EXTREME POINTS IN SPACES OF POLYNOMIALS

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Abstract. We determine the extreme points of the unit ball in spaces of complex polynomials (of a fixed degree), living either on the unit circle or on a subset of the real line and endowed with the supremum norm.

Introduction

Let $\mathcal{P}_n$ stand for the space of all polynomials with complex coefficients of degree not exceeding $n$. Given a compact set $E \subseteq \mathbb{C}$, one may treat $\mathcal{P}_n$ as a subspace of $C(E)$, the space of continuous functions on $E$, and equip it with the maximum norm

$$
\|P\|_\infty = \|P\|_{\infty,E} := \max_{z \in E} |P(z)| \quad (P \in \mathcal{P}_n).
$$

The resulting space will be denoted by $\mathcal{P}_n(E)$. We write

$$
\text{ball}(\mathcal{P}_n(E)) := \{P \in \mathcal{P}_n : \|P\|_{\infty,E} \leq 1\}
$$

for the unit ball of $\mathcal{P}_n(E)$, and we shall be concerned with the extreme points of this ball. (As usual, an element of a convex set $S$ is said to be its extreme point if it is not the midpoint of any nondegenerate segment contained in $S$.)

In this paper, we explicitly characterize the extreme points of ball$(\mathcal{P}_n(E))$ in the case where $E$ is either the circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ or a perfect compact subset of the real line $\mathbb{R}$. The description obtained is, perhaps, a bit more complicated than one could at first expect; however, the complexity seems to be in the nature of things.

Let us begin by recalling that the extreme points of the unit ball in $L^\infty(\mathbb{T})$ – or in $C(\mathbb{T})$ – are precisely the functions of modulus 1. (The same applies to other sets in place of $\mathbb{T}$.) Further, in the space $H^\infty$ of bounded analytic functions on $\{|z| < 1\}$, as well as in the disk algebra $H^\infty \cap C(\mathbb{T})$, the extreme points are known to be the unit-norm functions $f$ with $\int_\mathbb{T} \log(1 - |f(z)|^2) |dz| = -\infty$; see [H, Chap. 9].

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Yet another relevant example is provided by a theorem of Konheim and Rivlin [KR], dealing with the space \( P_n(R)(I) \) of all real polynomials of degree \( \leq n \) on the segment \( I := [-1,1] \). The theorem states that a unit-norm polynomial \( P \) is an extreme point of ball \((P_n(R)(I))\) if and only if \( N_I(1 - P^2) > n \); here \( N_I(f) \) is the total number of zeros (multiplicities included) that \( f \) has on \( I \). A similar result holds for real trigonometric polynomials on \( T \); see [R] or Proposition 1 in Section 1 below.

With these examples in mind, one might be tempted to believe that, in order to recognize the extreme points among all unit-norm elements \( P \) of the complex space \( P_n(E)(E) \) (say, with \( E = T \) or \( E = I \)), one only needs to know “how often” \( |P| \) takes the extremal value 1 on \( E \). In other words, one might seek to characterize the extreme points \( P \) in terms of the zeros – and their multiplicities – of the polynomial \( 1 - |P|^2 \). (Strictly speaking, \( 1 - |P|^2 \) is a trigonometric polynomial for \( P \in P_n(T) \) and a true polynomial when \( P \) lives on \( R \).)

However, no such thing can be done. Indeed, along with solving the two versions of the problem in Sections 1 and 2 (one of these deals with the circle, and the other with subsets of \( R \)), we also construct in each case a pair of unit-norm polynomials \( P_1, P_2 \) in \( P_n(E) \) satisfying

\[
1 - |P_1|^2 = 2 \left( 1 - |P_2|^2 \right),
\]

so that \( P_1 \) is a non-extreme point of ball\((P_n(E))\), while \( P_2 \) is extreme. In fact, the construction is carried out for the smallest possible value of \( n \), which equals 2 when \( E = T \), and 3 when \( E \) is a real segment.

In conclusion, we briefly mention the \( L^1 \) counterpart of the problem, i.e., the problem of determining the extreme points of the unit ball in certain \( L^1 \)-spaces of polynomials. Here, the real case was settled by Garkavi [G] and the complex case by the author [D]. Garkavi’s, as well as Konheim and Rivlin’s results were then rediscovered – or reproved – by Parnes in [P], where the current problem (the case of complex polynomials on \( T \) with the sup-norm) was also considered, but not solved.

