ARITHMETIC PROPERTIES OF PERIODIC MAPS

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Abstract. Let $\psi_1, \ldots, \psi_k$ be periodic maps from $\mathbb{Z}$ to a field of characteristic $p$ (where $p$ is zero or a prime). Assume that positive integers $n_1, \ldots, n_k$ not divisible by $p$ are their periods respectively. We show that $\psi_1 + \cdots + \psi_k$ is constant if $\psi_1(x) + \cdots + \psi_k(x)$ equals a constant for $|S|$ consecutive integers $x$ where $S = \bigcup_{s=1}^{k} \{r/n_s : r = 0, \ldots, n_s - 1\}$. We also present some new results on finite systems of arithmetic sequences.

1. Introduction

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ we call

$$a(n) = a + n\mathbb{Z} = \{a + nx : x \in \mathbb{Z}\}$$

an arithmetic sequence with modulus $n$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^{k}$$

(1.1)

of such sequences, the covering function $w_A: \mathbb{Z} \to \mathbb{Z}$ given by

$$w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}|$$

(1.2)

is obviously periodic modulo the least common multiple $[n_1, \ldots, n_k]$ of all the moduli $n_1, \ldots, n_k$. If $w_A(x) \leq 1$ for all $x \in \mathbb{Z}$ (i.e., $a_i(n_i) \cap a_j(n_j) = \emptyset$ if $1 \leq i < j \leq k$), then we say that (1.1) is disjoint. When $w_A(x) \geq 1$ for all $x \in \mathbb{Z}$ (i.e., $\bigcup_{s=1}^{k} a_s(n_s) = \mathbb{Z}$), (1.1) is called a cover of $\mathbb{Z}$.

A famous result of H. Davenport, L. Mirsky, D. Newman and R. Rado (cf. [NZ]) states that if (1.1) is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$, then we must have $n_{k-1} = n_k$. In 1958 S. K. Stein [St] conjectured that if (1.1) is disjoint with $1 < n_1 < \cdots < n_k$ then there exists an integer $x \notin \bigcup_{s=1}^{k} a_s(n_s)$ with $1 \leq x \leq 2^k$. In 1965 P. Erdős [E2] offered a prize for a proof of his
following stronger conjecture (see [E1]): (1.1) forms a cover of \( \mathbb{Z} \) if it covers those integers from 1 to \( 2^k \). (The above \( 2^k \) is best possible because \( \{2^{s-1}(2^s)\}^{s=1}_{s} \) covers 1, \ldots, \( 2^k - 1 \) but does not cover any multiple of \( 2^k \).) In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [CV1, CV2] supplied a long and awkward proof of the Erdős conjecture for \( k \geq 20 \).

Let \( m \) be a positive integer. In [Su4, Su5] the author called (1.1) an \( m \)-cover of \( \mathbb{Z} \) if \( w_A(x) \geq m \) for all \( x \in \mathbb{Z} \), and an exact \( m \)-cover of \( \mathbb{Z} \) if \( w_A(x) = m \) for all \( x \in \mathbb{Z} \). Recently the author [Su10] found that \( m \)-covers of \( \mathbb{Z} \) are closely related to subset sums in a field and zero-sum problems on abelian groups.

Here is a result of [Su4, Su5] stronger than Erdős’ conjecture: (1.1) forms an exact \( m \)-cover of \( \mathbb{Z} \) if it covers \( \{\sum_{s \in I} m_s/n_s\} : I \subseteq \{1, \ldots, k\} \) consecutive integers at least \( m \) times, where the given \( m_1, \ldots, m_k \in \mathbb{Z}^+ \) are relatively prime to \( n_1, \ldots, n_k \) respectively. (As usual the fractional part of a real number \( x \) is denoted by \( \{x\} \).) In [Su5] the author asked whether we have a similar result for exact \( m \)-covers of \( \mathbb{Z} \). The answer is actually negative, moreover there is no constant \( c(k, m) \in \mathbb{Z}^+ \) such that (1.1) forms an exact \( m \)-cover of \( \mathbb{Z} \) whenever it covers \( c(k, m) \) consecutive integers exactly \( m \) times. In fact, if (1.1) is an exact \( m \)-cover of \( \mathbb{Z} \), then for any integer \( N > 1 \) the system \( \{a_1(n_1), \ldots, a_k(n_k), 0(N)\} \) covers 1, \ldots, \( N - 1 \) exactly \( m \) times but it covers 0 exactly \( m + 1 \) times! (This observation is due to the author’s student H. Pan.)

