subcategory of $\mathcal{M}_K^{\mathcal{C}}$ containing $\mathcal{CV}_K^{\mathcal{C}\circ}$ and analogously for $\mathcal{M}_K^\circ \subset \mathcal{M}_K$.

If we make the same construction inside the category of Grothendieck motives, we get a category which is equivalent to the category $\mathrm{Rep}_{\mathbb{Q}}(G_K)$ of discrete $\mathbb{Q}$-representations of the group G_K. Thus, the example above gives, in particular, a functor $\mathrm{Rep}_{\mathbb{Q}}(G_K) \to \mathcal{M}_K^{\mathcal{C},\circ}$. This functor need not be fully faithful (see remark (c) of Lemma 2.4). If it is not fully faithful, it will typically also fail to be essentially surjective, since one may have extra projectors appearing in the right hand side.

We are now ready to state our main result

Theorem 1.3. *Suppose that M is an object of $\mathcal{M}_K$ such that $gr_F^\bullet H_{DR}(M) = gr_F^0 H_{DR}(M)$. Then $\Psi(M)$ is isomorphic to an Artin motive in $\mathcal{M}_K^{\{(\ell),(\mathrm{cris},v)\}}$.*

A motive satisfying the hypothesis is said to be of type $(0,0)$.

The proof of Theorem 1.3 will be given in the next section. In particular, the theorem implies that the G_K-action on the ℓ-adic realisations of M is finite, whence the connection with the Fontaine-Mazur conjecture. Conversely, most of the proof of the theorem consists of showing this finiteness.

2. Finiteness of $\rho_\ell(G_K)$

The main point in the proof of the theorem will be to show that the image of $\rho_\ell(G_K)$ is finite for all ℓ. Obviously it is enough to do this for a fixed ℓ. We begin with the following more general result.

Lemma 2.1. *Let V be a finite dimensional $\mathbb{Q}_p$-vector space with a continuous, solvable G_K-action, which is unramified outside finitely many places. Suppose that for every prime v of K lying over p, the inertia at v acts through a finite quotient. Then G_K acts on V through a finite quotient.*

Proof. If $\rho(G_K) \subset \mathrm{Aut}(V)$ is solvable then the Zariski closure $\overline{\rho(G_K)} \subset \mathrm{Aut}(V)$ is solvable as well. Let $\{0\} = H_0 \subset H_1 \cdots \subset H_k = \overline{\rho(G_K)}$ be the derived series, i.e. $H_{i-1} = (H_i, H_i)$ for all $1 \leq i \leq k$. Then $H_{i-1} \subset H_i$ is closed for each i [Bo, I.2.4]. We proceed by induction on the length $k = k(\rho)$.

If $k = 1$, $\overline{\rho(G_K)}$ is abelian and hence $\rho(G_K)$ is abelian. Let H be the kernel of the G_K-action on V, and let $K_H \subset \bar{K}$ be the corresponding subfield. We have to show that K_H/K is a finite extension. By class field theory, it is enough to show that for any prime w of K, the action of a decomposition group D_w at w takes the inertia onto a finite image. For primes over p we already have this. So let w be a prime not lying over p.

Let $L \subset V$ be a G_K-stable $\mathbb{Z}_p$-lattice. The map $GL(L) \to GL(L/pL)$ given by reduction modulo p has finite image, and its kernel is a pro-p-group. Thus it suffices to show that the maximal pro-p-quotient, $\mathbb{Z}_p(1)$, of the inertia $I_w \subset D_w$ acts on L through a finite quotient. In fact, since the image of D_w in $GL(L)$ is abelian, the image of this quotient in $GL(L)$ has order at most the number of p-power roots of unity in K_w.

Now let $k > 1$. By the induction hypothesis it is enough to show that the quotient H_k/H_{k-1} is finite. Indeed, assuming this, we have that $H_{k-1} \subset H_k$ is closed of finite index and hence open. Therefore the preimage $\rho^{-1}(H_{k-1}) \subset G_K$ is open and corresponds to a number field K'. Since $\rho(G_{K'}) \subset H_{k-1}$, we must have $k(\rho|_{G_{K'}}) \leq k-1$. Moreover, $\rho|_{G_{K'}}$ satisfies the assumptions on the image of the inertia subgroups at primes lying over p. It follows, by induction, that $\rho|_{G_{K'}}$ has finite image, whence so does ρ itself.