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1. Polynomials on the circle

Among the unit-norm polynomials in \( P_n(T) \), we single out the class of monomials; these are of the form \( cz^k \), where \( c \in C, |c| = 1 \) and \( 0 \leq k \leq n \). Of course, every monomial is an extreme point of ball\((P_n(T))\).

Now if \( P \in P_n(T) \) satisfies \( \|P\|_\infty = 1 \) and is distinct from a monomial, let \( z_1, \ldots, z_N \) be an enumeration of the (nonempty) set \( \{ z \in T : |P(z)| = 1 \} \). The points \( z_1, \ldots, z_N \) are thus the distinct zeros of \( 1 - |P|^2 \) lying on \( T \), and the multiplicities of these zeros will be denoted by \( 2\mu_1, \ldots, 2\mu_N \). The \( \mu_j \)'s are positive integers, and their sum

\[
\mu := \sum_{j=1}^N \mu_j
\]
does not exceed \( n \). To see why, note that the function \( z \mapsto 1 - |P(z)|^2 \) (living on \( \mathbb{T} \)) is a nonnegative trigonometric polynomial of degree \( \leq n \), not vanishing identically. Therefore, its zeros lying on \( \mathbb{T} \) are necessarily of even order, while the total number of its zeros (multiplicities included) is at most \( 2n \). Hence \( 2\mu_1 + \cdots + 2\mu_N \leq 2n \), so that \( \mu \leq n \), as claimed above.

Next, for \( z = e^{it} \in \mathbb{T} \) and \( k \in \mathbb{N} \), consider the Wronski-type matrix

\[
W(z; k) = \begin{pmatrix}
\bar{z}^{\mu/2}P(z) & \bar{z}^{\mu/2+1}P(z) & \ldots & \bar{z}^{n-\mu/2}P(z) \\
(\bar{z}^{\mu/2}P(z))' & (\bar{z}^{\mu/2+1}P(z))' & \ldots & (\bar{z}^{n-\mu/2}P(z))'
\end{pmatrix}.
\]

The exponent \( n - \mu/2 \) in the last column should be viewed as \( \mu/2 + (n - \mu) \); thus, \( W(z; k) \) is a \( k \times (n - \mu + 1) \) matrix. The derivatives involved are with respect to the real variable \( t = \arg z \).

Let \( W_R(z; k) \) and \( W_I(z; k) \) stand for the real and imaginary parts of \( W(z; k) \), respectively. Finally, we need the block matrix

\[
W_P = \begin{pmatrix}
W_R(z_1; \mu_1) & W_I(z_1; \mu_1) \\
W_R(z_2; \mu_2) & W_I(z_2; \mu_2) \\
\ldots & \ldots \\
W_R(z_N; \mu_N) & W_I(z_N; \mu_N)
\end{pmatrix}.
\]

Here, each “entry” \( W_R(z_j; \mu_j) \) or \( W_I(z_j; \mu_j) \) is actually a \( \mu_j \times (n - \mu + 1) \) submatrix, as defined above, where everything is computed at the point \( z_j \). In particular, \( W_P \) is a \( \mu \times 2(n - \mu + 1) \) matrix, and its rank is therefore bounded by \( \min(\mu, 2(n - \mu + 1)) \).

**Theorem 1.** Let \( P \in \mathcal{P}_n(\mathbb{T}) \), \( \|P\|_\infty = 1 \). The following are equivalent.

(i) \( P \) is an extreme point of \( \text{ball}(\mathcal{P}_n(\mathbb{T})) \).

(ii) Either \( P \) is a monomial, or \( \text{rank } W_P = 2(n - \mu + 1) \).

The proof will be preceded by a brief discussion.

