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ANALYTIC CAPACITY, BILIPSCHITZ MAPS AND CANTOR SETS

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Abstract. We show that for planar Cantor sets analytic capacity is a bilipschitz invariant.

1. Introduction.

Let $E \subset \mathbb{C}$ be a compact plane set. The *analytic capacity* of E is

$$\gamma(E) = \sup\{|f'(\infty)| : f \in A(E,1)\}$$

where

$$A(E,1) = \{f : f \text{ is analytic on } \mathbb{C} \setminus E, f(\infty) = 0 \text{ and } \sup_{\mathbb{C}\setminus\mathbb{E}} |f(z)| \leq 1\}$$

and $f'(\infty) = \lim_{z\to\infty} zf(z)$. Then $\gamma(E) > 0$ if and only if $A(E,1)$ contains a non-constant function [G2]. A homeomorphism

$$T : E \to T(E)$$

is *bilipschitz* if T and T^{-1} satisify Lipschitz conditions

$$\frac{1}{K}|z-w| \leq |T(z) - T(w)| \leq K|z-w| \tag{1.1}$$

for all $z, w \in E$. This paper is concerned with the

Conjecture. *If T is bilipschitz, then*

$$\gamma(T(E)) \leq C(K)\gamma(E),$$

where $C(K)$ depends only on the constant K in (1.1).

Because $f \circ T$ and f are seldom both analytic this conjecture may look foolhardy, but it has some supporting evidence. First, let $N(E)$ be the *Newtonian capacity* of E, which we define by

$$N(E) = \sup\Big\{\mu(E) : \mu \text{ Borel}, \mu > 0, \sup_z \int_E \frac{d\mu(w)}{|z-w|} \leq 1\Big\}. \tag{1.2}$$

Then $\gamma(E) \geq N(E)$ because

$$f(z) = \int_E \frac{d\mu(w)}{z-w} \in A(E,1)$$

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for all μ in (1.2), and it is clear from the definition (1.1) that

$$N(T(E)) \leq KN(E).$$

Second, suppose E has finite one dimensional Hausdorff measure $\Lambda_1(E) < \infty$. Then by a deep theorem of David [D], $\gamma(E) = 0$ if and only if $\Lambda_1(E \cap \Gamma) = 0$ for every rectifiable curve Γ. Therefore,

$$\gamma(T(E)) = 0 \text{ if and only if } \gamma(E) = 0$$

when $\Lambda_1(E) < \infty$. If the rectifiable curve Γ satisfies an Ahlfors condition:

$$A^{-1}r \leq \Lambda_1(\Gamma \cap D(z, r)) \leq Ar,\ z \in \Gamma,\ 0 < r \leq \operatorname{diam}(\Gamma),$$

then it is well known that for all $E \subset \Gamma$,

$$C(A)^{-1}\Lambda_1(E) \leq \gamma(E) \leq C(A)\Lambda_1(E),$$

and therefore

$$\gamma(T(E)) \leq C(A, K)\gamma(E),$$

because $T(\Gamma)$ is a rectifiable curve that also satisfies an Ahlfors condition. However, we do not have the preceding inequality with constant $C(K)$ independent of the curve Γ; indeed, that would be equivalent to the full conjecture.

Here we establish the conjecture for the Cantor sets with $\Lambda_1(E) = \infty$ that were studied in [E], [G2], [Ma] and especially [MTV] and for their bilipschitz images. Let E be a compact set of the form

$$E = \bigcap_{n=0}^{\infty} E_n, \tag{1.3}$$

$$E_n = \bigcup_{|J|=n} Q_J^n, \tag{1.4}$$

where Q_J^n is closed, $J = (j_1, j_2, \dots, j_n)$ is a multiindex of length $|J| = n$ with $j_k \in \{1, 2, 3, 4\}$, and

