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THE ASYMPTOTIC BEHAVIOUR OF HEEGAARD GENUS

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1. Introduction

Heegaard splittings have recently been shown to be related to a number of important conjectures in 3-manifold theory: the virtually Haken conjecture, the positive virtual b_1 conjecture and the virtually fibred conjecture [3]. Of particular importance is the rate at which the Heegaard genus of finite-sheeted covering spaces grows as a function of their degree. This was encoded in the following definitions.

Let M be a compact orientable 3-manifold. Let $\chi_-^h(M)$ be the negative of the maximal Euler characteristic of a Heegaard surface for M. Let $\chi_-^{sh}(M)$ be the negative of the maximal Euler characteristic of a strongly irreducible Heegaard surface for M, or infinity if such a surface does not exist. Define the *infimal Heegaard gradient* of M to be

$$\inf_i \{ \frac{\chi_-^h(M_i)}{d_i} : M_i \text{ is a degree } d_i \text{ cover of } M \}.$$

The *infimal strong Heegaard gradient* of M is

$$\liminf_i \{ \frac{\chi_-^{sh}(M_i)}{d_i} : M_i \text{ is a degree } d_i \text{ cover of } M \}.$$

The following conjectures were put forward in [3]. According to Theorem 1.7 of [3], either of these conjectures, together with a conjecture of Lubotzky and Sarnak [4] about the failure of Property (τ) for hyperbolic 3-manifolds, would imply the virtually Haken conjecture for hyperbolic 3-manifolds.

Heegaard gradient conjecture. *A compact orientable hyperbolic 3-manifold has zero infimal Heegaard gradient if and only if it virtually fibres over the circle.*

Strong Heegaard gradient conjecture. *Any closed orientable hyperbolic 3-manifold has positive infimal strong Heegaard gradient.*

Some evidence for these conjectures was presented in [3]. More precisely, suitably phrased versions of these conjectures were shown to be true when one restricts attention to cyclic covers dual to a non-trivial element of $H_2(M, \partial M)$, to reducible manifolds and (in the case of the strong Heegaard gradient conjecture) to congruence covers of arithmetic hyperbolic 3-manifolds.

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A less quantitative version of the conjectures is simply that $\chi_-^{sh}(M_i)$ cannot grow too slowly as a function of d_i, and that if $\chi_-^{h}(M_i)$ does grow sufficiently slowly, then M is virtually fibred. These expectations are confirmed in the following result, which is the main theorem of this paper.

Theorem 1. *Let M be a closed orientable 3-manifold that admits a negatively curved Riemannian metric. Let $\{M_i \to M\}$ be a collection of finite regular covers with degree d_i.*

1. *If $\chi_-^{h}(M_i)/\sqrt{d_i} \to 0$, then $b_1(M_i) > 0$ for all sufficiently large i.*
2. *$\chi_-^{sh}(M_i)/\sqrt{d_i}$ is bounded away from zero.*
3. *If $\chi_-^{h}(M_i)/\sqrt[4]{d_i} \to 0$, then M_i fibres over the circle for all sufficiently large i.*

A slightly weaker form of Theorem 1(1) appeared in [3] as Corollary 1.4, with essentially the same proof. It is included here in order to emphasise its connection to the other two results.

The following corollary of Theorem 1(3) gives a necessary and sufficient condition for M to be virtually fibred in terms of the Heegaard genus of its finite covers. We say that a collection $\{M_i \to M\}$ of finite covers has *bounded irregularity* if the normalisers of $\pi_1 M_i$ in $\pi_1 M$ have bounded index in $\pi_1 M$.

Corollary 2. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric, and let $\{M_i \to M\}$ be its finite-sheeted covers with degree d_i. Then the following are equivalent:*

1. *M_i is fibred for infinitely many i;*
2. *in some subsequence with bounded irregularity, $\chi_-^{h}(M_i)$ is bounded;*
3. *in some subsequence with bounded irregularity, $\chi_-^{h}(M_i)/\sqrt[4]{d_i} \to 0$.*

Proof. (1) $\Rightarrow$ (2). If some finite-sheeted cover $\tilde{M}$ is fibred, then so is any finite cyclic cover M_i of $\tilde{M}$ dual to the fibre. The normaliser $N(\pi_1 M_i)$ of $\pi_1 M_i$ in $\pi_1 M$ contains $\pi_1 \tilde{M}$, so $[\pi_1 M : N(\pi_1 M_i)]$ is bounded and hence these covers have bounded irregularity. Also, $\chi_-^{h}(M_i)$ is bounded by twice the modulus of the Euler characteristic of the fibre, plus four.