For an assertion \( P \) we adopt Iverson’s notation

\[
[P] = \begin{cases} 
1 & \text{if } P \text{ holds,} \\
0 & \text{otherwise.}
\end{cases}
\]

Observe that \( w_A(x) = \sum_{s=1}^{k} \psi_s(x) \) where \( \psi_s(x) = \left\lfloor n_s / x - a_s \right\rfloor \) is periodic modulo \( n_s \).

Our first result is completely new!

**Theorem 1.1.** Let \( F \) be a field of characteristic \( p \) where \( p \) is zero or a prime. Let \( n_1, \ldots, n_k \) be positive integers not divisible by \( p \), and let \( \psi_1, \ldots, \psi_k \) be maps from \( \mathbb{Z} \) to \( F \) with periods \( n_1, \ldots, n_k \) respectively. Then \( \psi_1 + \cdots + \psi_k = 0 \) if \( \psi_1(x) + \cdots + \psi_k(x) = 0 \) for \( \sum_{d \in D} \varphi(d) \) consecutive integers \( x \), where \( \varphi \) is Euler’s totient function, \( D = \bigcup_{s=1}^{k} D(n_s) \), and \( D(n) \) denotes the set of positive divisors of \( n \in \mathbb{Z}^+ \).

**Remark 1.1.** Clearly \( \sum_{d \in D} \varphi(d) \) in Theorem 1.1 equals the cardinality of the set

\[
\bigcup_{d \in D} \left\{ \frac{c}{d} : 0 \leq c < d \text{ and } (c, d) = 1 \right\} = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : r = 0, 1, \ldots, n_s - 1 \right\},
\]

where \( (c, d) \) is the greatest common divisor of \( c \) and \( d \). The result stated in the abstract is equivalent to Theorem 1.1 since a constant can be viewed as a function on \( \mathbb{Z} \) periodic mod 1.
Corollary 1.1. Let $w(x)$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ with period $n_0 \in \mathbb{Z}^+$. Then $w(x)$ is the covering function of (1.1) if $w_A(x) = w(x)$ for

$$\left| \bigcup_{s=0}^{k} \left\{ \frac{1}{n_s}, \ldots, \frac{n_s - 1}{n_s} \right\} \right| \leq n_0 + n_1 + \cdots + n_k - k$$

consecutive integers $x$. In particular, (1.1) forms an exact $m$-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^{k} \{r/n_s : r = 0, \ldots, n_s - 1\}|$ consecutive integers exactly $m$ times.

Proof. Let $D = \bigcup_{s=0}^{k} D(n_s)$. As

$$\psi(x) := w_A(x) - w(x) = -w(x) + \sum_{s=1}^{k} \left\lfloor \frac{n_s}{x - a_s} \right\rfloor$$

vanishes for $|\bigcup_{s=0}^{k} \{r/n_s : r = 0, \ldots, n_s - 1\}| = \sum_{d \in D} \varphi(d)$ consecutive integers $x$, we have $\psi(x) = 0$ for all $x \in \mathbb{Z}$ by Theorem 1.1. When $n_0 = 1$ and $w(x) = m \in \mathbb{Z}^+$, this yields the latter result in Corollary 1.1. □

Remark 1.2. The problem whether a given $A = \{a_s(n_s)\}_{s=1}^{k}$ forms a cover of $\mathbb{Z}$ is known to be co-NP-complete. (See, e.g. [GJ] and [T].) However, Corollary 1.1 indicates that we can check whether system $A$ has a given covering function in polynomial time! In 1997 the author [Su6] showed that if (1.1) covers all the integers the same number of times then

$$\left\{ \sum_{s=1}^{k} \frac{1}{n_s} : I \subseteq \{1, \ldots, k\} \right\} \supseteq \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : r = 0, \ldots, n_s - 1 \right\}.$$

Example 1.1. Let (1.1) be an exact $m$-cover of $\mathbb{Z}$, and let $n$ be an integer greater than $n_k$. Then the system

$$A' = \{a_1(n_1), \ldots, a_{k-1}(n_{k-1}), a_k + n_k(n)\}$$

covers each of the consecutive integers $a_k + 1, \ldots, a_k + 2n_k - 1$ exactly $m$ times but it does not cover $a_k$ or $a_k + 2n_k$ exactly $m$ times. For example, $B = \{1(2), 2(4), 0(4)\}$ is a disjoint cover of $\mathbb{Z}$, thus $B' = \{1(2), 2(4), 4(6)\}$ covers 1, 2, 3, 4, 5, 6, 7 exactly once but it is not a disjoint cover of $\mathbb{Z}$. Note that the set $\bigcup_{n \in \{2,4,6\}} \{r/n : r = 0, \ldots, n - 1\}$ just has 8 elements.

