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FUGLEDE'S CONJECTURE IS FALSE IN 5 AND HIGHER DIMENSIONS

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Abstract. We give an example of a set $\Omega \subset \mathbf{R}^5$ which is a finite union of unit cubes, such that $L^2(\Omega)$ admits an orthonormal basis of exponentials $\{\frac{1}{|\Omega|^{1/2}} e^{2\pi i \xi_j \cdot x} : \xi_j \in \Lambda\}$ for some discrete set $\Lambda \subset \mathbf{R}^5$, but which does not tile $\mathbf{R}^5$ by translations. This answers (one direction of) a conjecture of Fuglede [1] in the negative, at least in 5 and higher dimensions.

1. Introduction

Let Ω be a domain in $\mathbf{R}^n$, i.e., Ω is a Lebesgue measurable subset of $\mathbf{R}^n$ with finite non-zero Lebesgue measure. We say that a set $\Lambda \subset \mathbf{R}^n$ is a *spectrum* of Ω if $\{\frac{1}{|\Omega|^{1/2}} e^{2\pi i x \cdot \xi}\}_{\xi \in \Lambda}$ is an orthonormal basis of $L^2(\Omega)$. In this paper we use $|\Omega|$ to denote the Lebesgue measure of a set Ω, and $\#A$ to denote the cardinality of a finite set A.

Conjecture 1.1. [1] *A domain Ω admits a spectrum if and only if it is possible to tile $\mathbf{R}^n$ by a family of translates $\{t + \Omega : t \in \Lambda\}$ of Ω (ignoring sets of measure zero).*

Fuglede [1] proved this conjecture (also known as the spectral set conjecture) under the additional assumption that the tiling set or the spectrum are lattice subsets of $\mathbf{R}^n$. This conjecture arose from the study of commuting skew-adjoint extensions of the partial derivative operators $\frac{\partial}{\partial x_j}$, and has attracted much recent interest, see the references given in the bibliography for a partial list of papers relating to this conjecture, and [8] for a survey.

Our main result here is that Fuglede's conjecture is false in sufficiently high dimension.

Theorem 1.2. *Let $n \geq 5$ be an integer. Then there exists a compact set $\Omega_2 \subset \mathbf{R}^n$ of positive measure such that $L^2(\Omega_2)$ admits an orthonormal basis of exponentials $\{\frac{1}{|\Omega_2|^{1/2}} e^{2\pi i \xi_j \cdot x} : \xi_j \in \Lambda_2\}$ for some $\Lambda_2 \subset \mathbf{R}^n$, but such that Ω_2 does not tile $\mathbf{Z}^n$ by translations. In particular, Fuglede's conjecture is false in $\mathbf{R}^n$ for $n \geq 5$.*

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The counterexample Ω_2 is elementary - it is an explicit finite union of unit cubes, and is based on a counterexample to Fuglede's conjecture in a specific finite abelian group. Basically, the idea is to exploit the existence of Hadamard matrices (i.e. orthogonal matrices whose entries are all ± 1, or more generally a p^{th} root of unity) of order not equal to a power of p; when $p = 2$ the first example occurs at dimension 12, and when $p = 3$ the first example occurs at dimension 6. Such Hadamard matrices quickly lead to a counterexample to Fuglede's conjecture in the finite groups $\mathbf{Z}_2^{12}$ and $\mathbf{Z}_3^6$ (actually to $\mathbf{Z}_2^{11}$ and $\mathbf{Z}_3^5$), and one can use standard transference techniques to move this counterexample to $\mathbf{Z}^5$ and thence to $\mathbf{R}^5$.

Our arguments do not preclude the possibility that the conjecture may still be true in lower dimensions, and in particular in one dimension; see for instance [13] for some evidence in favor of the one-dimensional conjecture. For instance, the results of [13] show that Fuglede's conjecture is true for cyclic p-groups $\mathbf{Z}_{p^N}$, so one cannot directly replicate the above counterexample in one dimension.

It may still be true that Fuglede's conjecture still holds in higher dimensions under more restrictive assumptions on the domain Ω_2, for instance if one enforces convexity (our example is highly non-convex, although it does not fall into the class of non-convex objects studied on [10], or the near-cubic objects studied in [9]).