(2) $\Rightarrow$ (3). This is trivial, since d_i must tend to infinity.

(3) $\Rightarrow$ (1). Since $[\pi_1 M : N(\pi_1 M_i)]$ is bounded in this subcollection, we may pass to a further subsequence where $N(\pi_1 M_i)$ is a fixed subgroup of $\pi_1 M$. Let $\tilde{M}$ be the finite-sheeted cover of M corresponding to this subgroup. Then, the covers $\{M_i \to M\}$ in this subsequence give a collection $\{M_i \to \tilde{M}\}$ of finite regular covers such that $\chi_-^{h}(M_i)/[\pi_1 \tilde{M} : \pi_1 M_i]^{1/4} \to 0$. By Theorem 1(3), M_i is fibred for all sufficiently large i. □

2. Background material

Generalised Heegaard splittings

A Heegaard splitting of a closed orientable 3-manifold can be viewed as arising from a handle structure. If one builds the manifold by starting with a single 0-handle, then attaching some 1-handles, then some 2-handles and then a 3-handle, the manifold obtained after attaching the 0- and 1-handles is handlebody, as is the closure of its complement. Thus, the boundary of this submanifold is a Heegaard surface. Generalised Heegaard splittings arise from more general handle structures: one starts with some 0-handles, then adds some 1-handles, then some 2-handles, then 1-handles, and so on, in an alternating fashion, ending with some 3-handles. One then considers the manifold embedded in M consisting of the 0-handles and the first j batches of 1- and 2-handles. Let F_j be the boundary of this manifold, but discarding any 2-sphere components that bound 0- or 3-handles. After a small isotopy, so that these surfaces are all disjoint, they divide M into compression bodies. In fact, the surfaces $\{F_j : j \text{ odd}\}$ form Heegaard surfaces for the manifold $M - \bigcup\{F_j : j \text{ even}\}$. We term the surfaces F_j *even* or *odd*, depending on the parity of j.

More details about generalised Heegaard splittings can be found in [7] and [6]. The following theorem summarises some of the results from [7].

Theorem 3. *From any minimal genus Heegaard surface F for a closed orientable irreducible 3-manifold M, other than S^3, one can construct a generalised Heegaard splitting $\{F_1, \dots, F_n\}$ in M with the following properties:*

1. *F_j is incompressible and has no 2-sphere components, for each even j;*
2. *F_j is strongly irreducible for each odd j;*
3. *F_j and F_{j+1} are not parallel for any j;*
4. *$|\chi(F_j)| \leq |\chi(F)|$ for each j;*
5. *$|\chi(F)| = \sum(-1)^j \chi(F_j)$.*

Corollary 4. *Let M, F and $\{F_1, \dots, F_n\}$ be as in Theorem 3. Suppose that, in addition, M is not a lens space. Let $\overline{F}$ be the surface obtained from $\bigcup_j F_j$ by replacing any components that are parallel by a single component. Then*

1. *$|\chi(\overline{F})| \leq |\chi(\bigcup_j F_j)| < |\chi(F)|^2$;*
2. *$\overline{F}$ has at most $\frac{3}{2}|\chi(F)|$ components.*

Proof. Note first that no component of $\bigcup_j F_j$ is a 2-sphere. When j is even, this is (1) of Theorem 3. The same is true when j is odd, since the odd surfaces form Heegaard surfaces for the complement of the even surfaces, and M is not S^3. Hence, none of the compression bodies H in the complement of $\bigcup_j F_j$ is a 3-ball.