Corollary 1.2. Let (1.1) be a system of arithmetic sequences, and let $m$ be any integer greater than $k - f([n_1, \ldots, n_k])$. (The function $f$ is given by $f(1) = 0$ and $f(\prod_{i=1}^{r} p_i) = \prod_{i=1}^{r} (p_i - 1)$ where $p_1, \ldots, p_r$ are primes.) Then there is an $x \in \{0, 1, \ldots, |S| - 1\}$ such that $w_A(x) \neq m$ where $S = \bigcup_{s=1}^{k} \{r/n_s : r = 0, 1, \ldots, n_s - 1\}$.

Proof. If (1.1) is an exact $m$-cover of $\mathbb{Z}$, then $k \geq m + f([n_1, \ldots, n_k])$ by Corollary 4.5 of [Su7]. Thus, in view of the condition, (1.1) does not form an exact $m$-cover of $\mathbb{Z}$ and hence the desired result follows from Corollary 1.1. □

Our next theorem extends some earlier work in [Su4, Su5].
Theorem 1.2. Let $n_1, \ldots, n_k$ be positive integers, and let $R_1, \ldots, R_k$ be finite subsets of $\mathbb{Z}$. For $s = 1, \ldots, k$, let $c_{st}$ lie in the complex field $\mathbb{C}$ for each $t \in R_s$, and set

$$X_s = \left\{ x \in \mathbb{Z} : \sum_{t \in R_s} c_{st} e^{2\pi i t \frac{1}{n_s} x} = 0 \right\}. \quad (1.4)$$

If the system $\{X_s\}_{s=1}^k$ covers $W$ consecutive integers at least $m$ times where $1 \leq m \leq k$ and

$$W = \max_{I \subseteq \{1, \ldots, k\}} \left| \left\{ \sum_{s \in I} r_s \frac{1}{n_s} : r_s \in R_s \right\} \right| \leq \max_{|I| = k-m+1} \prod_{s \in I} |R_s|, \quad (1.5)$$

then it covers every integer at least $m$ times.

Corollary 1.3. Let (1.1) be a system of arithmetic sequences, and let $m_1, \ldots, m_k$ be integers relatively prime to $n_1, \ldots, n_k$ respectively. Let $l$ be any nonnegative integer with $w_A(x) \geq l$ for all $x \in \mathbb{Z}$, and set

$$W_l = \max_{I \subseteq \{1, \ldots, k\}} \left| \left\{ \sum_{s \in J} m_s \frac{1}{n_s} : J \subseteq I \right\} \right| \leq 2^{k-l}. \quad (1.6)$$

Then the covering function $w_A(x)$ takes its minimum on every set of $W_l$ consecutive integers.

Proof. Without loss of generality we may assume that $1 \leq m_s \leq n_s$ for all $s = 1, \ldots, k$. As $m(A) = \min_{x \in \mathbb{Z}} w_A(x) \geq l$ and $W_l \geq W_m(A)$, it suffices to work with $l = m(A)$ below.

The case $l = k$ is trivial, so we let $l < k$. Set $c_{s0} = 1$ and $c_{sm_s} = -e^{-2\pi i a_s m_s/n_s}$ for $s = 1, \ldots, k$. Since $m_s$ and $n_s$ are relatively prime,

$$X_s := \left\{ x \in \mathbb{Z} : c_{s0} e^{2\pi i \frac{a_s}{n_s} x} + c_{sm_s} e^{2\pi i \frac{m_s}{n_s} x} = 0 \right\} = a_s(n_s).$$

Applying Theorem 1.2 with $m = l + 1$ and $R_s = \{0, m_s\}$ ($1 \leq s \leq k$), we immediately get the desired result. \hfill \Box

Remark 1.3. (a) [Su9] contains some other interesting results on the covering function of (1.1). (b) $W_l$ in (1.6) might be smaller than its value in the case $m_1 = \cdots = m_k = 1$. Let $n_1 = 3$, $n_2 = 5$ and $n_3 = 15$. Set

$$W_0(m_1, m_2, m_3) = \left| \left\{ \sum_{s \in J} \frac{m_s}{n_s} : J \subseteq \{1, 2, 3\} \right\} \right|$$

for $m_1, m_2, m_3 \in \mathbb{Z}$. Then $W_0(1, 1, 2) = 7 < W_0(1, 1, 1) = 8$.