We do not address the issue as to whether the converse direction of Fuglede's conjecture might still hold; in other words, whether every set which tiles $\mathbf{R}^n$ by translations admits a spectrum. Again, it seems that one should first look at p-groups to determine the truth or falsity of this conjecture.

2. The finite model: failure of Fuglede in $\mathbf{Z}_2^{12}$, $\mathbf{Z}_3^6$, and $\mathbf{Z}_3^5$.

We begin with a finite version of Theorem 1.2, in the finite group $\mathbf{Z}_p^n$, where $p = 2, 3$ and $\mathbf{Z}_p := \mathbf{Z}/p\mathbf{Z}$. If $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ are elements of $\mathbf{Z}_p^n$, we define the dot product $\xi \cdot x \in \mathbf{Z}_p$ as

$$\xi \cdot x := \sum_{j=1}^{n} \xi_j x_j$$

and in particular we can define the quantity $e^{2\pi i(\xi \cdot x)/p}$, which is always a p^{th} root of unity.

To illustrate the method, we begin with a counterexample in $\mathbf{Z}_2^{12}$, although we will not directly use this example for our main result.

Theorem 2.1. *There exists a non-empty subset* $\Omega_0 \subset \mathbf{Z}_2^{12}$ *such that* $l^2(\Omega_0)$ *admits an orthonormal basis of exponentials* $\{\frac{1}{(\#\Omega_0)^{1/2}} e^{2\pi i(\xi_j \cdot x)/2} : \xi_j \in \Lambda_0\}$ *for some* $\Lambda_0 \subset \mathbf{Z}_2^{12}$*, but such that* Ω_0 *does not tile* $\mathbf{Z}_2^{12}$ *by translations.*

Proof. Let $e_1, \dots, e_{12}$ be the standard basis for $\mathbf{Z}_2^{12}$, thus e_j is the 12-tuple which equals 1 in the j^{th} entry and 0 everywhere else. We shall take Ω_0 to simply be this 12-element set:

$$\Omega_0 := \{e_1, e_2, \dots, e_{12}\}.$$

It is clear that Ω_0 does not tile $\mathbf{Z}_2^{12}$ by translations, since $\#\Omega_0 = 12$ does not divide evenly into $\#\mathbf{Z}_2^{12} = 2^{12}$. On the other hand, Ω_0 admits an orthonormal set of exponentials. To see this, we take any Hadamard matrix H of order 12, for instance[1]

$$H := \begin{pmatrix}
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
+1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 \\
+1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 \\
+1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 \\
+1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\
+1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 \\
+1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 \\
+1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1
\end{pmatrix}.$$

One can verify that all the rows of H are orthogonal to each other (this is part of what it means for a matrix to be Hadamard). We then define the set of 12 frequencies $\Lambda_0 := \{\xi_1, \dots, \xi_{12}\}$ by requiring that the 12-dimensional vector $(e^{2\pi i(e_j \cdot \xi_k)/2})_{j=1}^{12}$ matches the k^{th} row of H, thus for instance

$$\xi_1 := (0,1,1,1,1,1,1,1,1,1,1,1), \xi_2 := (0,0,1,0,1,1,1,0,0,0,1,0), \text{ etc.}$$

It is then clear that the twelve exponential functions $\frac{1}{\sqrt{12}} e^{2\pi i(\xi_k \cdot x)/2}$, $k = 1, \dots, 12$, form an orthonormal basis of $l^2(\Omega)$ as claimed. □

The above simple example may be compared with the example used to disprove Tijdeman's conjecture in [15], or to disprove Keller's conjecture in [14]; the three counterexamples are not directly related to each other, but they do share a similar flavor, in that the combinatorics of tiling and orthogonality in "higher-dimensional" situations may behave quite differently from what one might intuitively extrapolate from "low-dimensional" examples. These three examples also show that the main obstructions to tiling or spectral conjectures in Euclidean spaces in fact come from finite abelian groups of relatively small order.

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[1]We are grateful to Neil Sloane for making this example available on his web page `http://www.research.att.com/~njas/hadamard`.