We claim also that no H is a solid torus. For if it were, consider the compression body to which it is adjacent. If this were a product, some even surface would be compressible, contradicting (1) of Theorem 3. However, if it was not a product, then it is a solid torus, possibly with punctures. If it has no punctures, then M is a lens space, contrary to assumption. If it does, then some component of an even surface would be a 2-sphere, again contradicting (1). This proves the claim.

We expand (5) of Theorem 3 as follows:

$$|\chi(F)| = \frac{-\chi(F_1)}{2} + \frac{\chi(F_2) - \chi(F_1)}{2} + \frac{\chi(F_2) - \chi(F_3)}{2} + \cdots + \frac{-\chi(F_n)}{2}. \qquad (*)$$

For any compression body H, other than a 3-ball, with negative boundary $\partial_- H$ and positive boundary $\partial_+ H$, $\chi(\partial_- H) - \chi(\partial_+ H)$ is even and non-negative. It is zero if and only if H is a product or a solid torus. Since F_j and F_{j+1} are not parallel for any j, each term in $(*)$ is therefore at least one. So, $n+1$, the number of terms on the right-hand side of $(*)$, is at most $|\chi(F)|$. Hence,

$$|\chi(\bigcup_j F_j)| = \sum_j |\chi(F_j)| \leq n|\chi(F)| < |\chi(F)|^2.$$

The inequality $|\chi(\overline{F})| \leq |\chi(\bigcup_j F_j)|$ simply follows from the fact that we discard some components of $\bigcup_j F_j$ to form $\overline{F}$. This proves (1).

Now it is trivial to check that, for any compression body H, other than a 3-ball, solid torus or product, $|\partial H| \leq \frac{3}{2}(\chi(\partial_- H) - \chi(\partial_+ H))$. The number of components of $\overline{F}$ is half the sum, over all complementary regions H of $\bigcup_j F_j$ that are not products, of $|\partial H|$. This is at most $\frac{3}{4}(\chi(\partial_- H) - \chi(\partial_+ H))$. But the sum, over all complementary regions H of $\bigcup_j F_j$, of $\frac{1}{2}(\chi(\partial_- H) - \chi(\partial_+ H))$ is the right-hand side of $(*)$. Thus, we deduce that the number of components of $\overline{F}$ is at most $\frac{3}{2}|\chi(F)|$, proving (2). □

Realisation as minimal surfaces

One advantage of using generalised Heegaard splittings satisfying (1) and (2) of Theorem 3 is that minimal surfaces then play a rôle in the theory. The following theorem of Freedman, Hass and Scott [2] applies to the even surfaces.

Theorem 5. *Let S be an orientable embedded incompressible surface in a closed orientable irreducible Riemannian 3-manifold. Suppose that no two components of S are parallel, and that no component is a 2-sphere. Then there is an ambient isotopy of S so that afterwards each component is either a least area, minimal surface or the boundary of a regular neighbourhood of an embedded, least area, minimal non-orientable surface.*

We will apply the above result to the incompressible components of $\overline{F}$. If we cut M along these components, the remaining components form strongly irreducible Heegaard surfaces for the complementary regions. A theorem of Pitts and Rubinstein [5] now applies.

Theorem 6. *Let S_1 be a (possibly empty) embedded stable minimal surface in a closed orientable irreducible Riemannian 3-manifold M with a bumpy metric. Let S_2 be a strongly irreducible Heegaard surface for a complementary region of S_1. Then there is an ambient isotopy, leaving S_1 fixed, taking S_2 to a minimal surface, or to the boundary of a regular neighbourhood of a minimal embedded non-orientable surface, with a tube attached that is vertical in the I-bundle structure on this neighbourhood.*

Bumpy metrics were defined by White in [8]. After a small perturbation, any Riemannian metric can be made bumpy. Then we may ambient isotope $\overline{F}$ so that each component is as described in Theorems 5 and 6.

We will need some parts of the proof of Theorem 6, and not just its statement. Let X be the component of $M - S_1$ containing S_2. Then, as a Heegaard surface, S_2 determines a sweepout of X. In any sweepout, there is a surface of maximum area, although it need not be unique. Let a be the infimum, over all sweepouts in this equivalence class, of this maximum area. Then Pitts and Rubinstein showed that there is a sequence of sweepouts, whose maximal area surfaces tend to an embedded minimal surface, and that the area of these surfaces tends to a. This minimal surface, or its orientable double cover if it is non-orientable, is isotopic to S_2 or to a surface obtained by compressing S_2.