Our third theorem characterizes the least period of a covering function.
Theorem 1.3. Let $\lambda_s \in \mathbb{C}$, $a_s \in \mathbb{Z}$ and $n_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$. Then the smallest positive period $n_0$ of the (weighted) covering function

$$w(x) = \sum_{s=1}^{k} \lambda_s \left[ n_s \mid x - a_s \right]$$

is the least $n \in \mathbb{Z}^+$ such that $\alpha n \in \mathbb{Z}$ for all those $\alpha \in [0,1)$ with

$$\sum_{1 \leq s \leq k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \neq 0.$$  

Remark 1.4. Under the condition of Theorem 1.3, it can be easily checked that

$$\sum_{x=0}^{N-1} w(x)/N = \sum_{s=1}^{k} \lambda_s / n_s$$

where $N = [n_1, \ldots, n_k]$. If $w(x) = 0$ for all $x \in \mathbb{Z}$, then $n_0 = 1$ and hence

$$\sum_{\alpha n_s \in \mathbb{Z}} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \quad \text{for all } \alpha \in [0,1). \quad (1.7)$$

This was first obtained by the author [Su2] in 1991 via an analytic method, and the converse was proved in [Su3]. In [Su8] the author determined those functions $f: \bigcup_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ such that $\sum_{s=1}^{k} \lambda_s f(a_s + n_s \mathbb{Z})$ only depends on the covering function $w(x)$, this was announced by the author [Su1] in 1989.

Let $l$ be a positive integer, and let

$$\mathbb{Z}^l = \{ \vec{x} = (x_1, \ldots, x_l) : x_1, \ldots, x_l \in \mathbb{Z} \}$$

be the direct sum of $l$ copies of the ring $\mathbb{Z}$. For $\vec{x}, \vec{y} \in \mathbb{Z}^l$, we use $\vec{x} \mid \vec{y}$ to mean that $\vec{y} = \vec{q} \vec{x}$ (where $\vec{q} \in \mathbb{Z}^l$). A function $\Psi: \mathbb{Z}^l \to \mathbb{C}$ is said to be periodic modulo $\vec{n} \in \mathbb{Z}^l$ if $\Psi(\vec{x}) = \Psi(\vec{y})$ whenever $\vec{x} - \vec{y} = (x_1 - y_1, \ldots, x_l - y_l)$ is divisible by $\vec{n}$. For $x_1, \ldots, x_l \in \mathbb{Z}$, we also use $[x_i]_{1 \leq i \leq l}$ to denote the least common multiple of $x_1, \ldots, x_l$.

Theorem 1.4. Let $\lambda_s \in \mathbb{C}$, $\vec{a}_s \in \mathbb{Z}^l$ and $\vec{n}_s \in (\mathbb{Z}^+)^l$ for $s = 1, \ldots, k$ where $l \in \mathbb{Z}^+$. Suppose that the function

$$w(\vec{x}) = \sum_{s=1}^{k} \lambda_s \left[ \vec{n}_s \mid \vec{x} - \vec{a}_s \right] \quad (1.8)$$

is periodic modulo $\vec{n}_0 \in (\mathbb{Z}^+)^l$. Let $\vec{d} \in (\mathbb{Z}^+)^l$, $\vec{d} \nmid \vec{n}_0$ and

$$I(\vec{d}) = \{ 1 \leq s \leq k : \vec{d} \mid \vec{n}_s \neq \emptyset \}.$$
If \( \sum_{s \in I(\vec{d})} \lambda_s/(n_{s1} \cdots n_{sl}) \neq 0 \), then

\[
|I(\vec{d})| \geq \left| \left\{ \left\{ \sum_{t=1}^{l} \frac{a_{st}}{d_t} \right\} : s \in I(\vec{d}) \right\} \right| \geq \min_{0 \leq s \leq k \atop d_t \nmid n_s} \left[ \frac{d_t}{(d_t, n_{st})} \right]_{1 \leq t \leq l} \geq p(d_1 \cdots d_l)
\]

where we use \( p(m) \) to denote the least prime divisor of an integer \( m > 1 \).