Now let $\omega := e^{2\pi i/3}$ be a cube root of unity. Observe that the 6×6 matrix

$$H := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \end{pmatrix}$$

is orthogonal, with all its entries equal to a cube root of unity. By repeating the above argument, we thus see that the six-element set $\Omega_0' := \{e_1, \dots, e_6\}$ in $\mathbf{Z}_3^6$ has a spectrum Λ_0', also with six elements, but does not tile $\mathbf{Z}_3^6$ since 6 does not go evenly into 3^6.

In fact one can descend from this example to $\mathbf{Z}_3^5$ by the following simple observation. We may translate Ω_0' by $-e_1$ to create a new set

$$\{0, e_2 - e_1, \dots, e_6 - e_1\};$$

this of course does not affect the property that Λ_0' is a spectrum. This set is contained in the 5-dimensional space

$$\Gamma := \{(x_1, \dots, x_6) \in \mathbf{Z}_3^6 : x_1 + \dots + x_6 = 0\};$$

one can then project the spectrum onto Γ^*, the dual space of Γ. But Γ is clearly isomorphic to $\mathbf{Z}_3^5$. We have thus proved

Theorem 2.2. *There exists a set $\Omega_0'' \subset \mathbf{Z}_3^5$ of six elements, such that $l^2(\Omega_0'')$ admits an orthonormal basis of exponentials $\{\frac{1}{(\#\Omega_0'')^{1/2}} e^{2\pi i(\xi_j \cdot x)/3} : \xi_j \in \Lambda_0''\}$ for some $\Lambda_0'' \subset \mathbf{Z}_3^5$, but such that Ω_0'' does not tile $\mathbf{Z}_3^5$ by translations.*

Note that the non-tiling property follows since 6 does not go evenly into 3^5. A similar argument allows one to replace $\mathbf{Z}_2^{12}$ with $\mathbf{Z}_2^{11}$ in Theorem 2.1, but we will not need to do so here. We also remark that the above theorem clearly also holds if $\mathbf{Z}_3^5$ is replaced by $\mathbf{Z}_3^n$ for any $n \geq 5$.

An explicit example for Theorem 2.2 can be given by

$$\Omega_0'' := \{0, e_1, e_2, e_3, e_4, e_5\}$$

and

$$\Lambda_0'' := \{(0,0,0,0,0), (0,1,2,2,1), (1,0,1,2,2), (2,1,0,1,2), (2,2,1,0,1), (1,2,2,1,0)\}.$$

Observe that this example is invariant under the cyclic shift $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_4, x_5, x_1)$; this is useful in reducing the amount of computation required to verify that Λ_0'' is indeed a spectrum for Ω_0''.

3. The discrete model: failure of Fuglede in $\mathbf{Z}^5$

We now modify the construction in the previous section to prove a discrete version of Theorem 1.2, in the lattice $\mathbf{Z}^5$. The dual group to this lattice is the torus $\mathbf{R}^5/\mathbf{Z}^5$; if $x \in \mathbf{Z}^5$ and $\xi \in \mathbf{R}^5/\mathbf{Z}^5$ we can define the expression $e^{2\pi i \xi \cdot x}$ in the obvious manner.

Theorem 3.1. *There exists a non-empty finite subset* $\Omega_1 \subset \mathbf{Z}^5$ *such that* $l^2(\Omega_1)$ *admits an orthonormal basis of exponentials* $\{\frac{1}{(\#\Omega_1)^{1/2}} e^{2\pi i \xi_j \cdot x} : \xi_j \in \Lambda_1\}$ *for some* $\Lambda_1 \subset \mathbf{R}^5/\mathbf{Z}^5$*, but such that* Ω_1 *does not tile* $\mathbf{Z}^5$ *by translations.*

Proof. Heuristically, this theorem follows immediately from Theorem 2.2 by pulling Ω_0'' back under the obvious homomorphism $\mathbf{Z}^5 \to \mathbf{Z}_3^5$. Unfortunately this has the problem of making the pre-image of Ω_0'' infinite; however this can be rectified by truncating this pre-image at a sufficiently large scale, and noting that boundary effects of the truncation will be negligible if the scale is large enough.