To show the finiteness of H_k/H_{k-1} consider the Tannakian category of G_K-representations generated by V. The Tannakian formalism gives the existence of a G_K-representation $\rho_W : G_K \to \mathrm{Aut}(W)$ which factorises through ρ such that $\overline{\rho_W(G_K)} = H_k/H_{k-1}$. This representation is an abelian representation satisfying the finiteness conditions at the inertia subgroups over p. Therefore the case $k = 1$ shows that ρ_W has finite image, which must be equal to $\overline{\rho_W(G_K)}$. ☐

Now we consider the p-adic étale realisation $H_p(M)$ of M for some rational prime p.

Lemma 2.2. *Suppose that M is as in Theorem 1.3, and p a rational prime. For almost every prime v of K, the eigenvalues of Φ_v acting on $H_p(M)$ are algebraic integers, whose absolute values under any embedding into $\mathbb{C}$ are bounded independently of v. (In fact their absolute value is 1.)*

Proof. Firstly, let us remark that the condition on the de Rham filtration of M and the comparison isomorphisms imply that all Hodge-Tate weights of $H_p(M)$ have to be 0. Now Sen's result [Se, cor. of thm. 11] implies that for all $v \mid p$ the inertia groups I_v act through a finite quotient.

We may assume that M is simple, and hence pure of some fixed weight. Here, we are thinking of weights in terms of the eigenvalues of Frobenii, and this makes sense, since the Weil conjecture holds for M. Then $\det(M)$, is also pure, and it must be pure of weight 0, for example by Lemma 2.1. Thus, we have $wt(M) = (\dim M)^{-1} wt(\det M) = 0$. Now the second claim and the fact that the eigenvalues are algebraic numbers follows immediately from the Weil conjectures for M.

It remains to check that they are algebraic *integers*. This is certainly true for the cohomology of algebraic varieties. Now M has to be a direct summand of some $h^n(X)(r)$ of weight 0, i.e. $n = 2r$, for some smooth projective variety X.

Since twisting multiplies the eigenvalues by q^{-r}, the eigenvalues remain v'-adic units for primes v' of $\bar{\mathbb{Q}}$ not lying above the rational prime dividing v. Thus it is enough to check that the eigenvalues are v'-adic units at the other primes v' of $\bar{\mathbb{Q}}$. Moreover, since the characteristic polynomial of Φ_v acting on $H_p(M)$ has $\mathbb{Q}$-coefficients, it is enough to prove this for any such v'.

By the condition $(*_{\mathrm{char}})$ for M the eigenvalues of interest are equal to those of φ_v^m on $H_{\mathrm{cris},v}(M)$, where m is defined via $\#\kappa(v) = q_v^m$. Since $H_{\mathrm{cris},v}(M)$ is a direct summand of $H^{2r}_{\mathrm{cris},v}(X)(r)$ (compatible with Frobenius actions and filtrations !), it has the structure of a weakly admissible module in the sense of [Fo]. Since all Hodge-Tate weights of M are 0, the Hodge polygon of this module is a line of slope 0. For weakly admissible modules, the Newton polygon lies on or above the Hodge polygon and has the same endpoints, so convexity implies that the Newton polygon is also a line of slope 0. □

Proposition 2.3. *Let M be as in the theorem. The action of G_K on $H_p(M)$ factors through a finite quotient.*

Proof. By Lemma 2.2, for any prime v of K outside some finite set, the eigenvalues of Φ_v acting on $H_p(M)$ are algebraic integers, and they have absolute values bounded independently of v. Since the characteristic polynomial of Φ_v acting on $H_p(M)$ has $\mathbb{Q}$-coefficients and fixed degree, these eigenvalues take only finitely many values as v varies. Now consider the function on G_K which takes an element $g \in G_K$ to the characteristic polynomial of its action on $H_p(M)$. This is a locally constant function. Indeed any g can be approximated by a sequence of Frobenius elements. By the above, such a sequence contains an infinite subsequence all of whose members have the same characteristic polynomial, which must then be equal to that of g.

