ON A QUESTION OF LOUIS NIRENBERG

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Abstract. This note proves that if $A, B$ are $C^\infty$ real vector fields in an open set $\Omega \subset \mathbb{R}^3$ such that $A, B$ and $[A, B]$ are linearly independent then, given any $C^\infty$ real vector field $C$ in $\Omega$ and any function $\varphi \in C^\infty(\Omega)$, the second order operator $L = AB + C + \varphi$ is locally solvable at every point of $\Omega$. The result can be extended to first-order real pseudodifferential operators with simple real characteristics.

1. Statement and Proof of Theorem 1

Theorem 1. Let $A, B$ be $C^\infty$ real vector fields in an open set $\Omega \subset \mathbb{R}^3$ such that $A, B$ and $[A, B]$ are linearly independent. Let the $C^\infty$ real vector field $C$ in $\Omega$ and the function $\varphi \in C^\infty(\Omega)$ be arbitrary and call $L$ the second order operator $L = AB + C + \varphi$. Given any point $x^0 \in \Omega$ and any number $\varepsilon > 0$ there is an open neighborhood $U_{x^0, \varepsilon} \subset \Omega$ of $x^0$ with the following property: there is a bounded linear operator $G_{x^0, \varepsilon} : H^{-1}(U_{x^0, \varepsilon}) \longrightarrow H^{-1}(U_{x^0, \varepsilon})$ with norm $\leq \varepsilon$ and such that $LG_{x^0, \varepsilon} f = f$ in $U_{x^0, \varepsilon}$ for every $f \in H^{-1}(U_{x^0, \varepsilon})$.

In the statement $H^{-1}(U_{x^0})$ denotes the standard Sobolev space. We indicate in the second section of this note how right-inverses of $L$ acting from the Sobolev space $H^s(U_{x^0})$ to itself can be found, for each $s \in \mathbb{R}$ (after some contraction of $U_{x^0}$ about $x^0$). The proof will also make clear under which hypotheses one can get right-inverses acting from $H^s(U_{x^0})$ to $H^{s+1}(U_{x^0})$ (cf. Corollary 1).

Theorem 1 answers a question of Louis Nirenberg originating in joint work, currently in progress, with I. Ekeland. At the microlocal level Theorem 1 is closely related to the works [Ha1], [Ha2].

Below we use systematically the notation $\| \cdot \|$ and $(\cdot, \cdot)$ for the $L^2$ norm and the $L^2$ inner product, respectively; but we shall use the notation $\| \cdot \|_s$ for the norm in the Sobolev space $H^s$, $s \neq 0$. The letters $K, K_1, \ldots$ will denote various constants that depend solely on the (pseudo)differential operators being considered.

Proof. Call $L^*$ the adjoint of $L$. Let $x^0 \in \Omega$ be arbitrary; actually we take it to be the origin in the coordinates $x_i$, $i = 1, 2, 3$. We select a number $\delta > 0$ such that the closure of the ball $\mathcal{B}_\delta = \{ x \in \mathbb{R}^3; |x| < \delta \}$ is contained in $\Omega$.

By our hypothesis we can write

\begin{equation}
C = \alpha(x) A + \beta(x) B + \gamma(x) T
\end{equation}

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where \( T = [A, B] \) and \( \alpha, \beta, \gamma \in C^\infty(\Omega) \). For most vector fields \( C \) the claim in Theorem 1 is a consequence of the following

**Lemma 1.** Under the hypotheses of Theorem 1, if \( \gamma(0) \neq -\frac{1}{2} \) then there are constants \( K, \delta > 0 \) such that

\[
\|ABu\| + \|BAu\| + \|Tu\| \leq K \|L^*u\| \tag{1.2}
\]

for all \( u \in C^\infty_c(\mathcal{B}_\delta) \).

**Proof.** Below, given two quadratic functionals \( Q_1(u) \) and \( Q_2(u) \), we shall write

\[
Q_1(u) \sim Q_2(u) \quad \text{if to each } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that}
\]

\[
|Q_1(u) - Q_2(u)| \leq \varepsilon \left( \|ABu\|^2 + \|BAu\|^2 + \|Tu\|^2 \right)
\]

for all \( u \in C^\infty_c(\mathcal{B}_\delta) \).