Now, when M is negatively curved, one may use Gauss-Bonnet to bound the area of this surface. Suppose that $\kappa < 0$ is the supremum of the sectional curvatures of M. Then, as the surface is minimal, its sectional curvature is at most κ. Hence, by Gauss-Bonnet, its area is at most $2\pi|\chi(S_2)|/|\kappa|$. Thus, we have the following result.

Addendum 7. *Let S_1, S_2 and M be as in Theorem 6. Suppose that the sectional curvature of M is at most $\kappa < 0$. Then, for each $\epsilon > 0$, there is a sweepout of the component of $M - S_1$ containing S_2, equivalent to the sweepout determined by S_2, so that each surface in this sweepout has area at most $(2\pi|\chi(S_2)|/|\kappa|) + \epsilon$.*

One has a good deal of geometric control over minimal surfaces when M is negatively curved. As observed above, their area is bounded in terms of their Euler characteristic and the supremal sectional curvature of M. In fact, by ruling out the existence of long thin tubes in the surface, one has the following.

Theorem 8. *There is a function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following property. Let M be a Riemannian 3-manifold, whose injectivity radius is at least $\epsilon/2 > 0$, and whose sectional curvature is at most $\kappa < 0$. Let S be a closed minimal surface in M. Then there is a collection of at most $f(\kappa, \epsilon)|\chi(S)|$ points in S, such that the balls of radius $f(\kappa, \epsilon)$ about these points cover S. (Here, distance is measured using the path metric on S.)*

This is proved in Proposition 6.1 of [3]. More precisely, formulas (1) and (2) there give the result.

The Cheeger constant of manifolds and graphs

The *Cheeger constant* of a compact Riemannian manifold M is defined to be

$$h(M) = \inf \left\{ \frac{\text{Area}(S)}{\min\{\text{Volume}(M_1), \text{Volume}(M_2)\}} \right\},$$

where S ranges over all embedded codimension one submanifolds that divide M into M_1 and M_2.

A central theme of [3] is that the Cheeger constant of a 3-manifold and its Heegaard splittings are intimately related. One example of this phenomenon is the following result.

Theorem 9. *Let M be a closed Riemannian 3-manifold. Let $\kappa < 0$ be the supremum of its sectional curvatures. Then*

$$h(M) \leq \frac{4\pi\, \chi_-^h(M)}{|\kappa| \mathit{Volume}\ (M)}.$$

This is essentially Theorem 4.1 of [3]. However, there, $\chi_-^h(M)$ is replaced by $c_+(M)$, which is an invariant defined in terms of the generalised Heegaard splittings of M. But the above inequality follows from an identical argument. We briefly summarise the proof.

From a minimal genus Heegaard splitting of M, construct a generalised Heegaard splitting $\{F_1, \dots, F_n\}$ satisfying (1) to (5) of Theorem 3. Let $\overline{F}$ be the surface obtained from $\bigcup_j F_j$ by discarding multiple copies of parallel components. Apply the isotopy of Theorem 5 to the incompressible components of $\overline{F}$. Each complementary region corresponds to a component of the complement of the even surfaces, and therefore contains a component of some odd surface F_j. Label this region with the integer j, and let M_j be the union of the regions labelled j. There is some odd j such that the volumes of $M_1 \cup \cdots \cup M_{j-2}$ and $M_{j+2} \cup \cdots \cup M_n$ are each at most half the volume of M. Now, $F_j \cap M_j$ forms a strongly irreducible Heegaard surface for M_j. Applying Addendum 7, we find for each $\epsilon > 0$, a sweepout of M_j, equivalent to that determined by F_j, by surfaces with area at most $(2\pi|\chi(F_j)|/|\kappa|) + \epsilon$. But, $|\chi(F_j)| \leq \chi_-^h(M)$, by (4) of Theorem 3. Some surface in this sweepout divides M into two parts of equal volume. So, as ϵ was arbitrary,

$$h(M) \leq \frac{4\pi\, \chi_-^h(M)}{|\kappa| \mathrm{Volume}\ (M)}.$$

In this paper, we will consider the Cheeger constants of regular finite-sheeted covering spaces M_i of M. Here, M_i is given the Riemannian metric lifted from M. It is possible to estimate $h(M_i)$ in terms of graph-theoretic data, as follows.