It follows that after replacing K by a finite extension, we may assume that the action of G_K on $H_p(M)$ is unipotent. In particular, this means that G_K acts through a solvable quotient. As was remarked above, the inertia subgroups of G_K act through a finite quotient for all $v \mid p$, we may conclude using Lemma 2.1. □

Proof of Theorem 1.3. If K'/K is a finite Galois extension, then the image of $\mathrm{Spec}(K')$ in $\mathcal{M}_K^{\mathcal{C}_\circ}$ decomposes into a direct sum corresponding to the decomposition of the regular representation of $\mathrm{Gal}(K'/K)$ over the field $\mathbb{Q}$.

We have shown that the action of G_K on the compatible system of representations $(H_\ell(M))_\ell$ factors through a finite quotient $\mathrm{Gal}(K'/K)$, where K' is a finite extension of K. The compatibility condition also insures that K' can be chosen independently of ℓ. For example, for each ℓ, the minimal such field K' is determined by the condition that $G_{K'}$ is the maximal normal open subgroup of G_K such that for almost all places v of K, such that $\Phi_v \in G_{K'}$, the characteristic polynomial of Φ_v on $H_\ell(M)$ is a power of $T-1$.

The ℓ-adic realisation of $\mathrm{Spec}(K')$ is the regular representation of $\mathrm{Gal}(K'/K)$. The action of $\mathrm{Gal}(K'/K)$ on the $H_\ell(M)$ corresponds to a character of this group, and because the representation theory of $\mathrm{Gal}(K'/K)$ is semi-simple, $\Psi(M)$ is a direct summand in the image in $\mathcal{M}^{\{\ell,(\mathrm{cris},v)\}}$ of $(\mathrm{Spec}(K'))^s$ for some integer

s. Indeed, for the ℓ-adic realisations this is immediate, and the crystalline realisations can be recovered from the ℓ-adic ones, by taking D_v invariants in COMP-4. □

Remarks.

(a) We do not know how to prove that in the situation of Theorem 1.3, M is actually an Artin motive in $\mathcal{M}_K$. The problem is that one does not have a characterisation of the image of $H_{DR}(M) \subset H_{\mathrm{cris},v}(M)$. This seems to be a reasonable conjecture, however, which, when $K = \mathbb{Q}$, can be viewed as a generalisation of Deligne's conjecture on rank 1-motives for the category $\mathcal{M}_K$ (see conjecture 3.2 below). In fact, we will easily check that Deligne's conjecture actually holds in this context.

(b) Sometimes, one can improve Theorem 1.3 a little bit, and show that at least $\Psi(M)$ is the image of a Grothendieck motive. This amounts to asking whether the character of G_K corresponding to M corresponds to a $\mathbb{Q}$-representation. A partial answer is given by the following

Lemma 2.4. *Let G be a finite group, and V an irreducible $\bar{\mathbb{Q}}$-representation of G. Suppose that*

(i) The traces of V lie in $\mathbb{Q}$.

(ii) For every prime ℓ of $\mathbb{Q}$, V is defined over $\mathbb{Q}_\ell$.

Then V is defined over $\mathbb{Q}$.

Proof. Suppose that V is defined over a subfield K of $\bar{\mathbb{Q}}$, which is finite and Galois over $\mathbb{Q}$ (i.e there exists a K-representation V_K with $V = V_K \otimes_K \bar{\mathbb{Q}}$). For every embedding s of K into $\bar{\mathbb{Q}}$, we denote by V^s, the representation obtained (from V_K) by extension of scalars. Set S' equal to the direct sum of the V^s. Then S' descends to a $\mathbb{Q}$-representation S. The condition (i) ensures that the V^s are all isomorphic. This shows that if $W \subset \mathbb{Q}[G]$ is an irreducible factor such that $W \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ contains V, then $W \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ is actually a direct sum of copies of V.

Set $D = \mathrm{End}_{\mathbb{Q}[G]}(W)$. Since W is irreducible this is a division algebra. Since the character of W is an integer multiple of that of V, (ii) implies

$$D \otimes \mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_l[G]}(W \otimes \mathbb{Q}_\ell) \xrightarrow{\sim} M_r(\mathbb{Q}_l)$$

Thus D splits at every finite prime, whence it must be split. This implies that $D = \mathbb{Q}$. In other words, $W \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ is irreducible, so that $W = V$. □

Remarks.