**Remark 1.5.** Theorem 1.4 is a generalization of the main result of [Su2] which corresponds to the case \( l = 1 \) and improves the Znám–Newman result [N].

**Corollary 1.4.** Let \( \lambda_s \in \mathbb{C} \setminus \{0\} \), \( \vec{a}_s \in \mathbb{Z}^l \) and \( \vec{n}_s \in (\mathbb{Z}^+)^l \) for \( s = 1, \ldots, k \) where \( l \in \mathbb{Z}^+ \). Suppose that all those moduli \( \vec{n}_s \) which are maximal with respect to divisibility are distinct. Then the function \( w(\vec{x}) \) given by (1.8) is periodic modulo \( \vec{n}_0 \in (\mathbb{Z}^+)^l \) if and only if \( \vec{n}_0 \) is divisible by all the moduli \( \vec{n}_1, \ldots, \vec{n}_k \).

**Proof.** If \( \vec{n}_s \mid \vec{n}_0 \) for all \( s = 1, \ldots, k \), then the function \( w(\vec{x}) \) is obviously periodic modulo \( \vec{n}_0 \).

Now suppose that \( w(\vec{x}) \) is periodic modulo \( \vec{n}_0 \) but not all the moduli divide \( \vec{n}_0 \). Then there exists a maximal modulus \( \vec{n}_r \) with respect to divisibility such that \( \vec{n}_r \mid \vec{n}_0 \). By the condition,

\[
I(\vec{n}_r) := \{1 \leq s \leq k : \vec{n}_r \mid \vec{n}_s \} = \{1 \leq s \leq k : \vec{n}_s = \vec{n}_r \} = \{r\}.
\]

On the other hand, by Theorem 1.4 we should have \( |I(\vec{n}_r)| \geq p(n_{r1} \cdots n_{rl}) \). The contradiction ends our proof.

**Remark 1.6.** Corollary 1.4 in the case \( l = 1 \) was essentially established by Š. Porubský [P].

## 2. Proofs of Theorems 1.1–1.4

**Lemma 2.1.** Let \( c_1, \ldots, c_n \) lie in a field \( F \), and let \( z_1, \ldots, z_n \) be distinct elements of \( F \setminus \{0\} \). If \( \sum_{j=1}^{n} c_j z_j^x \) vanishes for \( n \) consecutive integers \( x \), then it vanishes for all \( x \in \mathbb{Z} \).

**Proof.** Suppose that \( \sum_{j=1}^{n} c_j z_j^{h+i-1} = 0 \) for every \( i = 1, \ldots, n \) where \( h \in \mathbb{Z} \). Since the Vandermonde determinant

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
z_1 & z_2 & \cdots & z_n \\
\vdots & \vdots & \ddots & \vdots \\
z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1}
\end{vmatrix} = \prod_{1 \leq i < j \leq n} (z_j - z_i)
\]

does not vanish, by Cramer’s rule we have \( c_j z_j^h = 0 \) and hence \( c_j = 0 \) for all \( j = 1, \ldots, n \). Therefore \( \sum_{j=1}^{n} c_j z_j^x = 0 \) for any \( x \in \mathbb{Z} \). \( \square \)
Proof of Theorem 1.1. As \( p \) does not divide \( N = [n_1, \ldots, n_k] \), the equation \( x^N - 1 = 0 \) has \( N \) distinct roots in the algebraic closure \( E \) of the field \( F \). The multiplicative group \( \{ \zeta \in E : \zeta^N = 1 \} \) of order \( N \) is cyclic, so \( E \) contains an element \( \zeta \) of multiplicative order \( N \). For \( a \in \mathbb{Z} \) and \( 1 \leq s \leq k \), we have the geometric series

\[
\frac{1}{n_s} \sum_{r=0}^{n_s-1} \zeta^{n_s ar} = [n_s \mid a].
\]