By analogy with the Cheeger constant for a Riemannian manifold, one can define the *Cheeger constant* $h(X)$ of a finite graph X. If A is a subset of the vertex set $V(X)$, ∂A denotes those edges with precisely one endpoint in A. Then $h(X)$ is defined to be

$$\inf \left\{ \frac{|\partial A|}{|A|} : A \subset V(X) \text{ and } 0 < |A| \leq |V(X)|/2 \right\}.$$

Proposition 10. *Let M be a compact Riemannian manifold. Let $\mathcal{X}$ be a finite set of generators for $\pi_1 M$. Then there is a constant $k_1 \geq 1$ with the following property. If X_i is the Cayley graph of $\pi_1 M/\pi_1 M_i$ with respect to the generators $\mathcal{X}$, then*

$$k_1^{-1}\, h(X_i) \leq h(M_i) \leq k_1\, h(X_i).$$

This is essentially contained in [1], but we outline a proof. Lemma 2.3 of [3] states that, if $\mathcal{X}$ and $\mathcal{X}'$ are two finite sets of generators for $\pi_1 M$, then there is a constant $k \geq 1$ with the following property. If X_i and X_i' are the Cayley graphs of $\pi_1 M/\pi_1 M_i$ with respect to $\mathcal{X}$ and $\mathcal{X}'$, then

$$k^{-1}\, h(X_i) \leq h(X_i') \leq k\, h(X_i).$$

Thus, for the purposes of proving Proposition 10, we are free to choose $\mathcal{X}$. We do this as follows. We pick a connected fundamental domain in the universal cover of M. The translates of this domain to which it is adjacent correspond to a finite set $\mathcal{X}$ of generators for $\pi_1 M$. There is an induced fundamental domain in any finite regular cover M_i of M. Its translates are in one-one correspondence with the group $\pi_1 M/\pi_1 M_i$. Two translates are adjacent if and only if one is obtained from the other by right-multiplication by an element in $\mathcal{X}$. Thus, the Cayley graph X_i should be viewed as a coarse approximation to M_i. Any subset A of $V(X_i)$, as in the definition of $h(X_i)$, therefore determines a decomposition of M_i. After a further modification, we may assume that this is along a codimension one submanifold. The existence of a constant k_1 such that $h(M_i) \leq k_1\, h(X_i)$ is then clear. The other inequality is more difficult to establish. One needs to control the geometry of a codimension one submanifold S in M_i that is arbitrarily close to realising the Cheeger constant of M_i. This is achieved in the proof of Lemma 2 of [1].

Constructing non-trivial cocycles

Some new machinery has been developed in [3] that gives necessary and sufficient conditions on a finitely presented group to have finite index subgroups with infinite abelianisation. We describe some of the ideas behind this now.

Let C be a finite cell complex with a single 0-cell and in which every 2-cell is a triangle. Let G be its fundamental group, and let $\mathcal{X}$ be the generators arising from the 1-cells. Associated with any finite index normal subgroup H_i of G, there is a finite-sheeted covering space C_i of C. Its 1-skeleton X_i is the Cayley graph of G/H_i with respect to $\mathcal{X}$. The following theorem is an expanded form of Lemma 2.4 of [3] and has exactly the same proof. It will play a key rôle in this paper.

Theorem 11. *Suppose that $h(X_i) < \sqrt{2/(3|V(X_i)|)}$. Let A be any non-empty subset of $V(X_i)$ such that $|\partial A|/|A| = h(X_i)$ and $|A| \leq |V(X_i)|/2$. Then there is a 1-cocycle c on C_i that is not a coboundary. Its support is a subset of the edges of ∂A, and it takes values in $\{-1, 0, 1\}$. As a consequence, H_i has infinite abelianisation.*

3. The proof of the main theorem

We start with a closed orientable 3-manifold M admitting a negatively curved Riemannian metric. After a small perturbation, we may assume that the metric is bumpy. Let $\kappa < 0$ be the supremum of its sectional curvatures. Pick a 1-vertex triangulation T of M. The edges of T, when oriented in some way, form a set $\mathcal{X}$ of generators for $\pi_1(M)$. Let K be the 2-skeleton of the complex dual to T.