$$Q_{J,j_{n+1}}^{n+1} \subset Q_J^n$$

for all n and J. We assume there are constants

$$0 < a_1 < a_2 < 1/2$$

and

$$c_1, c_2 > 0$$

and a sequence $\sigma = (\sigma_n)$ such that $\sigma_0 = 1$ and

$$a_1 \leq \frac{\sigma_{n+1}}{\sigma_n} \leq a_2, \tag{1.5}$$

$$\operatorname{diam}(Q_J^n) \leq c_1\sigma_n \tag{1.6}$$

and

$$\operatorname{dist}(Q_J^n, Q_{J'}^n) \geq c_2\sigma_n,\ J \neq J'. \tag{1.7}$$

A paradigm for the set E is obtained by letting Q_J^n be a square of side σ_n with sides parallel to the axes and requiring that $Q_{J,j}^{n+1}, j = 1, 2, 3, 4$, be the four corner subsquares of Q_J^n. In this case E is the square Cantor set $E(\sigma)$ from [MTV], where it was proved that

$$C^{-1}\Big(\sum \frac{1}{4^{2n}\sigma_n^2}\Big)^{-1/2} \leq \gamma(E(\sigma)) \leq C\Big(\sum \frac{1}{4^{2n}\sigma_n^2}\Big)^{-1/2}$$

with constant C independent of σ.

Now it is clear from (1.6) and (1.7) that if the sets E and E' are defined by (1.3) and (1.4) for the same sequence (σ_n), then

$$T(E \cap Q_J^n(E)) = E' \cap Q_J^n(E') \tag{1.8}$$

defines a bilipschitz map from E onto E' with constant $K = K(a_1, a_2, c_1, c_2)$. In particular, (1.3) - (1.7) describe all bilipschitz images of the Cantor set $E(\sigma)$.

Theorem. *If E is defined by* (1.3), (1.4), (1.5), (1.6) *and* (1.7), *then there is constant*

$$C = C(c_1, c_2, a_1, a_2)$$

such that

$$C^{-1}\Big(\sum \frac{1}{4^{2n}\sigma_n^2}\Big)^{-1/2} \leq \gamma(E) \leq C\Big(\sum \frac{1}{4^{2n}\sigma_n^2}\Big)^{-1/2}.$$

Corollary. *There is a constant $C = C(K, a_1, a_2, c_1, c_2)$ such that*

$$C^{-1}\gamma(E) \leq \gamma(T(E)) \leq C\gamma(E)$$

whenever E is a Cantor set $E(\sigma)$ and T is a bilipschitz map on E satisfying (1.1) *with constant K.*

The Corollary follows immediately from the Theorem and the above discussion.

2. Proof of Theorem

The proof of the theorem depends on some exciting recent work of Tolsa [T1] and [T2]. Define the maximal function of a positive Borel measure μ as

$$M_\mu(z) = \sup_r \frac{\mu(B(z,r))}{r}$$

where $B(z,r) = \{w : |w - z| < r\}$. Let $R(z, w, \zeta)$ be the radius of the circle through z, w and $\zeta \in \mathbb{C}$. Then $R(z, w, \zeta)^{-1}$ is called the *Menger curvature* of the triple (z, w, ζ). See Melnikov [Me]. Define the *pointwise Menger curvature* of μ at z as

$$c_\mu^2(z) = \int\int \frac{1}{R(z,w,\zeta)^2} d\mu(w) d\mu(\zeta)$$

and as in [V] define the *Menger Potential* of μ by

$$U_\mu(z) = M_\mu(z) + c_\mu(z).$$

Then the results we need from Tolsa [T1] and [T2] can be expressed as two inequalities:

$$\gamma(E) \geq C_1 \sup\bigl\{\mu(E) : \sup_{z\in E} U_\mu(z) \leq 1\bigr\}, \tag{2.1}$$

and

$$\gamma(E) \leq C_2 \inf\bigl\{\mu(E) : \inf_{z\in E} U_\mu(z) \geq 1\bigr\} \tag{2.2}$$

with absolute constants C_1 and C_2. Let E satisfy the hypothesis of the Theorem and define $\mu = \mu_E$ by

$$\mu(Q_J^n \cap E) = 4^{-n}.$$

Then for all $z \in E$

$$M_\mu(z) \simeq \sup_n \frac{1}{4^n \sigma_n}. \tag{2.3}$$

with constants depending only on c_1 and c_2. Note that $M_\mu(z) = \infty$ is possible for all $z \in E$.