(a) Below we will introduce motives with coefficients. It is then almost a tautology that in the situation of Theorem 1.3 $\Psi(M) \otimes_{\mathbb{Q}} E$ is the image of a Grothendieck motive with E-coefficients for some finite extension E of $\mathbb{Q}$.

(b) The above lemma gives a rather elegant criterion for an irreducible representation to be defined over $\mathbb{Q}$. There is a more general criterion which works for any representation: If V is a $\bar{\mathbb{Q}}$-representation of G whose traces are contained in a number field K, and which is defined over K_v for every place v of K, (including the infinite ones) then V is defined over K. We leave the proof as an exercise in the theory of the Brauer group.

(c) If one leaves out the infinite places, this result fails in general - even when $K = \mathbb{Q}$ (without assuming irreducibility). We are grateful to Ofer Gabber for providing a counter example. He also pointed out that the above result (b) holds more generally for representations of finite dimensional, semi-simple K-algebras.

3. Remarks on some conjectures

3.1. Motives with coefficients. We briefly recall the definition of motives with coefficients in some number field E. As remarked in [De] the construction is valid for any additive pseudo-abelian category in which the morphisms are $\mathbb{Q}$-vector spaces. In the definition of motives sketched in 1.2 one takes the homomorphisms to be $\mathrm{Hom}_{\mathcal{CV}^{\mathcal{C}}_{K,E}}(h(X)_E, h(Y)_E) = \mathrm{Hom}_{\mathcal{CV}^{\mathcal{C}}_K}(h(X), h(Y)) \otimes_{\mathbb{Q}} E$ and proceeds as before (language B in [De]). Equivalently one may define a motive over K with coefficients in E to be a motive M over K equipped with an E-module structure $E \to \mathrm{End}(M)$ (language A of [De]). In particular a motive M gives rise to a motive with coefficients M_E via the "language B".

The rank $rk_E(M)$ of a motive M is defined to be $\dim_E(H_\sigma(M))$ if one has a Betti-realisation. More generally one takes the rank to be $rk_E(M) = \dim_{E_\alpha}(H_\alpha(M))$. This is only well defined if $\mathcal{C}$ is big enough.

For E as above we define $\mathcal{M}_{K,E}$ to be the full sub-category of the category of motives in $\mathcal{M}_K$ with coefficients in E, consisting of objects which satisfy the following condition:

($*_{\mathrm{char},E}$) The characteristic polynomials of Φ_v on all the ℓ-adic (regarded as $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-modules) and crystalline realisations have E-coefficients and one has

$$P^r_{E,\ell}(T) = \det{}_E(1 - T\Phi_v; H^r_\ell(X)) = \det{}_E(1 - T\varphi_v^m; H^r_{\mathrm{cris},v}(X)) = P^r_{E,\mathrm{cris},v}(T).$$

This condition does not seem to be guaranteed by ($*_{\mathrm{char}}$).

The reader should note that although one of the main points of motives with coefficients is that one then has a slew of new realisations (for example λ-adic realisations for λ a prime of E,) we will consider only the realisations defined previously for motives without coefficients. In "language A" this means we consider the usual realisations of the motive M, which are equipped with an E-action by functoriality. If M is a motive without coefficients then passing from M to M_E has the effect of applying $\otimes_{\mathbb{Q}} E$ on the level of realisations.

3.2. A conjecture of Deligne. In [De] the following conjecture was stated. It can be regarded as a special case of the conjecture that motives of $(0,0)$ are Artin motives.

Conjecture. *Every motive over $\mathbb{Q}$ with coefficients in some finite extension $E/\mathbb{Q}$ which is of rank* 1, *is of the form $E(\varepsilon)(n)$ for some character ε of $G_{\mathbb{Q}}$ which takes values in the roots of unity of E and some $n \in \mathbb{Z}$.*

Theorem 3.1. *Deligne's conjecture is true for motives in the category $\mathcal{M}_{\mathbb{Q},E}$*