First of all, the condition \( \text{rank } W_P = 2(n - \mu + 1) \) can only be met if \( \mu \geq 2(n - \mu + 1) \), i.e., if \( \mu \geq \frac{2}{3}(n + 1) \). (The weaker condition \( \mu > n/2 \) was pointed out in \([P]\) as necessary in order that \( P \) be an extreme point.) The inequalities \( \frac{2}{3}(n+1) \leq \mu \leq n \) being incompatible for \( n = 0 \) and \( n = 1 \), there are no nontrivial extreme points for these \( n \). (Here and below, “nontrivial” means “distinct from a monomial”.) Now for \( n = 2, 3, 4 \), the two inequalities – in conjunction with the fact that \( \mu \in \mathbb{N} \) – reduce to the condition \( \mu = n \), which must be therefore fulfilled by each nontrivial extreme point \( P \) of \( \text{ball}(\mathcal{P}_n(\mathbb{T})) \).

On the other hand, for \( n \geq 2 \), the nontrivial extreme points \( P \) with \( \mu = n \) are characterized by the condition \( \text{rank } W_P = 2 \), which means that the two columns of \( W_P \) are linearly independent. This, in turn, is equivalent to saying that there
is no straight line in $\mathbb{C}$ passing through the origin and containing the set

$$\bigcup_{j=1}^{N} \left\{ \mathbf{z}_{j}/2 P(z_{j}), \left(\mathbf{z}_{j}/2 P\right)'(z_{j}), \ldots, \left(\mathbf{z}_{j}/2 P\right)^{(\mu_{j}-1)}(z_{j}) \right\}.$$ 

Now let us consider an example.

**Example 1.** Put $P_{0}(z) := \frac{1}{2}(z + z^{-1})$, so that $P_{0}(e^{it}) = \cos t$; then define

$$P_{1}(z) := zP_{0}(z) = \frac{1}{2}(z^{2} + 1)$$

and

$$P_{2}(z) := \frac{z}{\sqrt{2}} (P_{0}(z) + i) = \frac{1}{2\sqrt{2}}(z^{2} + 2iz + 1).$$

Clearly, $P_{1}$ and $P_{2}$ are unit-norm polynomials in $\mathcal{P}_{2}(\mathbb{T})$. In fact, for $z = e^{it} \in \mathbb{T}$,

$$|P_{1}(z)|^{2} = P_{0}^{2}(z) = \cos^{2} t$$

and

$$|P_{2}(z)|^{2} = \frac{1}{2} (P_{0}^{2}(z) + 1) = \frac{1}{2} (\cos^{2} t + 1).$$

In particular,

$$1 - |P_{1}(z)|^{2} = 2 \left(1 - |P_{2}(z)|^{2}\right), \quad z \in \mathbb{T}$$

(indeed, both sides equal $\sin^{2} t$), and so the two polynomials have the same $z_{j}$'s and $\mu_{j}$'s. Specifically, these are $z_{1} = 1$, $z_{2} = -1$ (or vice versa) and $\mu_{1} = \mu_{2} = 1$, so that $N = \mu = n = 2$.

However, while $P_{1}$ is the arithmetic mean of two monomials, $z^{2}$ and 1, and hence a non-extreme point of ball($\mathcal{P}_{2}(\mathbb{T})$), it turns out that $P_{2}$ is an extreme point thereof. This last fact follows by Theorem 1, since the matrix

$$W_{P_{2}} = \begin{pmatrix}
\Re(P_{2}(1)) & \Im(P_{2}(1)) \\
\Re(-P_{2}(-1)) & \Im(-P_{2}(-1))
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}$$

has rank 2.

The proof of Theorem 1 will rely on two elementary observations. The first of these, stated for an arbitrary compact set $E \subset \mathbb{C}$, will also be used when proving Theorem 2 in the next section.

**Observation 1.** Clearly, a given unit-norm polynomial $P \in \mathcal{P}_{n}$ is an extreme point of ball($\mathcal{P}_{n}(E)$) if and only if the only polynomial $Q \in \mathcal{P}_{n}$ satisfying

$$\|P + Q\|_{\infty} \leq 1 \quad \text{and} \quad \|P - Q\|_{\infty} \leq 1$$

(1.1)
is $Q \equiv 0$. Rewriting (1.1) as

$$|P \pm Q|^2 = |P|^2 \pm 2\Re(PQ) + |Q|^2 \leq 1$$

and noting that $\max(x, -x) = |x|$ for all $x \in \mathbb{R}$, we see that $P$ is extreme iff there is no nontrivial $Q \in \mathcal{P}_n$ for which

(1.2) $$2|\Re(PQ)| + |Q|^2 \leq 1 - |P|^2$$
everywhere on $E$.