We will consider a collection $\{M_i \to M\}$ of finite regular covers of M, having the properties of Theorem 1. In particular, we will assume (at least) that $\chi_-^h(M_i)/\sqrt{d_i} \to 0$. (Note that this is justified when proving Theorem 1(2), by passing to a subsequence, and using the fact that $\chi_-^h(M_i) \leq \chi_-^{sh}(M_i)$.) The triangulation T and 2-complex K lift to T_i and K_i, say, in M_i. The 1-skeleton of T_i forms the Cayley graph X_i of $G_i = \pi_1 M/\pi_1 M_i$ with respect to $\mathcal{X}$.

According to Theorem 9,

$$h(M_i) \leq \frac{4\pi}{|\kappa|}\frac{\chi_-^h(M_i)}{\text{Volume}(M_i)} = \frac{4\pi}{|\kappa|\text{Volume}(M)}\frac{\chi_-^h(M_i)}{d_i},$$

By Proposition 10, there is a constant $k_1 \geq 1$ independent of i such that $h(X_i) \leq k_1 h(M_i)$. Setting

$$k_2 = \frac{4\pi k_1}{|\kappa|\text{Volume}(M)},$$

we deduce that

$$h(X_i) \leq k_2 \frac{\chi_-^h(M_i)}{d_i}.$$

Let $V(X_i)$ be the vertex set of X_i. Let A be a non-empty subset of $V(X_i)$ such that $|\partial A|/|A| = h(X_i)$ and $|A| \leq |V(X_i)|/2 = d_i/2$. By Theorem 11, when $h(X_i) < \sqrt{2/(3d_i)}$, T_i admits a 1-cocycle c that is not a coboundary. Since $h(X_i) \leq k_2\chi_-^h(M_i)/d_i$, and we are assuming (at least) that $\chi_-^h(M_i)/\sqrt{d_i} \to 0$, then such a cocycle exists for all sufficiently large i. This establishes (1) of the Theorem 1.

Theorem 11 states that c takes values in $\{-1, 0, 1\}$, and its support is a subset of the edges of ∂A. Dual to this cocycle is a transversely oriented normal surface S in T_i which is homologically non-trivial. Remove any 2-sphere components from S. This is still homologically non-trivial, since all 2-spheres in M_i are inessential, as M_i is negatively curved. The intersection of S with the 2-skeleton of T_i is a graph in S whose complementary regions are triangles and squares. Let $V(S)$ and $E(S)$ be its vertices and edges. Its vertices are in one-one correspondence with the edges of T_i in the support of c. So,

$$|V(S)| \leq |\partial A| = |A|h(X_i) \leq d_i h(X_i)/2 \leq k_2\chi_-^h(M_i)/2.$$

The valence of each vertex is at most the maximal valence of an edge in T, k_3, say. So,

$$|\chi(S)| < |E(S)| \leq |V(S)|k_3/2 \leq k_2k_3\chi_-^h(M_i)/4.$$

Setting $k_4 = k_2k_3/4$, we have deduced the existence of a homologically non-trivial, transversely oriented, properly embedded surface S with $|\chi(S)| \leq k_4\chi_-^h(M_i)$ and with no 2-sphere components. By compressing S and removing components if necessary, we may assume that S is also incompressible and connected. Thus, we have proved the following result.

Theorem 12. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric. Then there is a constant $k_4 > 0$ with the following property. Let $\{M_i \to M\}$ be a collection of finite regular covers, with degree d_i. If $\chi_-^h(M_i)/\sqrt{d_i} \to 0$, then, for all sufficiently large i, M_i contains an embedded, connected, oriented, incompressible, homologically non-trivial surface S such that $|\chi(S)| \leq k_4 \chi_-^h(M_i)$.*

By a theorem of Freedman, Hass and Scott [2] (Theorem 5 in this paper), there is an ambient isotopy taking S to a minimal surface. We therefore investigate the coarse geometry of minimal surfaces in M_i.