The main difficulty in proving the Theorem comes from the obvious fact that a bilipschitz mapping may transform triples with positive Menger curvature into triples with zero curvature. For example the vertices of an equilateral triangle of side length 1 may be mapped into three collinear points. In the next example we will see that this may happen at all scales and locations, at least on a set of Hausdorff dimension less than 1.

Define a Cantor set as follows. Start with the interval $[0,1]$ and take 4 subintervals of length $1/5$ forming three equal gaps in $[0,1]$. Perform the same operation on each of these 4 intervals obtaining at the second step 16 intervals of length $1/25$. Proceeding inductively we obtain at the n-th step 4^n intervals Q_J^n of length 5^{-n}. Then (1.3) and (1.4) define a Cantor set E associated to the sequence $\sigma_n = 5^{-n}$. Define another Cantor set E' with the same sequence by starting with the unit square, taking 4 corner squares of side length $1/5$ at the first step and then proceeding inductively. As we pointed out before, there is a bilipschitz mapping T from E onto E' satisfying (1.8). Therefore the measure $\mu = \mu_E$ is transformed into the measure $\mu' = \mu_{E'}$. Notice that $c^2_\mu(z) = 0, z \in E$, but $c^2_{\mu'}(z) = \infty, z \in E'$, as shown in [T1]. Nevertheless, it can be easily seen that $U_\mu(z) = \infty$ for all $z \in E$ and $U_{\mu'}(z) = \infty$ for all $z \in E'$.

Lemma 1. *If E satisfies* (1.3), (1.4), (1.5), (1.6) *and* (1.7)*, then*

$$c^2_\mu(z) \leq C(c_1, c_2) \sum_{n=1}^{\infty} \frac{1}{4^{2n}\sigma_n^2}.$$

Note that by (2.1), (2.3) and Lemma 1,

$$\gamma(E) \geq C'(c_1, c_2)\Bigl(\sum \frac{1}{4^{2n}\sigma_n^2}\Bigr)^{-1/2},$$

which gives the leftmost inequality in the Theorem.

Proof of Lemma 1. The argument is from Mattila [Ma], and depends only on the trivial estimate

$$\frac{1}{R(z,w,\zeta)} \le \frac{2}{|z-w|}. \tag{3.1}$$

By symmetry we have

$$c_\mu^2(z) = 2\iint_{|\zeta-z|\le|w-z|} \frac{1}{R(z,w,\zeta)^2} d\mu(\zeta)d\mu(w) \ .$$

Set

$$A_n = \{(\zeta,w) : |\zeta-z| \le |w-z| \text{ and } c_1\sigma_n \le |w-z| < c_1\sigma_{n-1}\},$$

for $n \ge 1$. Then clearly

$$2\iint_{|\zeta-z|\le|w-z|} \frac{1}{R(z,w,\zeta)^2} d\mu(\zeta)d\mu(w)$$
$$\le C + \sum_{n=1}^{\infty} \iint_{A_n} \frac{8}{|w-z|^2} d\mu(\zeta)d\mu(w) \le C\sum_{n=1}^{\infty} \frac{1}{4^{2n}\sigma_n^2}.$$

□

To prove the reverse inequality it is enough by (2.2) to show that

$$U_\mu(z) \ge C\left(\sum \frac{1}{4^{2n}\sigma_n^2}\right)^{\frac{1}{2}}, \tag{3.2}$$

for all $z \in E$.

Take $z \in E$. For each n define $Q_J^n(z)$ as the Q_J^n such that $z \in Q_J^n$ and following [J] define the Jones number

$$\beta_n(z) = \inf\Big\{ \frac{\sup_{w\in E\cap Q_J^n(z)} \operatorname{dist}(w,L)}{\sigma_n} : \ L \text{ is a line}\Big\}.$$

Thus $2\beta_n(z)\sigma_n$ is the width of the narrowest strip containing $Q_J^n(z)$ and $\beta_n(z)$ is small if the inequality reverse to the trivial estimate (3.1) fails on $Q_J^n(z)$.