**Observation 2.** If $z = e^{it}$ and $\zeta_j = e^{it_j}$ ($j = 1, \ldots, N$) are points of $T$, and if $k_1, \ldots, k_N$ are positive integers with $\sum_{j=1}^N k_j = k$, then the identities

$$z - \zeta_j = 2ie^{it_j/2}e^{it/2}\sin \frac{t-t_j}{2}$$
yield

(1.3) $$\prod_{j=1}^N (z - \zeta_j)^{k_j} = ce^{ikt/2} \prod_{j=1}^N \left( \sin \frac{t-t_j}{2} \right)^{k_j},$$

where

$$c = (2i)^k \prod_{j=1}^N \exp \left( \frac{ik_j t_j}{2} \right).$$

**Proof of Theorem 1.** $(ii) \implies (i)$. We shall assume that $P$ is distinct from a monomial (otherwise, it is obviously extreme) and that rank $W_P = 2(n - \mu + 1)$. Now suppose (1.2) holds for some $Q \in \mathcal{P}_n$. In particular, we have then

$$|Q(z)|^2 \leq 1 - |P(z)|^2, \quad z \in T,$$

and so, for $j = 1, \ldots, N$, the polynomial $Q$ vanishes at $z_j$ with multiplicity at least $\mu_j$. (Recall that the multiplicity of $z_j$ as a zero of $1 - |P|^2$ is $2\mu_j$.) Hence $Q$ is divisible by $\prod_{j=1}^N (z - z_j)^{\mu_j}$ and takes the form

(1.4) $$Q(z) = Q_0(z) \cdot z^{\mu/2} \prod_{j=1}^N \left( \sin \frac{t-t_j}{2} \right)^{\mu_j}, \quad z = e^{it} \in T,$$

for some $Q_0 \in \mathcal{P}_{n-\mu}$; by $t_j$ we now denote arg $z_j$. Here, to arrive at (1.4), we have used (1.3) with $z_j$ in place of $\zeta_j$ and with $\mu_j$ (resp., $\mu$) in place of $k_j$ (resp., $k$). From (1.4) we get

(1.5) $$\Re \left( \overline{P(z)}Q(z) \right) = \prod_{j=1}^N \left( \sin \frac{t-t_j}{2} \right)^{\mu_j} \Re \left( z^{\mu/2}\overline{P(z)}Q_0(z) \right).$$
Combining this with the fact that
\[ |\Re (P(z)Q(z))| \leq 1 - |P(z)|^2 \]
(which is contained in (1.2)) yields
\[ (1.6) \quad |\Re \left( z^{\mu/2} P(z) Q_0(z) \right) | \leq (1 - |P(z)|^2) \prod_{j=1}^{N} \left| \sin \frac{t - t_j}{2} \right|^{-\mu_j}, \quad z = e^{it} \in \mathbb{T}. \]

The right-hand side of (1.6) being \( O(|z - z_j|^\mu_j) \) as \( z \to z_j \), the left-hand side must also vanish at each \( z_j \) with multiplicity at least \( \mu_j \). In other words, for each \( j = 1, \ldots, N \), one has
\[ R \left( z^{\mu/2} P(z) Q_0(z) \right) \left( l \right) (z_j) = 0 \quad (l = 0, 1, \ldots, \mu_j - 1). \]

Putting
\[ Q_0(z) = \sum_{k=0}^{n-\mu} (c_k + id_k) z^k \]
and substituting this into (1.7), we obtain
\[ (1.8) \quad \sum_{k=0}^{n-\mu} c_k \Re \left( z^{\mu/2 + k} P \right) \left( l \right) (z_j) - \sum_{k=0}^{n-\mu} d_k \Im \left( z^{\mu/2 + k} P \right) \left( l \right) (z_j) = 0 \]
\[ (j = 1, \ldots, N; \ l = 0, 1, \ldots, \mu_j - 1), \]
which can be viewed as a homogeneous system of \( \mu_1 + \cdots + \mu_N = \mu \) linear equations with \( 2(n-\mu+1) \) real unknowns \( c_0, \ldots, c_{n-\mu}, d_0, \ldots, d_{n-\mu} \). The matrix of this system is precisely \( WP \), and the hypothesis \( \text{rank} WP = 2(n-\mu+1) \) implies that the only solution is
\[ c_0 = \cdots = c_{n-\mu} = d_0 = \cdots = d_{n-\mu} = 0. \]
Thus \( Q_0 \equiv 0 \), whence also \( Q \equiv 0 \), and \( P \) is an extreme point.