Therefore

\[
\sum_{s=1}^{k} \psi_s(x) = \sum_{s=1}^{k} \sum_{a=0}^{n_s-1} [n_s \mid a-x] \psi_s(a)
= \sum_{s=1}^{k} \sum_{a=0}^{n_s-1} \frac{1}{n_s} \sum_{r=0}^{n_s-1} \zeta^{n_s (a-x) r} \psi_s(a)
= \sum_{s=1}^{k} \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \sum_{0 \leq \alpha < 1} \zeta^{\alpha N (a-x)}
= \sum_{\alpha \in S} (\zeta^{-\alpha N})^x \left( \sum_{s=1}^{k} [\alpha n_s \in \mathbb{Z}] \sum_{a=0}^{n_s-1} \psi_s(a) \zeta^{\alpha N a} \right),
\]

where \( S \) is the set

\[
\{ \alpha \in [0, 1) : \alpha_n \in \mathbb{Z} \text{ for some } 1 \leq s \leq k \} = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : r = 0, \ldots, n_s - 1 \right\}.
\]

As those \( \zeta^{-\alpha N} \) with \( \alpha \in S \) are distinct, applying Lemma 2.1 we find that \( \sum_{s=1}^{k} \psi_s(x) = 0 \) for \( |S| \) consecutive integers \( x \) if and only if \( \sum_{s=1}^{k} \psi_s(x) = 0 \) for all \( x \in \mathbb{Z} \). By Remark 1.1, \( |S| = \sum_{d \in D} \varphi(d) \). This concludes the proof. \( \square \)

Proof of Theorem 1.2. Clearly an integer \( x \) is covered by \( \{ X_s \}_{s=1}^{k} \) at least \( m \) times if and only if \( x \) is covered by \( \{ X_s \}_{s \in I} \) for all \( I \subseteq \{ 1, \ldots, k \} \) with \( |I| = k - m + 1 \).

Now let \( I \subseteq \{ 1, \ldots, k \} \) and \( |I| = k - m + 1 \). For any \( x \in \mathbb{Z} \), we have

\[
\prod_{s \in I} \sum_{t \in R_s} e^{2 \pi i \frac{t}{n_s} x} = \sum_{R_s \in \mathcal{R}_s} \left( \prod_{s \in I} c_{R_s} \right) e^{2 \pi i x \sum_{s \in I} r_s / n_s}
= \sum_{\theta \in R(I)} C_{I, \theta} e^{2 \pi i \theta x}
\]

where

\[
R(I) = \left\{ \left\{ \sum_{s \in I} r_s / n_s \right\} : r_s \in R_s \right\} \quad \text{and} \quad C_{I, \theta} = \sum_{r_s \in R_s \text{ for } s \in I} \prod_{s \in I} c_{R_s}.
\]
Since those $e^{2\pi i \theta}$ with $\theta \in R(I)$ are distinct, by Lemma 2.1 the system \(\{X_s\}_{s \in I}\) covers \(|R(I)|\) consecutive integers \(x\) if and only if it covers all \(x \in \mathbb{Z}\).

In view of the above, we immediately obtain the desired result. \(\square\)

**Proof of Theorem 1.3.** Let \(S = \{0 \leq \alpha < 1 : \alpha n_s \in \mathbb{Z} \text{ for some } 1 \leq s \leq k\}\) and

\[
T = \left\{ 0 \leq \alpha < 1 : c_\alpha = \sum_{1 \leq s \leq k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \neq 0 \right\}.
\]

For each \(s = 1, \ldots, k\) the arithmetical function \(\psi_s(x) = \lambda_s [n_s | x - a_s]\) is periodic modulo \(n_s\). By the proof of Theorem 1.1, for any \(x \in \mathbb{Z}\) we have

\[
w(x) = \sum_{s=1}^{k} \lambda_s [n_s | x - a_s] = \sum_{\alpha \in S} e^{-2\pi i \alpha x} c_\alpha = \sum_{\alpha \in T} e^{-2\pi i \alpha x} c_\alpha.
\]

Let \(n\) be the least positive integer such that \(\alpha n \in \mathbb{Z}\) for all \(\alpha \in T\). By the above, \(w(x) = w(x + n)\) for all \(x \in \mathbb{Z}\). Thus \(n_0 | n\).

If \(T = \emptyset\), then \(n = 1\) and hence \(n_0 = n\). In the case \(T \neq \emptyset\), we have

\[0 = w(x) - w(x + n_0) = \sum_{\alpha \in T} e^{-2\pi i \alpha x} (1 - e^{-2\pi i \alpha n_0}) c_\alpha
\]

for every \(x = 0, \ldots, |T| - 1\), and hence \((1 - e^{-2\pi i \alpha n_0}) c_\alpha = 0\) for any \(\alpha \in T\) (Vandermonde). Now that \(\alpha n_0 \in \mathbb{Z}\) (i.e., \(e^{-2\pi i \alpha n_0} = 1\)) for all \(\alpha \in T\), we have \(n_0 \geq n\) and thus \(n_0 = n\).