Lemma 2. *Let* $\delta = \frac{c_2}{2\sqrt{2}}$. *If*

$$\beta_n(z) \le \delta\frac{\sigma_{n+p}}{\sigma_n}, \tag{3.3}$$

for some $p \ge 1$, *then*

$$\sum_{k=1}^{p} 4^{n+k}\sigma_{n+k} \le \frac{4\ c_1}{c_2}\ 4^n\sigma_n. \tag{3.4}$$

Proof of Lemma 2. By the definition of $\beta_n(z)$ there is a rectangle $R \supset Q_J^n(z)$ such that $Q_J^n(z)$ meets each of the four sides of R and such that the smallest side of R has length $2\beta_n(z)\sigma_n$. Let P denote the orthogonal projection onto the midline L of R. By (1.7), the definition of δ and trigonometry we have for $j \ne k$

$$\operatorname{dist}(P(Q_{J,j}^{n+1}), P(Q_{J,k}^{n+1})) \ge \frac{c_2}{2}\sigma_{n+1}.$$

Then because $R \cap L$ is connected,

$$R \cap L \setminus \bigcup_{j=1}^{4} P(Q_{J,j}^{n+1})$$

contains three intervals each having endpoints in two distinct $P(Q_{J,j}^{n+1})$ and each having length at least $\frac{c_2}{2}\sigma_{n+1}$.

Similarly, for $k = 1, 2, \ldots, p$ and for each $Q_K^{n+k-1} \subset Q_J^n(z)$, $R \cap L$ contains three intervals having endpoints in two distinct $P(Q_{K,j}^{n+k})$ and having length at least $\frac{c_2}{2}\sigma_{n+k}$. Since there are 4^{k-1} distinct $Q_K^{n+k-1} \subset Q_J^n(z)$, we obtain at least $3 \cdot 4^{k-1}$ pairwise disjoint intervals of length at least $\frac{c_2}{2}\sigma_{n+k}$ and furthermore, for $k > j$ these intervals are disjoint from the $3 \cdot 4^{j-1}$ intervals having endpoints in distinct $P(Q_{K'}^{n+j})$. The sum of the lengths of all these intervals is not larger than $\sqrt{2}\ \mathrm{diam}(Q_J^n(z)) \leq \sqrt{2}\ c_1\ \sigma_n$. Thus (3.4) follows. □

Set

$$a_n = \frac{1}{4^{2n}\sigma_n^2}$$

and for each positive integer p

$$S = S(p) = \{n : 2a_n \geq \mathrm{Max}_{1 \leq j \leq p}\ a_{n+j}\}.$$

We need the following reformulation of Lemma 2.

Lemma 3. *There exist a large positive integer $p = p(c_1, c_2)$ and a small positive number $\eta = \eta(a_1, p)$ such that if $n \in S(p)$ then*

$$\beta_n(z) \geq \ \eta. \tag{3.5}$$

Proof of Lemma 3. If $\beta_n(z) \leq \delta \frac{\sigma_{n+p}}{\sigma_n}$ and $n \in S(p)$, then by Lemma 2

$$\frac{1}{\sqrt{2}}\ p\ 4^n \sigma_n \leq \sum_{k=1}^{p} 4^{n+k}\sigma_{n+k} \leq \frac{4\ c_1}{c_2}\ 4^n \sigma_n,$$

which gives an upper bound on p. If p is chosen to be larger than $\sqrt{2}\ \frac{4\ c_1}{c_2}$, then

$$\beta_n(z) \geq \delta \frac{\sigma_{n+p}}{\sigma_n} \geq \delta a_1^p \equiv \eta,$$

whenever $n \in S(p)$. □

The next lemma gives a relation between $\beta_n(z)$ and $c_\mu^2(z)$. See [P] for further results of this type. Assume from now on that p and η are given by Lemma 3.