Proof. First, after twisting by an integer, we may assume that the H_{DR}, which is a 1-dimensional E-vector space, has the trivial filtration (this is where we use the fact that $K = \mathbb{Q}$.). Theorem 1.3 implies that the action of $G_{\mathbb{Q}}$ on the ℓ-adic realisations factors through a finite quotient. Since M has rank 1 $(*_{\mathrm{char},E})$ implies that these representations correspond to a finite character $\varepsilon : G_{\mathbb{Q}} \to E^*$. To see that M is isomorphic to the motive $E(\varepsilon)$ corresponding to ε we simply choose any E-linear isomorphism $H_{DR}(M) \xrightarrow{\sim} H_{DR}(E(\varepsilon))$. These then induce isomorphisms on the ℓ-adic and crystalline realisations via COMP-3 and COMP-4. These isomorphisms are automatically compatible with the Frobenius actions, since the latter are just multiplications by (the same) elements of E. □

3.3. The unramified Fontaine-Mazur conjecture. Our finiteness result in §2 is related to the following conjecture made in [FM].

Conjecture (Conjecture 5a)**.** *Let S be a finite set of primes. If p is distinct from all of the residual characteristics of S, then any p-adic representation of $G_{K,S}$ factors through a finite quotient group of $G_{K,S}$.*

Another conjecture made in the above mentioned paper is the following one.

Conjecture (Conjecture 1)**.** *An irreducible p-adic representation is geometric (i.e. it is unramified outside a finite set of places of K and its restriction to the decomposition groups D_v for $v \mid p$ is potentially semi-stable) if and only if it comes from algebraic geometry (i.e. if it is isomorphic to a subquotient of the p-adic cohomology of a smooth projective variety.)*

Using our result we can show the next proposition.

Proposition 3.2. *Assume that the following assumptions hold*

(i) Conjecture 1 is true;

(ii) the Tate conjecture on algebraic cycles holds, i.e. $H^{2j}_{ét}(\bar{X}, \mathbb{Q}_\ell)(j)^{G_K}$ is generated by the images of algebraic cycles of codimension j;

(iii) the action of G_K on $H_\ell(X)$ is semi-simple for every smooth projective variety X/K. (Grothendieck-Serre conjecture)

Then Conjecture 5a) holds.

Remark. In [FM] an even stronger statement - without proof - is made, namely that conjecture 1 together with the Tate conjecture implies conjecture 5a). We were, however, not able to give a proof without the additional assumption of the Grothendieck-Serre conjecture.

Proof. It suffices to show the statement for semi-simple representations. Indeed if we know that semi-simple representations have finite image, we apply this to the semi-simplification and after a finite extension we may assume that the Galois action on $H_p(M)$ is unipotent. An argument as in the proof of Proposition

2.3 gives the statement. (So it is not necessary to assume the Grothendieck-Serre conjecture here.)

Let N_p be a semi-simple geometric p-adic representation of $G_{K,S}$. Conjecture 1 of [FM] gives that any irreducible geometric representation comes from algebraic geometry. This means we have a smooth projective variety X such that N_p is a subquotient of $H^n_{ét}(X \times_K \bar{K}, \mathbb{Q}_p)(r) = H_p(M)$, i.e. of the p-adic realisation of the (Grothendieck-)motive $M = h^n(X)(r)$. As the Grothendieck-Serre conjecture predicts that the action of G_K is semi-simple, we can assume that our representation is a direct summand of $H_p(M)$. In other words, we have an idempotent e in $\mathrm{End}_{G_K}(H_p(M))$ such that $eH_p(M) = N_p$.

Now - making use of the Tate-conjecture, comp. [Ta, § 3] - we have the surjectivity of the map $\mathrm{End}_{\mathcal{M}_K^{Groth}}(M) \otimes \mathbb{Q}_p \to \mathrm{End}_{G_K}(H_p(M))$. If we had $e \in \mathrm{End}_{\mathcal{M}_K^{Groth}}(M)$ then we could conclude that the $G_{K,S}$-action on N_p was finite by using Theorem 1.3. (The condition on the grading being assured by Sen's theorem.) A priori we do not know this, but it suffices to show the lemma below. □

Lemma 3.3. *There is a direct summand N' of M such that $N_p \subset N'_p = H_p(N')$, and the inertia groups I_v for $v \mid p$ act on N'_p through finite quotients.*

Proof. We start with the following two observations

(i) For any finite extension $E/\mathbb{Q}$ we may replace M by M_E. (Here we leave $N_p \subset H_p(M) \subset H_p(M_E)$ unchanged.)
(ii) If $M = M_1 \oplus M_2$ it is enough to show the statement for M_1 or M_2, with N_p replaced by its projections to $(M_1)_p$ and $(M_2)_p$ respectively.