The proof of Theorem 1.3 is now complete. \(\square\)

**Proof of Theorem 1.4.** Let \(\vec{c}\) be any vector in \(\mathbb{Z}^l\) with \(\vec{d} \nmid \vec{c} \vec{n}_0\). Then, for some \(1 \leq r \leq l\) we have \(d_r \nmid c_r n_0\). Note that \(\vec{n}_0\) divides the vector \((0, \ldots, 0, n_0 r, 0, \ldots, 0)\).

For any \(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_l \in \mathbb{Z}\), since

\[
\sum_{s=1}^{k} \left( \lambda_s \prod_{t=1 \atop t \neq r}^{l} [n_{st} | x_t - a_{st}] \right) [n_{sr} | x_r - a_{sr}] = w(\vec{x})
\]

is periodic mod \(n_0\) as a function of \(x_r\), by Theorem 1.3 we must have

\[
\sum_{s=1}^{k} \left( \lambda_s \prod_{t=1 \atop t \neq r}^{l} [n_{st} | x_t - a_{st}] \right) e^{2\pi i (c_r/d_r) a_{sr}} \frac{n_{sr}}{n_0} = 0.
\]

(Recall that \((c_r/d_r)n_0 \notin \mathbb{Z}\).)
Let $J = \{1 \leq s \leq k : d_r \mid c, n_{sr}\}$ and $\lambda_{sr}' = \lambda_{sr} n_{sr}^{-1} e^{2\pi i a_{sr} c_r / d_r}$ for $s \in J$. Given $r' \in \{1, \ldots, l\} \setminus \{r\}$ and $x_t \in \mathbb{Z}$ with $t \neq r, r'$, we have

$$\sum_{s \in J} \left( \lambda_{sr}' \prod_{t=1, t \neq r, r'}^l \frac{[n_{st} \mid x_t - a_{st}]}{[n_{sr'} \mid x_{sr'} - a_{sr']}} \right) = 0$$

for all $x_{r'} \in \mathbb{Z}$. By applying Remark 1.4 $l - 1$ times we finally obtain that

$$\sum_{s=1}^k \frac{\lambda_{sr}}{n_{s1} \cdots n_{sl}} e^{2\pi i \sum_{t=1}^l a_{st} c_t / d_t} = 0. \quad (2.2)$$

Set $m = \min_{0 \leq s \leq k} d_t \mid n_{st}$, $[d_t / (d_t, n_{st})] \mid 1 \leq t \leq l$. Clearly $m \geq p(d_1 \cdots d_l)$. Let $c$ be any positive integer less than $m$. For $s = 0, 1 \ldots, k$ we have

$$d_r \mid c n_{st} \iff d_t \mid n_{st} \text{ for all } t = 1, \ldots, l \iff \left[ \frac{d_t}{(d_t, n_{st})} \right]_{1 \leq t \leq l} \mid c \iff d_r \mid n_{sr}. \quad (2.2)$$

In other words, $d_r \mid c n_{sr}$ if and only if $s \in I(d_r)$. (2.2) in the case $\vec{c} = (c, \ldots, c)$ yields that

$$\sum_{s \in I(d_r)} \frac{\lambda_{sr}}{n_{s1} \cdots n_{sl}} e^{2\pi i \sum_{t=1}^l a_{st} / d_t} = 0.$$

Let $\Theta = \{ \sum_{t=1}^l a_{st} / d_t : s \in I(d) \}$. Suppose that $|\Theta| < m$. Then for each $c = 1, \ldots, |\Theta|$ we have

$$\sum_{\theta \in \Theta} e^{2\pi i c \theta} \sum_{s \in I(d)} \frac{\lambda_{sr}}{n_{s1} \cdots n_{sl}} = \sum_{s \in I(d)} \frac{\lambda_{sr}}{n_{s1} \cdots n_{sl}} e^{2\pi i \sum_{t=1}^l a_{st} / d_t} = 0.$$

By Lemma 2.1 this holds for all integers $c$, in particular $c = 0$:

$$\sum_{s \in I(d)} \frac{\lambda_{sr}}{n_{s1} \cdots n_{sl}} = 0.$$

This directly contradicts one of the hypotheses, whence $|\Theta| \geq m$. \qed
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