(i) \( \implies \) (ii). The above argument can be essentially reversed. Indeed, suppose that (ii) fails, so that \( P \) is distinct from a monomial and \( \text{rank} WP < 2(n-\mu+1) \). The homogeneous system (1.8) has then a nontrivial solution, and the equations (1.7) hold for \( j = 1, \ldots, N \) with some \( Q_0 \in P_{n-\mu}, Q_0 \not\equiv 0 \). Multiplying \( Q_0 \) by a number \( \varepsilon > 0 \) (if necessary), we may assume in addition that the norm \( \|Q_0\|_\infty \) is appropriately small; we shall specify our choice later.

Now that we have such a \( Q_0 \) at our disposal, let us define \( Q \) by (1.4), where, as before, it is understood that \( z_j = e^{it_j} \). By Observation 2, we have \( Q \in P_n \); we also remark that \( Q \not\equiv 0 \), because \( Q_0 \not\equiv 0 \), and that (1.5) holds true.
We further claim that
\[(1.9) \quad |Q(z)|^2 = O \left( 1 - |P(z)|^2 \right), \quad z \in \mathbb{T},\]
and
\[(1.10) \quad |\Re(\overline{P}(z)Q(z))| = O \left( 1 - |P(z)|^2 \right), \quad z \in \mathbb{T}.\]
Indeed, (1.9) is fulfilled because \(Q\) is divisible by \(\prod_{j=1}^{N} (z - z_j)^{\mu_j}\), and so \(|Q(z)|^2\) vanishes at those points of \(\mathbb{T}\) (viz., \(z_1, \ldots, z_N\)) where \(1 - |P(z)|^2\) does, with at least the same multiplicities (viz., \(2\mu_1, \ldots, 2\mu_N\)). Similarly, to verify (1.10), one checks that its left-hand side has a zero at each \(z_j\) of multiplicity \(\geq 2\mu_j\); this is due to (1.5) and (1.7).

In view of the above discussion, we could have started with a \(Q_0\) for which the quantity \(\|Q_0\|_{\infty}\), and hence also the “big oh” constants in (1.9) and (1.10), are as small as desired. In particular, a suitable choice ensures that
\[|Q(z)|^2 \leq \frac{1}{2} \left( 1 - |P(z)|^2 \right), \quad z \in \mathbb{T},\]
and
\[2 |\Re(\overline{P}(z)Q(z))| \leq \frac{1}{2} \left( 1 - |P(z)|^2 \right), \quad z \in \mathbb{T}.\]
Summing, we arrive at (1.2) and conclude that \(P\) is not an extreme point. The proof is complete. \(\square\)

One might also consider the space \(T_n\) of all trigonometric polynomials of degree \(\leq n\); these are, by definition, functions of the form \(\sum_{k=-n}^{n} c_k z^k\) living on \(\mathbb{T}\). A trigonometric polynomial \(T \in T_n\) is an extreme point of \(\text{ball}(T_n)\) if and only if \(z^n T\) is an extreme point of \(\text{ball}(P_{2n})\). Thus, the extreme points \(T\) of \(\text{ball}(T_n)\) are actually described by Theorem 1, where obvious adjustments are needed: one should first replace \(n\) by \(2n\), and then \(P\) by \(z^n T\). (Of course, the monomials in the theorem’s statement should now include those with negative exponents, too.)

Finally, we briefly discuss the subspace \(T_n^\mathbb{R}\) of real-valued functions in \(T_n\); a trigonometric polynomial \(\sum_{k=-n}^{n} c_k z^k\) is thus in \(T_n^\mathbb{R}\) iff \(c_{-k} = \overline{c}_k\) for \(|k| \leq n\). As before, given a nonconstant \(P \in T_n^\mathbb{R}\) with \(\|P\|_{\infty} = 1\), we let \(z_j = e^{it_j}\) (\(j = 1, \ldots, N\)) be the distinct zeros that the (nonnegative) trigonometric polynomial \(1 - P^2\) happens to have on \(\mathbb{T}\); the (even) multiplicity of the zero \(z_j\) is again denoted by \(2\mu_j\), and we write \(\mu = \sum_{j=1}^{N} \mu_j\). This time, however, \(1 - P^2\) is of degree \(\leq 2n\), so the only \(a \text{ priori}\) estimate on \(\mu\) is that \(\mu \leq 2n\). As to the constant polynomials \(P \equiv 1\) and \(P \equiv -1\), for each of these we put \(\mu = +\infty\).