We turn to the details. Let $\Omega_0'' \subset \mathbf{Z}_3^5$ and $\Lambda_0'' \subset \mathbf{Z}_3^5$ be the counterexample to Fuglede's conjecture in $\mathbf{Z}_3^5$ constructed in Theorem 2.2. We need three large integers

$$1 \ll L \ll M \ll N,$$

for instance we may pick $L := 10^{10}$, $M := L^2$, and $N := M^2$.

We define $\Omega_1 \subset \mathbf{Z}^5$ to be the set

$$\Omega_1 := \bigcup_{k \in [0,M)^5} (3k + \Omega_0'')$$

where we identify $\mathbf{Z}_3^5$ with the set $\{0,1,2\}^5 \subset \mathbf{Z}^5$ in the obvious (non-homomorphic!) manner, and $[0,M)^5$ is the discrete cube

$$[0,M)^5 := \{(x_1, \dots, x_5) \in \mathbf{Z}^5 : 0 \leq x_j < M \text{ for all } j = 1, \dots, 5\}.$$

Observe that Ω_1 is a finite set in $[0,3M)^5$ consisting of $6M^5$ elements. We define the spectrum $\Lambda_1 \subset \mathbf{R}^5/\mathbf{Z}^5$ in a similar fashion by

$$\Lambda_1 := \bigcup_{l \in [0,M)^5} (\frac{l}{3M} + \frac{1}{3}\Lambda_0''),$$

where the homomorphism $\xi \mapsto \frac{1}{3}\xi$ from $\mathbf{Z}_3^5$ to $\{0 + \mathbf{Z}, \frac{1}{3} + \mathbf{Z}, \frac{2}{3} + \mathbf{Z}\}^5 \subset (\mathbf{R}/\mathbf{Z})^5$ is defined in the obvious manner. Note that Λ_1 is also a finite set consisting of $6M^5$ elements.

We now verify that the set of exponentials $\{\frac{1}{(\#\Omega_1)^{1/2}} e^{2\pi i \xi_j \cdot x} : \xi_j \in \Lambda_1\}$ form an orthonormal basis of $l^2(\Omega_1)$. The normalization property is obvious. Since the number of exponentials equals the dimension of $l^2(\Omega_1)$, it will suffice to prove orthogonality, i.e. that

$$\sum_{x \in \Omega_0''} e^{2\pi i (\xi - \xi') \cdot x} = 0 \text{ for all distinct } \xi, \xi' \in \Lambda_1. \quad (1)$$

We write $\xi = \frac{l}{3M} + \frac{1}{3}\xi_0$ and $\xi' = \frac{l'}{3M} + \frac{1}{3}\xi_0'$ for some $l, l' \in [0,M)^5$ and $\xi_0, \xi_0' \in \Lambda_0''$; since $\xi \neq \xi'$, we observe that at least one of $l \neq l'$ or $\xi_0 \neq \xi_0'$ must hold. We similarly write $x = 3k + x_0$ where k ranges over $[0,M)^5$ and x_0 ranges over Ω_0''. We can then rewrite the left-hand side of (1) as

$$\sum_{k \in [0,M)^5} \sum_{x_0 \in \Omega_0''} e^{2\pi i (\frac{l-l'}{3M} + \frac{1}{3}(\xi_0 - \xi_0')) \cdot (3k + x_0)}.$$

We may expand this as

$$\sum_{k\in[0,M)^5}\sum_{x_0\in\Omega_0''} e^{2\pi i(l-l')\cdot k/M} e^{2\pi i(l-l')\cdot x_0/3M} e^{2\pi i(\xi_0-\xi_0')\cdot x_0/3}$$

since the dot product of $\frac{1}{3}(\xi_0-\xi_0')$ and $3k$ is an integer and hence negligible.

The sum $\sum_{k\in[0,M)^5} e^{2\pi i(l-l')\cdot k/M}$ vanishes unless $l=l'$ (this is basically the Fourier inversion formula for $(\mathbf{Z}/M\mathbf{Z})^5$). Thus we may assume $l=l'$, and hence $\xi_0\neq\xi_0'$. But then the previous expression simplifies to

$$M^5\sum_{x_0\in\Omega_0''} e^{2\pi i(\xi_0-\xi_0')\cdot x_0/3}$$

which vanishes since the frequencies in Λ_0'' were chosen to give an orthogonal basis of $l^2(\Omega_0'')$. This proves the existence of a spectrum.