Since the origin is not a critical point of the vector fields \( A \) and \( B \) the following is true:

- \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that, for all } u \in C^\infty_c(\mathcal{B}_\delta), \)

\[
\|u\| \leq \varepsilon \|Bu\| \leq \varepsilon^2 \|ABu\|, \quad \|u\| \leq \varepsilon \|Au\| \leq \varepsilon^2 \|BAu\|. \tag{1.3}
\]

We have

\[
L^* = BA - \gamma(x)T + p(x)A + q(x)B + r(x) + \varphi(x)
\]

with \( p, q, r \in C^\infty(\Omega) \). It follows at once from (1.3) that

\[
\|L^*u\|^2 \cong \|(BA - \gamma T)u\|^2. \tag{1.4}
\]

We use the fact that

\[
\|(BA - \gamma T)u\|^2 = \|BAu\|^2 + \|\gamma Tu\|^2 - 2\Re (BAu, \gamma Tu).
\]

and

\[
\|(BA - \gamma T)u\|^2 = \|(AB - (1 + \gamma)T)u\|^2 = \|ABu\|^2 + \|((1 + \gamma)T)u\|^2 - 2\Re (ABu, (1 + \gamma)Tu)
\]

to derive

\[
\|(BA - \gamma T)u\|^2 = \frac{1}{2} \|ABu\|^2 + \frac{1}{2} \|BAu\|^2 + \frac{1}{2} \|\gamma Tu\|^2 + \frac{1}{2} \|(1 + \gamma)Tu\|^2 - \Re ((BAu, \gamma Tu) + (ABu, \gamma Tu)) - \Re (ABu, Tu). \tag{1.5}
\]

We claim that

\[
(BAu, \gamma Tu) + (\gamma Tu, ABu) \cong 0. \tag{1.6}
\]
Indeed,
\[(BAu, \gamma Tu) = (Au, (B^* + B) (\gamma Tu)) - (Au, \gamma TBu) - (Au, [B, \gamma T] u) =
(Au, (B^* + B) (\gamma Tu)) - (u, (A^* + A) (\gamma TBu)) - (Au, [B, \gamma T] u) + (u, A (\gamma TBu)) =
(Au, (B^* + B) (\gamma Tu)) - (T^* (\gamma (A^* + A) u), Bu) - (Au, [B, \gamma T] u)
+ ([A, \gamma T]^* u, Bu) + ((\gamma T)^* u, ABu) =
(Au, (B^* + B) (\gamma Tu)) - (T^* (\gamma (A^* + A) u), Bu) - (Au, [B, \gamma T] u)
+ ([A, \gamma T]^* u, Bu) + ((\gamma T)^* + \gamma T) u, ABu) - (\gamma Tu, ABu).
\]

Putting (1.6) into (1.5) yields
\[2 \| (BA - \gamma T) u \|^2 \cong \| ABu \|^2 + \| BAu \|^2 + \| \gamma Tu \|^2
+ \| (1 + \gamma) Tu \|^2 - 2 \text{Re} (ABu, Tu).
\]

We have
\[(ABu, [A, B] u) \cong - (Bu, [A, B] Au) \cong (B [A, B] u, Au) \cong - ([A, B] u, BAu)
\]
and therefore
\[2 \text{Re} (ABu, [A, B] u) \cong \| [A, B] u \|^2.
\]
Combining (1.7) and (1.8) yields
\[2 \| (BA - \gamma T) u \|^2 \cong \| ABu \|^2 + \| BAu \|^2 + 2 \int \gamma (1 + \gamma) |Tu|^2 \, dx.
\]
But for any \(0 < \theta < 1\),
\[\frac{1}{2} (1 - \theta) \| (AB - BA) u \|^2 \leq (1 - \theta) \| ABu \|^2 + (1 - \theta) \| BAu \|^2
\]
whence
\[\| ABu \|^2 + \| BAu \|^2 + 2 \int \gamma (1 + \gamma) |Tu|^2 \, dx \geq \theta \left( \| ABu \|^2 + \| BAu \|^2 \right) + 2 \int \left( \gamma (1 + \gamma) + \frac{1}{4} (1 - \theta) \right) |Tu|^2 \, dx
\]
The hypothesis \(\gamma (0) \neq -\frac{1}{2}\) is equivalent to
\[\gamma (0) (1 + \gamma (0)) + \frac{1}{4} > 0.
\]
We can find \(\theta\) and \(\delta > 0\) such that
\[\forall x \in \mathcal{B}_\delta, \gamma(x) (1 + \gamma (x)) + \frac{1}{4} (1 - \theta) \geq \theta
\]
and therefore such that
\[\| ABu \|^2 + \| BAu \|^2 + 2 \int \gamma (1 + \gamma) |Tu|^2 \, dx \geq \theta \left( \| ABu \|^2 + \| BAu \|^2 + \| Tu \|^2 \right).\]
Combining this with (1.4) and possibly further reducing $\delta$ yields (1.2).