The following proposition is a trigonometric version of the Konheim–Rivlin result that can be found in [R]; a short self-contained proof will be given here for the sake of completeness.
Proposition 1. Let $P \in \mathcal{T}_n^\mathbb{R}$, $\|P\|_\infty = 1$. Then $P$ is an extreme point of ball $(\mathcal{T}_n^\mathbb{R})$ if and only if $\mu > n$.

Proof. To prove the “if” part, assume that $\mu > n$ and that (1.1) holds for some $Q \in \mathcal{T}_n^\mathbb{R}$. We have then $\pm P \pm Q \leq 1$ on $\mathbb{T}$, where the signs can be chosen in the four possible ways. Consequently,

$$|Q| \leq 1 - |P| \leq 1 - P^2.$$  

Now since the right-hand side has in total $2\mu (> 2n)$ zeros on $\mathbb{T}$, while $Q$ is of degree $\leq n$, it follows that $Q \equiv 0$ and $P$ is an extreme point.

To establish the “only if” part, assume that $\mu \leq n$ and put

$$Q(e^{it}) := \varepsilon \prod_{j=1}^N \left( \sin \left( \frac{t - t_j}{2} \right) \right)^{2\mu_j},$$

with a suitable $\varepsilon > 0$. Then $Q \in \mathcal{T}_n^\mathbb{R}$, and making $\varepsilon$ sufficiently small we can arrange it so that

$$|Q| \leq \frac{1}{2} \left( 1 - P^2 \right) \leq 1 - |P|.$$  

From this, (1.1) follows immediately, and $P$ fails to be extreme. \qed

We remark, in conclusion, that every nonconstant trigonometric polynomial in ball $(\mathcal{T}_n^\mathbb{R})$ is a non-extreme point of ball$(\mathcal{T}_n)$, the unit ball of the complex space $\mathcal{T}_n$.

2. Polynomials on subsets of $\mathbb{R}$

Let $K$ be a perfect compact subset of $\mathbb{R}$ (as usual, “perfect” means “having no isolated points”), and let $P$ be a nonconstant polynomial in $\mathcal{P}_n(K)$ with $\|P\|_\infty = 1$. Here and throughout this section, $\|P\|_\infty$ stands for $\|P\|_{\infty,K} := \max_{x \in K} |P(x)|$. (Likewise, some of the other symbols below should not be confused with their namesakes in Section 1.)

Further, let $x_1, \ldots, x_N$ be the distinct elements of the set $\{x \in K : |P(x)| = 1\}$, and let $m_1, \ldots, m_N$ denote the respective multiplicities of these points, regarded as zeros for $1 - |P|^2$. (We remark that $m_j$ need not be even, unless $x_j$ is an interior point for $K$.) The function $x \mapsto 1 - |P(x)|^2$ being a polynomial of degree $\leq 2n$, we have $m_1 + \cdots + m_N \leq 2n$. Next, we introduce the numbers

$$\mu_j := \left\lfloor \frac{m_j + 1}{2} \right\rfloor \quad (j = 1, \ldots, N),$$

where $\lfloor \cdot \rfloor$ denotes the integral part, and their sum $\mu := \sum_{j=1}^N \mu_j$. Finally, let $M$ be the number of those $j$’s for which $m_j \geq 2$. Thus $0 \leq M \leq N$, and we may assume that the inequality $m_j \geq 2$ holds precisely for $1 \leq j \leq M$. 

For a constant polynomial $P \equiv c$ with $|c| = 1$, we put $\mu = +\infty$.