Lemma 4. *If $\beta_n(z) \geq \ \eta$, then*

$$\iint_{F_n} \frac{1}{R(z, w, \zeta)^2}\ d\mu(w) d\mu(\zeta) \geq \frac{\epsilon_0}{4^{2n}\sigma_n^2},$$

where

$$F_n = F_n(z) = \{(w_1, w_2) \in Q_J^n(z) : \mid w_j - z \mid \geq \frac{\eta}{8}\ \sigma_n, j = 1, 2\}$$

and ϵ_0 *is a positive constant depending on* η.

Proof of Lemma 4. Take a point b_1 in $E \cap Q_J^n(z)$ such that $|b_1 - z| \geq c_2 \sigma_{n+1}$. By (1.5) and the definition of η we then have $|b_1 - z| \geq \eta \sigma_n$. Let L be the line through z and b_1. Since $\beta_n(z) \geq \eta$ there is a point $b_2 \in E \cap Q_J^n(z)$ such that the distance from b_2 to L is larger than $(\frac{\eta}{2})\sigma_n$. Let B_j denote the disc centered at b_j of radius $(\frac{\eta}{8})\sigma_n$. It is then clear that for some positive number ϵ_1 depending on η we have

$$\mu(B_j) \geq \frac{\epsilon_1}{4^n}, j = 1, 2,$$

and

$$R(z, w_1, w_2) \leq \epsilon_1^{-1} \ \sigma_n, \ \ w_j \in B_j, j = 1, 2.$$

Thus

$$\iint_{B_1 \times B_2} \frac{1}{R(z, w, \zeta)^2} \ d\mu(w) d\mu(\zeta) \geq \frac{\epsilon_1^4}{4^{2n}\sigma_n^2},$$

which proves the lemma. □

The next lemma shows that if $\sum a_n < \infty$ then $n \in S = S(p)$ for many values of n. Recall that $a_n = \frac{1}{4^{2n}\sigma_n^2}$.

Lemma 5. *We have*

$$\sum_{n=1}^{\infty} a_n \leq 2p \sum_{n \in S} a_n + p \ M,$$

where $M = \sup_n a_n$.

Proof of Lemma 5. Set

$$b_n = \max\{a_j : (p-1)n < j \leq pn\}, \ n = 1, 2, \cdots$$

Let N be a large integer and let q be the positive integer such that $(p-1)q < N \leq pq$. Denote by G the set of integers n such that $1 \leq n \leq q$ and $2b_n \geq b_{n+1}$. Notice that an index $n \in G$ is good, in the sense that $b_n = a_m$ for some $m \in S$. Let B stand for the set of indexes between 1 and q which are not in G . Since

$$\sum_{n \in B} b_n \leq \frac{1}{2} \sum_{n=0}^{q} b_{n+1},$$

we have

$$\sum_{n \in G} b_n \geq \frac{1}{2} \sum_{n=1}^{q} b_n - \frac{1}{2} \ b_{q+1}.$$

Therefore

$$\sum_{n=1}^{N} a_n \leq p \ \sum_{n=1}^{q} b_n \leq 2p \ \sum_{n \in G} b_n + p \ b_{q+1} \leq 2p \ \sum_{n \in S} a_n + pM,$$

and the lemma follows by sending $N \to \infty$. □

We can now complete easily the proof of (3.2). Since the domains of integration F_n in Lemma 4 have bounded overlap, we get

$$c^2_\mu(z) \geq \frac{\epsilon_0}{C} \sum_{n\in S} \frac{1}{4^{2n}\sigma_n^2},$$

where C is some constant larger than 1. By Lemma 5 and (2.3) we then have, with another constant C,

$$U^2_\mu(z) \geq \frac{\epsilon_0}{C} \,(\sum_{n\in S} \frac{1}{4^{2n}\sigma_n^2} + M) \geq \frac{\epsilon_0}{C} \sum_{n=1}^{\infty} \frac{1}{4^{2n}\sigma_n^2},$$

which is (3.2). □

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