Since $A, B, C$ are linearly independent we see that (1.2) has the following consequence

\[(1.9) \quad \|u\|_1 \leq K_1 \|L^*u\|, \; u \in C_c^\infty (\Gamma_\delta),\]

where $\|\cdot\|_1$ is the norm in the Sobolev space $H^1 (\mathfrak{B}_\delta)$. We may state:

**Corollary 1.** Suppose the hypotheses of Theorem 1 satisfied and $\gamma (0) \neq -\frac{1}{2}$. Then, if the number $\delta > 0$ is sufficiently small there is a bounded linear operator $G_\delta : H^{-1} (\mathfrak{B}_\delta) \rightarrow L^2 (\mathfrak{B}_\delta)$ such that $LG_\delta f = f$ for every $f \in H^{-1} (\mathfrak{B}_\delta)$.

It remains to prove Theorem 1 when $\gamma (0) = -\frac{1}{2}$. To simplify notation it is convenient to assume $A = -A^*$; to achieve this it suffices to choose the local coordinates $x_i$ ($i = 1, 2, 3$) in such a way that $A = \frac{\partial}{\partial x_1}$.

Still under the hypothesis that $\gamma (0) \neq -\frac{1}{2}$ we apply (1.9) with $Au$ substituted for $u$, thus obtaining, for all $u \in C_c^\infty (\mathfrak{B}_\delta)$,

\[\|u\|_1 \leq 4\delta K \|L^*A^*u\| . \]

Let $D$ be any first-order linear differential operator in $\Omega$ with smooth coefficients and let $D^*$ denote its formal adjoint. We derive from the preceding inequality:

\[\|u\|_1 \leq 4\delta K_1 \|L^*A^*u + D^*u\| + 8K_2 \|u\|_1 \quad (1.10)\]

whence, provided $\delta K_2 \leq \frac{1}{2}$,

This last inequality has the following implication:

**Corollary 2.** Suppose the hypotheses of Theorem 1 satisfied and $\gamma (0) \neq -\frac{1}{2}$. Let $D$ be any first-order linear differential operator in $\Omega$ with smooth coefficients. Then, to each number $\varepsilon > 0$ there is a number $\delta > 0$ and a bounded linear operator $G_{\varepsilon,D} : H^{-1} (\mathfrak{B}_\delta) \rightarrow L^2 (\mathfrak{B}_\delta)$ whose norm does not exceed $\varepsilon$ and which is such that $(AL + D)G_{\varepsilon,D}f = f$ for every $f \in H^{-1} (\mathfrak{B}_\delta)$.

At this juncture we assume $\gamma (0) = -\frac{1}{2}$. We form

\[(AB + C + \varphi)A = AB - [A,B] + C + \varphi - [A,C + \varphi].\]

and we apply Corollary 2 with $L = AB - [A,B] + C + \varphi$ and $D = -[A,C + \varphi]$. This is permitted since, at the origin,

\[-[A,B] + C = -\frac{3}{2} T \mod (A,B).\]

Corollary 2 states that if $\delta > 0$ is sufficiently small then

\[(AB + C + \varphi)AG_{\varepsilon,D}f = f\]

for every $f \in H^{-1} (\mathfrak{B}_\delta)$. To complete the proof of Theorem 1 it suffices to observe that $AG_{\varepsilon,D}$ is a bounded linear operator $H^{-1} (\mathfrak{B}_\delta) \rightarrow H^{-1} (\mathfrak{B}_\delta)$ whose norm

\[\|AG_{\varepsilon,D}\| \leq C \varepsilon .\]
does not exceed $\varepsilon \|A\|$ where $\|A\|$ is the norm of the operator $A : L^2(\mathfrak{B}_\delta) \rightarrow H^{-1}(\mathfrak{B}_\delta)$.

**Remark 1.** Inspection of the proof of Theorem 1 shows that the requirement that $C$ be real can be slightly weakened: for instance the coefficients $\alpha$ and $\beta$ in (1.1) need not be real. It is also clear that the regularity requirements on all the coefficients can be weakened, to $C^3$ and possibly further.