Now suppose $P$ is a unit-norm polynomial in $\mathcal{P}_n(K)$ with the property $\mu \leq n$. To such a $P$, we associate the Wronski-type matrix

$$W(x; k) = \begin{pmatrix} P(x) & xP(x) & \ldots & x^{n-\mu}P(x) \\ P'(x) & (xP(x))' & \ldots & (x^{n-\mu}P(x))' \\ \vdots & \vdots & \ddots & \vdots \\ P^{(k-1)}(x) & (xP(x))^{(k-1)} & \ldots & (x^{n-\mu}P(x))^{(k-1)} \end{pmatrix},$$

where $x \in \mathbb{R}$ and $k \in \mathbb{N}$. The real and imaginary parts of $W(x; k)$ will be denoted by $W_\Re(x; k)$ and $W_\Im(x; k)$. This said, we form the block matrix

$$W_P = \begin{pmatrix} W_\Re(x_1; m_1 - \mu) & W_\Im(x_1; m_1 - \mu) \\ W_\Re(x_2; m_2 - \mu) & W_\Im(x_2; m_2 - \mu) \\ \vdots & \vdots \\ W_\Re(x_M; m_M - \mu) & W_\Im(x_M; m_M - \mu) \end{pmatrix},$$

which has $\sum_{j=1}^N m_j - \mu$ rows and $2(n - \mu + 1)$ columns. In the case that $M = 0$ (i.e., when $m_j = \mu_j = 1$ for all $j$), it is understood that $W_P$ is the zero matrix (of any order), so that rank $W_P = 0$.

**Theorem 2.** Let $P \in \mathcal{P}_n(K)$ and $\|P\|_\infty = 1$. The following are equivalent.

(i) $P$ is an extreme point of $\text{ball}(\mathcal{P}_n(K))$.

(ii) Either $\mu > n$, or rank $W_P = 2(n - \mu + 1)$.

One easily checks that for $n \leq 2$, condition (ii) reduces to just saying that $\mu > n$. It is for $n \geq 3$ that things become more complicated, as the following example shows.

**Example 2.** Let $K = [-1, 2]$, and put

$$P_1(x) := \frac{1}{2}(x^3 - 3x), \quad P_2(x) := \frac{1}{\sqrt{2}}(P_1(x) + i).$$

One easily verifies that $|P_1(x)| \leq 1$ for $x \in K$, the equality being attained at the points

$$(2.1) \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = 2.$$

Then one deduces a similar fact for $P_2$ by noting that

$$|P_2(x)|^2 = \frac{1}{2} (P_1^2(x) + 1).$$

Thus, $P_1$ and $P_2$ are unit-norm elements of $\mathcal{P}_3(K)$. Furthermore,

$$1 - P_2^2(x) = 2(1 - |P_2(x)|^2) = -\frac{1}{4}x(x + 1)^2(x - 1)^2(x - 2).$$
The zeros of this last polynomial belonging to $K$ (i.e., the common $x_j$’s for $P_1$ and $P_2$) are given by (2.1), and the corresponding (common) multiplicities are

$$m_1 = 2, \quad m_2 = 2, \quad m_3 = 1.$$ 

Hence $\mu_1 = \mu_2 = \mu_3 = 1$, so that $N = \mu = n = 3$ and $M = 2$. Theorem 2 tells us now that $P_1$ is a non-extreme point of ball $(P_3(K))$, while $P_2$ is extreme. Indeed, the polynomial $P_1$ being real-valued, the second column of the $(2 \times 2)$-matrix $W_{P_1}$ is null, whence rank $W_{P_1} = 1$, whereas the matrix

$$W_{P_2} = \begin{pmatrix} \Re (P_2(-1)) & \Im (P_2(-1)) \\ \Re (P_2(1)) & \Im (P_2(1)) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has rank 2.

**Proof of Theorem 2.** (ii) $\Rightarrow$ (i). Suppose (1.1) is fulfilled for some $Q \in \mathcal{P}_n$. Then (1.2) holds everywhere on $K$, whence in particular

$$|Q(x)|^2 \leq 1 - |P(x)|^2, \quad x \in K.$$ 

Here, the right-hand side is $O((x - x_j)^{m_j})$ as $x \to x_j$, and so

$$Q(x) = O \left( |x - x_j|^{m_j/2} \right) \quad \text{as} \quad x \to x_j, \quad x \in K. \quad (2.2)$$

Since $Q$ is a polynomial, while $\mu_j$ is the smallest integer in the interval $[m_j/2, \infty)$, it actually follows from (2.2) that $Q$ has a zero of multiplicity $\geq \mu_j$ at $x_j$. Hence

$$Q(x) = Q_0(x) \prod_{j=1}^{N} (x - x_j)^{\mu_j} \quad (2.3)$$

for some polynomial $Q_0$.