Set $\epsilon/2$ to be the injectivity radius of M. Let $f(\kappa, \epsilon)$ be the function from Theorem 8. Let $\tilde{K}$ be the lift of the 2-complex K to the universal cover of M. Let k_5 be the maximum number of complementary regions of $\tilde{K}$ that lie within a distance $f(\kappa, \epsilon)$ of any point, and let $k_6 = f(\kappa, \epsilon) k_5$.

Lemma 13. *Let S be a minimal surface in M_i. Then S intersects at most $k_6|\chi(S)|$ complementary regions of K_i. Hence, running through any such region, there are at most $k_6|\chi(S)|$ translates of S under the covering group action of G_i.*

Proof. By Theorem 8, the number of balls of radius $f(\kappa, \epsilon)$ required to cover S is at most $f(\kappa, \epsilon)|\chi(S)|$. The centre of each of these balls has at most k_5 complementary regions of K_i within a distance $f(\kappa, \epsilon)$. So, S intersects at most $k_6|\chi(S)|$ complementary regions of K_i. Each such region corresponds to an element of G_i. To prove the second half of the lemma, we may concentrate on the region corresponding to the identity. Then a translate gS runs through here, for some g in G_i, if and only if S runs through the region corresponding to g^{-1}. Thus, there can be at most $k_6|\chi(S)|$ such g. □

Proof of Theorem 1(2). Let F be a strongly irreducible Heegaard surface in M_i with $|\chi(F)| = \chi_-^{sh}(M_i)$. By Theorem 6, there is an ambient isotopy taking it either to a minimal surface or to the double cover of a minimal non-orientable surface, with a small tube attached. So, by Lemma 13, F intersects at most $k_6 \chi_-^{sh}(M_i)$ complementary regions of K_i. Hence, by Lemma 13, the number of copies of S that F intersects is at most $(k_6 \chi_-^{sh}(M_i))(k_6 k_4 \chi_-^h(M_i))$. This is less than d_i if $\chi_-^{sh}(M_i)/\sqrt{d_i}$ is sufficiently small. So there is a translate of S which misses F. It then lies in a complementary handlebody of F. But this is impossible, since S is incompressible. So, $\chi_-^{sh}(M_i)/\sqrt{d_i}$ is bounded away from zero. □

Proof of Theorem 1(3). For ease of notation, let $x = \chi_-^h(M_i)$. Let $\{F_1, \dots, F_n\}$ be a generalised Heegaard splitting for M_i, satisfying (1) - (5) of Theorem 3, obtained from a minimal genus Heegaard splitting. Replace any components of $F_1 \cup \dots \cup F_n$ that are parallel by a single component, and let $\overline{F}$ be the resulting surface. Isotope $\overline{F}$ so that each component is as in Theorems 5 or 6. Corollary 4 states that $|\chi(\overline{F})| < x^2$. By Lemma 13, the number of complementary regions of K_i that can intersect $\overline{F}$ is at most $k_6 x^2$. Let D be the corresponding subset of G_i.

Similarly, let C be the subset of G_i that corresponds to those complementary regions of K_i which intersect S. By Lemma 13, $|C| \leq k_6|\chi(S)| \leq k_6 k_4 x$.

We claim that, when i is sufficiently large, there are at least $9x/2$ disjoint translates of S under G_i that are also disjoint from $\overline{F}$. Let $m = 9x/2$. If the claim is not true, then for any m-tuple $(g_1 S, \dots, g_m S)$ of copies of S (where $g_j \in G_i$ for each j), either at least two intersect or one copy intersects $\overline{F}$. In the former case, $g_j c_1 = g_k c_2$, for some c_1 and c_2 in C, and for $1 \leq j < k \leq m$. Hence, $g_k^{-1} g_j \in CC^{-1}$. In the latter case, $g_j c_1 = d$ for some c_1 in C and d in D, and so $g_j \in DC^{-1}$. Thus, the sets $q_{jk}^{-1}(CC^{-1})$ and $p_j^{-1}(DC^{-1})$ cover $(G_i)^m$, where q_{jk} and p_j are the maps