The Tate conjectures imply that $\mathrm{End}(M)$ is a semi-simple $\mathbb{Q}$-algebra. Let $E/\mathbb{Q}$ be an extension such that all idempotents of $\mathrm{End}(M)\otimes\bar{\mathbb{Q}}$ (resp. $\mathrm{End}_{G_K}(N_p)\otimes\bar{\mathbb{Q}}$) are already contained in $\mathrm{End}(M)\otimes E$ (resp. $\mathrm{End}_{G_K}(N_p)\otimes E$). This extension is finite because $\mathrm{End}(M)$ is finite-dimensional. Let $M_{i,E}$ be a direct summand of M_E such that $\mathrm{End}_E(M_{i,E})$ does not contain nontrivial idempotents. Using the two observations above, it is enough to prove the lemma with $M_{i,E}$ in place of M. (This means in particular, that we replace N_p by its projection to $(M_{i,E})_p$.) We claim that we can take $N' = M_{i,E}$. Our assumptions on E imply that $\mathrm{End}_E(M_{i,E}) \xrightarrow{\sim} E$. Thus, we have

$$\mathrm{End}_{G_K,E}(N'_p) \simeq E \otimes \mathbb{Q}_p \simeq \oplus_{\mathfrak{p}|p} E_{\mathfrak{p}}.$$

As $N_p \subset N_p \otimes E \subset N'_p$ there is an idempotent $e_{\mathfrak{p}} \in E \otimes \mathbb{Q}_p$ (corresponding to the projection to $E_{\mathfrak{p}}$ above) such that $e_{\mathfrak{p}} N'_p \subset e_{\mathfrak{p}}(N_p \otimes_{\mathbb{Q}} E)$.

Fix a prime $v \mid p$ of K. Using the comparison isomorphisms (for N') we compute

$$\begin{aligned} e_{\mathfrak{p}}(H_{DR}(N') \otimes_K K_v) &\xrightarrow{\sim} e_{\mathfrak{p}} H_{\mathrm{cris},v}(N') \\ &\xrightarrow{\sim} e_{\mathfrak{p}}(N'_p \otimes B_{\mathrm{cris}})^{D_v} \\ &\subset e_{\mathfrak{p}}(N_p \otimes E \otimes B_{\mathrm{cris}})^{D_v} \end{aligned}$$

The representation N_p is unramified at v and this implies that the filtration on the $e_{\mathfrak{p}}(E \otimes_{\mathbb{Q}} K_v)$-module on the left hand side is concentrated in degree 0. Note that we have

$$e_{\mathfrak{p}}(E \otimes_{\mathbb{Q}} K_v) \xrightarrow{\sim} e_{\mathfrak{p}}(E \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K_v \xrightarrow{\sim} E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} K_v.$$

On the other hand

$$\begin{aligned} \oplus_{v|p} E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} K_v &= E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} (\oplus_{v|p} K_v) \\ &= E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p \otimes_{\mathbb{Q}} K) \\ &= E_{\mathfrak{p}} \otimes_{\mathbb{Q}} K \\ &= E_{\mathfrak{p}} \otimes_E (E \otimes_{\mathbb{Q}} K). \end{aligned}$$

Thus we find that

$$\begin{aligned} \oplus_{v|p} e_{\mathfrak{p}}(H_{DR}(N') \otimes_K K_v) &\xrightarrow{\sim} \oplus_{v|p}(E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} K_v) \otimes_{E \otimes_{\mathbb{Q}} K} H_{DR}(N') \\ &\xrightarrow{\sim} E_{\mathfrak{p}} \otimes_E H_{DR}(N'), \end{aligned}$$

so the filtration on the term on the right is concentrated in degree 0. and hence this is also true for $H_{DR}(N')$ itself. Now Sen's theorem [Se] implies that for $v \mid p$, I_v acts on N'_p through a finite quotient. $\square$

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