2. Further Remarks

2.1. **Meaning of the condition on $\gamma(0)$**. The meaning of the value $\gamma(0) = -\frac{1}{2}$ (cf. Corollaries 1, 2) becomes clearer if we write $AB + C = \frac{1}{2} (AB + BA) + C + \frac{i}{2} T$. The best way to understand this meaning is through the subprincipal symbol of the operator $L = AB + C + \varphi$. Call $A(x, \xi)$ the symbol of $A$; $A(x, \xi)$ is purely imaginary; likewise for $B$ and $C$. The symbol of $AB + C$ is

$$A(x, \xi)B(x, \xi) - i\nabla_\xi A(x, \xi) \cdot \nabla_x B(x, \xi) + C(x, \xi).$$

The subprincipal symbol of $L$ is

$$\sigma_{\text{sub}}(x, \xi) = C(x, \xi) - i\nabla_\xi A(x, \xi) \cdot \nabla_x B(x, \xi) - \frac{1}{2i} (\nabla_x \cdot \nabla_\xi)(A(x, \xi)B(x, \xi)).$$

Using the notation $\{,\}$ for the Poisson bracket we see that

$$\sigma_{\text{sub}}(x, \xi) \cong C(x, \xi) + \frac{1}{2i} \{A(x, \xi), B(x, \xi)\}$$

mod $(A(x, \xi), B(x, \xi))$. The right-hand side is the principal symbol of $C + \frac{i}{2} T$. The hypothesis that $\gamma(0) \neq -\frac{1}{2}$ is equivalent to the **ellipticity** of $C + \frac{i}{2} T$ on the double characteristics of $L$. For those values we get the best possible estimates, i.e., the estimates (1.2), yielding solutions $u \in L^2$ of the equation $Lu = f \in H^{-1}$. When the ellipticity of $C + \frac{i}{2} T$ fails, i.e., when $\gamma(0) = -\frac{1}{2}$, solvability still holds but we have only obtained solutions in $H^{-1}$. Considering that $L$ is a second-order differential operator and comparing to the elliptic case, one could say that there is local solvability with loss of one derivative when $\gamma(0) \neq -\frac{1}{2}$ and loss of two derivatives when $\gamma(0) = -\frac{1}{2}$.

2.2. **The pseudodifferential case and solvability in $H^s$**. It remains to prove the local solvability of $Lu = f$ in the sense of the Sobolev space $H^s$ for an arbitrary real number $s$. We shall do this through the extension of Theorem 1 to classical pseudodifferential operators of principal type in $\Omega \subset \mathbb{R}^n$ ($n \geq 2$ arbitrary). Inspection of the proof of Theorem 1 shows that the extension is valid:

**Theorem 2.** If $P_1$ and $P_2$ are two first-order classical pseudodifferential operators of principal type in $\Omega \subset \mathbb{R}^n$, with real principal symbols and such that

$$P_1^2 + P_2^2 + \{P_1, P_2\}^2$$
is elliptic, then \( L = P_1P_2 + \sqrt{-1}Q \) is locally solvable whatever the first-order classical pseudodifferential operator \( Q \) in \( \Omega \) having a real principal symbol.

More precisely, given any point \( x^0 \in \Omega \) and any number \( \varepsilon > 0 \) there is an open neighborhood \( U_{x^0, \varepsilon} \subset \Omega \) of \( x^0 \) with the following property: there is a bounded linear operator \( G_{x^0, \varepsilon} : H^{-1}(U_{x^0, \varepsilon}) \to H^{-1}(U_{x^0, \varepsilon}) \) with norm \( \leq \varepsilon \) such that

\[
\begin{align*}
L G_{x^0, \varepsilon} f = f & \text{ in } U_{x^0, \varepsilon} \text{ for every } f \in H^{-1}(U_{x^0, \varepsilon}).
\end{align*}
\]

If moreover the subprincipal symbol of \( L \) does not vanish on the common characteristics of \( P_1 \) and \( P_2 \) then \( G_{x^0, \varepsilon} \) can be taken to be a continuous linear operator \( H^{-1}(U_{x^0, \varepsilon}) \to L^2(U_{x^0, \varepsilon}) \) with norm \( \leq \varepsilon \).

The proof duplicates that of Theorem 1 replacing \( A \) by \( \sqrt{-1}P_1 \), \( B \) by \( \sqrt{-1}P_2 \) and \( C \) by \( \sqrt{-1}Q \). Its “pivot” is the analogue of Estimate (1.10), valid when the subprincipal symbol of \( L \) does not vanish on the double characteristics of \( L \):

\[
\|u\|_1 \leq \delta K_2 \|L^* P_1^* u + D^* u\|, \; u \in C_0^\infty(\mathfrak{B}_\delta),
\]

where now \( D \) is an arbitrary first-order pseudodifferential operator with real principal symbol (the positive constant \( K_2 \) depends on \( L \) and \( D \) but not on \( \delta \), nor, of course, on \( u \)).