We now show that the set Ω_1 does not tile $\mathbf{Z}^5$, if L,M,N were chosen sufficiently large; this will be a volume packing argument that relies once again on the fact that 6 does not go evenly into 3^5. Suppose for contradiction that we could find a subset $\Sigma\subset\mathbf{Z}^5$ such that the translates $\{t+\Omega_1 : t\in\Sigma\}$ tiled $\mathbf{Z}^5$.

The idea is to exploit the intuitive observation that Ω_1 has local density either equal to 0 or to $\frac{6}{3^5}$, except on the boundary. To make this rigorous[2] we now use the other numbers L,N chosen earlier. let $\Sigma_N := \Sigma\cap[0,N)^5$. Observe that the sets $\{t+\Omega_1 : t\in\Sigma_N\}$ are disjoint and each have cardinality $6M^5$, while their union contains the cube $[2M,N-2M)^5$, and is contained in the cube $[-2M,N+2M)^5$. Since both of these cubes have a cardinality of $N^5+O(MN^4)$, we thus have the cardinality estimate

$$6M^5\#\Sigma_N = N^5+O(MN^4).$$

Let A denote the annulus

$$A := [-5L,2M+5L)^5\backslash[5L,2M-5L)^5;$$

this A represents the boundary effects of our restriction of Ω_1 to $[0,M)^5$. We shall now work in a region of the tiling where A can be ignored. Observe that A has cardinality $\#A=O(M^4L)$. In particular, we have

$$\#\bigcup_{t\in\Sigma_M}(t+A)\leq(\#\Sigma_N)(\#A)=O(\frac{N^5}{6M^5}M^4L)=O(\frac{L}{M})N^5.$$

Thus, if M is chosen sufficiently large with respect to L, we see that

$$\#\bigcup_{t\in\Sigma_M}(t+A)<\#[\frac{N}{3},\frac{2N}{3})^5,$$

and thus we may find a point $x_0\in[\frac{N}{3},\frac{2N}{3})^5$ which is not contained in $t+A$ for any $t\in\Sigma_M$.

[2] Another approach, which is basically equivalent to this one, is to compute the convolution $\chi_{\Omega_1}*\frac{1}{L^n}\chi_{[0,L)^n}$ and observe that this is approximately equal to the function $\frac{6}{3^5}\chi_{[0,M)^n}$, which cannot tile $\mathbf{Z}^n$.

Fix this point x_0, and consider the quantity

$$f(t) := \#\big((t+\Omega_1) \cap (x_0 + [0,L)^5)\big)$$

defined for all $t \in \Sigma$. Since the sets $\{t + \Omega_1 : t \in \Sigma\}$ tile $\mathbf{Z}^5$, we must have

$$\sum_{t \in \Sigma} f(t) = \#(x_0 + [0,L)^5) = L^5. \tag{2}$$

On the other hand, we shall shortly show that for every $t \in \Sigma$, we have either

$$f(t) = 0 \text{ or } f(t) = L^5\Big(\frac{6}{3^5} + O\Big(\frac{1}{L}\Big)\Big). \tag{3}$$

Since 6 does not go evenly into 3^5, the estimates (2), (3) will cause the desired contradiction if L is chosen sufficiently large.

It remains to prove (3). We may assume of course that $f(t) \neq 0$; so that $t + \Omega_1$ intersects $x_0 + [0,L)^5$, which implies that

$$x_0 \in t + [-L, 2M + L)^5. \tag{4}$$

Since $x_0 \in [\frac{N}{3}, \frac{2N}{3}]^5$, this implies that $t \in [0,N]^5$, since N is much larger than L or M. In particular, $t \in \Sigma_N$. By construction of x_0, we thus see that $x_0 \notin t + A$. Combining this (4) we see that