Now if $\mu > n$, then (2.3) is only possible for $Q \equiv 0$, which implies that $P$ is an extreme point.

It remains to consider the case where $\mu \leq n$ and rank $W_P = 2(n - \mu + 1)$. In this case, (2.3) holds for some $Q_0 \in \mathcal{P}_{n-\mu}$, and we write

$$Q_0(x) = \sum_{k=0}^{n-\mu} (c_k + i d_k)x^k \quad (2.4)$$

with $c_k, d_k \in \mathbb{R}$. Also, (2.3) yields

$$\Re (\mathcal{P}(x)Q(x)) = \prod_{j=1}^{N} (x - x_j)^{\mu_j} \Re (\mathcal{P}(x)Q_0(x)). \quad (2.5)$$
Substituting this into the inequality
\[ |\Re (P(x)Q(x))| \leq 1 - |P(x)|^2, \quad x \in K \]
(which is a consequence of (1.2)), we get
\[ (2.6) \quad \prod_{j=1}^{N} |x - x_j|^\mu_j |\Re (P(x)Q_0(x))| \leq 1 - |P(x)|^2, \quad x \in K. \]
The right-hand side of (2.6) being \( O(|x - x_j|^{m_j}) \) as \( x \to x_j \), we deduce that
\[ (2.7) \quad \Re (P(x)Q_0(x)) = O\left(|x - x_j|^{m_j - \mu_j}\right) \quad \text{as} \quad x \to x_j, \quad x \in K. \]
Here, the restriction \( x \in K \) can be actually dropped (i.e., replaced by \( x \in \mathbb{R} \)), since \( \Re (PQ_0) \) is a polynomial. Thus (2.7) tells us that \( \Re (PQ_0) \) vanishes at \( x_j \) with multiplicity at least \( m_j - \mu_j \); of course, this is only meaningful for \( 1 \leq j \leq M \), since otherwise \( m_j = \mu_j = 1 \). Therefore,
\[ (2.8) \quad \Re (PQ_0)^{(l)}(x_j) = 0 \quad (1 \leq j \leq M, \quad 0 \leq l \leq m_j - \mu_j - 1). \]
With (2.4) plugged in, (2.8) becomes a homogeneous system of linear equations with respect to the unknowns \( c_0, \ldots, c_{n-\mu}, d_0, \ldots, d_{n-\mu} \). The matrix of the system is \( W_P \), and the hypothesis \( \text{rank } W_P = 2(n - \mu + 1) \) ensures that the only solution is the trivial one. Hence \( Q_0 \equiv 0 \), which implies \( Q \equiv 0 \) and proves that \( P \) is an extreme point.

(i) \( \implies \) (ii). Conversely, if \( \mu \leq n \) and \( \text{rank } W_P < 2(n - \mu + 1) \), then the homogeneous system just mentioned has a nontrivial solution, so that (2.8) holds with some \( Q_0 \in \mathcal{P}_{n-\mu}, Q_0 \not\equiv 0 \). Now if the norm \( \|Q_0\|_\infty \) is appropriately small (which can be safely assumed), then the nontrivial polynomial \( Q \in \mathcal{P}_n \) defined by (2.3) will satisfy
\[ (2.9) \quad |Q|^2 \leq \frac{1}{2} \left(1 - |P|^2\right) \]
and
\[ (2.10) \quad 2 |\Re (PQ)| \leq \frac{1}{2} \left(1 - |P|^2\right) \]
everywhere on \( K \). Indeed, for \( j = 1, \ldots, N \), the left-hand sides of (2.9) and (2.10) vanish at \( x_j \) with multiplicity at least \( m_j \) each. (To see why, recall that \( 2\mu_j \geq m_j \) and use the relations (2.5) and (2.8).)

Taken together, (2.9) and (2.10) yield (1.2), and we conclude that \( P \) fails to be extreme in ball \( (\mathcal{P}_n(K)) \). \( \square \)
References


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