Now let \( E \) be a properly supported, classical, elliptic, self-adjoint pseudodifferential operator in \( \Omega \) of order \( s \in \mathbb{R} \). If we write \( L = L_2 + L_1 \) modulo pseudodifferential operators of order zero and use the notation \( \sigma(\cdot) \) for the principal symbol, the subprincipal symbol of \( E^{-1}LE = L + E^{-1}[L, E] \) is equal to

\[
\sigma(L_1) - \frac{1}{2i}(\nabla_x \cdot \nabla_\xi) \sigma(L) - \sigma(E)^{-1}\{\sigma(L), \sigma(E)\}.
\]

But \( \{\sigma(L), \sigma(E)\} \equiv 0 \) on the double characteristics of \( L \). This allows us to apply (2.1) with \( E^{-1}LE \) in the place of \( L \):

\[
\|u\|_1 \leq \delta K_2 \|EL^* E^{-1}P_1^* u + D^* u\|, \; u \in C_0^\infty(\mathfrak{B}_\delta).
\]

Let \( \chi \in C_0^\infty(\mathfrak{B}_\delta), \; \chi \equiv 1 \) in \( \mathfrak{B}_{\delta/2} \). Take \( u \in C_0^\infty(\mathfrak{B}_{\delta/2}) \) and apply (2.2) with \( \chi Eu \) in the place of \( u \). We obtain

\[
\|u\|_{s+1} \leq \|\chi Eu\|_1 + \|u\|_s \leq
\]

\[
\delta K_3 \|E(L^* P_1^* u + D^* u)\| + \delta K_3 \|EL^* E^{-1}[P_1^*, E] u\| + \delta K_3 \|[D^*, E] u\| + \delta K_3 \|(EL^* E^{-1}P_1^* + D^*)[E, \chi] u\|.
\]

On the one hand the order of \([D^*, E]\) is \( \leq s \) while the operator \((L^* P_1^* + D^*)[E, \chi]\) acting on compactly supported distributions in \( \mathfrak{B}_{\delta/2} \) is regularizing. On the other hand the order of \( L^* E^{-1}[P_1^*, E] \) is \( \leq 1 \). By taking \( \delta > 0 \) suitably small we conclude that

\[
\|u\|_{s+1} \leq \delta K_4 \|L^* P_1^* u + D^* u\|_s + \|u\|_s, \; , u \in C_0^\infty(\mathfrak{B}_{\delta/2})
\]

From (2.3) we proceed as we did in the proof of Theorem 1 starting from Corollary 2. When the subprincipal symbol of \( L \) vanishes on the double characteristics of \( L \) we obtain an inequality of the following kind (after a redefinition
of $s$ and $\delta$):

\[ \|u\|_s \leq \delta K_5 \|L^* u\|_s + \|u\|_{s-1}, \quad u \in C_\infty^\infty(\mathcal{B}_\delta). \]

Such an inequality implies (by standard arguments) the solvability of $Lu = f$ in $H^s(\mathcal{B}_\delta)$ after further decreasing of $\delta$. We can state

**Corollary 3.** Let $L$ be as in Theorem 2 and let $s \in \mathbb{R}$ be arbitrarily given. Given any point $x^0 \in \Omega$ and any real number $s$ there is an open neighborhood $U_{x^0,s} \subset \Omega$ of $x^0$ with the following property: there is a bounded linear operator $G_{x^0,s} : H^s(U_{x^0,s}) \to H^s(U_{x^0,s})$ such that $LG_{x^0,s} f = f$ in $U_{x^0,s}$ for every $f \in H^s(U_{x^0,s})$. If moreover the subprincipal symbol of $L$ does not vanish on the common characteristics of $P_1$ and $P_2$ then $G_{x^0,\varepsilon}$ can be taken to be a continuous linear operator $H^s(U_{x^0,\varepsilon}) \to H^{s+1}(U_{x^0,\varepsilon})$.

### 2.3. Open questions.
Questions related to Theorems 1 and 2 that come to mind are the following.

1. Is there a convenient symbolic calculus specifically adapted to the construction of a parametrix for operators $L$ like those in Theorem 2 (cf. [Ha1]?)?
2. What is the geometry of the null bicharacteristics of the symbols $p_1, p_2$ or of the “double” half bicharacteristics of $p_1 p_2$ defined by the sign of $\{p_1, p_2\}$ ensuring semiglobal solvability of the operator $L$ in Theorem 2?
3. What are the generalizations of Theorem 1 to real, smooth vector fields $X_1, \ldots, X_r$ with $r \geq 3$ satisfying Hörmander’s condition? This is of course related to the local solvability of left-invariant differential operators on a nilpotent Lie group (see e.g. [MüR]).

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### References


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