$$\begin{aligned} q_{jk} \colon (G_i)^m &\to G_i \\ (g_1, \dots, g_m) &\mapsto g_k^{-1} g_j \end{aligned}$$

$$\begin{aligned} p_j \colon (G_i)^m &\to G_i \\ (g_1, \dots, g_m) &\mapsto g_j, \end{aligned}$$

for $1 \leq j < k \leq m$. The former sets $q_{jk}^{-1}(CC^{-1})$ each have size $|G_i|^{m-1}|CC^{-1}|$, and the latter sets $p_j^{-1}(DC^{-1})$ have size $|G_i|^{m-1}|DC^{-1}|$. So,

$$|G_i|^m \leq \binom{m}{2} |G_i|^{m-1}|C|^2 + m|G_i|^{m-1}|C||D|.$$

This implies that

$$d_i = |G_i| \leq \binom{m}{2} (k_6 k_4 x)^2 + m(k_6 k_4 x)(k_6 x^2).$$

The right-hand side has order x^4 as $i \to \infty$. However, $x/\sqrt[4]{d_i} \to 0$, which is a contradiction, proving the claim.

Consider these $9x/2$ copies of S. Each lies in the complement of $\overline{F}$, which is a collection of compression bodies. Since S is incompressible and connected, each copy of S must be parallel to a component of $\overline{F}$. By Corollary 4(2), $\overline{F}$ has at most $3x/2$ components. So, at least 3 copies of S are parallel, and at least 2 of these are coherently oriented. The proof is now completed by the following lemma. □

Lemma 14. *Let S be a connected, embedded, oriented, incompressible, non-separating surface in a closed orientable 3-manifold M_i. Suppose that the image of S under some finite order orientation-preserving homeomorphism h of M_i is disjoint from S, parallel to it and coherently oriented. Then M_i fibres over the circle with fibre S.*

Proof. Let Y be the manifold lying between S and $h(S)$. It is copy of $S \times I$, with S and $h(S)$ corresponding to $S \times \{0\}$ and $S \times \{1\}$. Take a countable collection

$\{Y_n : n \in \mathbb{Z}\}$ of copies of this manifold. Glue $S \times \{1\}$ in Y_n to $S \times \{0\}$ in Y_{n+1}, via h^{-1}. The resulting space Y_∞ is a copy of $S \times \mathbb{R}$. Let H be the automorphism of this space taking Y_n to Y_{n+1} for each n, via the 'identity'. Let $p\colon Y_0 \to Y$ be the identification homeomorphism. Extend this to a map $p\colon Y_\infty \to M_i$ by defining $p|Y_n$ to be $h^n p H^{-n}$.

We claim that this is a covering map. It may expressed as a composition $Y_\infty \to Y_\infty/\langle H^N \rangle \to M_i$, where N is the order of h. The first of these maps is obviously a covering map. The second is also, since it is a local homeomorphism and $Y_\infty/\langle H^N \rangle$ is compact. Hence, p is a covering map.

By construction, $h^n(S)$ lifts homeomorphically to $Y_{n-1} \cap Y_n$, for each n. Hence, the inverse image of S in Y_∞ includes all translates of $Y_{-1} \cap Y_0$ under $\langle H^N \rangle$. These translates divide Y_∞ into copies of $S \times I$. Since $p^{-1}(S)$ is incompressible and any closed embedded incompressible surface in $S \times I$ is horizontal, we deduce that $p^{-1}(S)$ divides Y_∞ into a collection of copies of $S \times I$. The restriction of p to one of these components Z is a covering map to a component of $M_i - S$. But $M_i - S$ is connected, as S is connected and non-separating. So, p maps Z surjectively onto $M_i - S$. By examining this map near S, we see that it is degree one and hence a homeomorphism. Therefore, M_i is obtained from a copy of $S \times I$ by gluing its boundary components homeomorphically. So, M_i fibres over the circle with fibre S. □

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