$$x_0 \in t + [5L, 2M - 5L)^5.$$

In particular, we have

$$x_0 + [0,L)^5 \subset t + [4L, 2M - 4L)^5.$$

If one then covers $x_0 + [0,L)^5$ by $\frac{1}{3^5}L^5 + O(L^4)$ cubes of the form $2k + \{0,1\}^5$ for various $k \in \mathbf{Z}^5$ - and observe that all but $O(L^4)$ of these cubes will be strictly contained inside $x_0 + [0,L)^5$ - then each of these cubes will lie inside $t + [3L, 2M - 3L)$, and thus will intersect $t + \Omega_1$ in exactly $\#\Omega_0'' = 6$ points. Thus we have

$$\#\big((t+\Omega_1) \cap (x_0 + [0,L)^5)\big) = [\frac{1}{3^5}L^5 + O(L^4)]6 + O(L^4)$$

which is (3) as desired. This concludes the proof that Ω_1 does not tile $\mathbf{Z}^5$, and we are done. □

4. The continuous model: failure of Fuglede in $\mathbf{R}^5$

We can now prove Theorem 1.2 by a standard transference argument. Let Ω_1 and Λ_1 be defined as in the previous section. Then we simply set

$$\Omega_2 := \Omega_1 + [0,1)^5$$

and

$$\Lambda_2 := \Lambda_1 + \mathbf{Z}^5.$$

Since Λ_1 was a spectrum for Ω_1, and $\mathbf{Z}^5$ is a spectrum for $[0,1)^5$, it is easy to see that Λ_2 is a spectrum for Ω_2. The proof that Ω_2 does not tile $\mathbf{R}^5$ proceeds almost exactly the same as in the previous section, with the obvious changes that cubes such as $[0,N)^5$ should now be solid cubes in $\mathbf{R}^5$ rather than discrete cubes in $\mathbf{Z}^5$, and that cardinality $\#(A)$ of sets should mostly be replaced by Lebesgue measure $|A|$ instead (except when dealing with Σ_N, which remains discrete). We omit the details.

The above argument can be extended from $\mathbf{R}^5$ to $\mathbf{R}^n$ for any $n \geq 5$, mainly becase Theorem 2.2 also extends in this manner.

References

[1] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal. **16** (1974), 101–121.

[2] A. Iosevich, N. Katz, T. Tao, *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. **123** (2001), 115–120.

[3] ———, *The Fuglede conjecture holds for convex planar domains*, to appear.

[4] A. Iosevich, S. Pedersen, *Spectral and tiling properties of the unit cube*, Internat. Math Research Notices **16** (1998), 819–828.

[5] P. E. T. Jorgensen, S. Pedersen, *Orthogonal harmonic analysis of fractal measures*, ERA Amer. Math. Soc. **4** (1998), 35–42

[6] M. Kolountzakis, *Non-symmetric convex domains have no basis of exponentials*, Illinois J. Math. **44** (2000), 542–550.

[7] ———, *Packing, tiling, orthogonality, and completeness*, Bull. London Math. Soc. **32** (2000), 589–599.

[8] ———, *The study of translational tiling with Fourier Analysis*, Proceedings of the Milano Conference on Fourier Analysis and Convexity, to appear.

[9] M. Kolountzakis, I. Łaba, *Tiling and spectral properties of near-cubic domains*, Studia Math. **160** (2004), 287–299.

[10] M. Kolountzakis, M. Papadimitrakis, *A class of non-convex polytopes which admit no orthogonal basis of exponentials*, Illinois J Math. **46** (2004). 1227–1232.

[11] S. Konyagin, I. Łaba, *Spectra of certain types of polynomials and tiling of integers with translates of finite sets*, J. Number Theory, **103** (2003), 267–280.

[12] I. Łaba, *Fuglede's conjecture for a union of two intervals*, Proc. Amer. Math. Soc. **129** (2001), 2965–2972.

[13] ———, *The spectral set conjecture and multiplicative properties of roots of polynomials*, J. London Math. Soc. **65** (2002), 661–671.

[14] J. Lagarias, P. Shor, *Keller's cube tiling conjecture is false in high dimensions*, Bull. Amer. Math. Soc. **27** (1992), 279–283.

[15] J. Lagarias, S. Szabó, *Universal spectra and Tijdeman's conjecture on factorization of cyclic groups*, J. Fourier Anal. Appl. **7** (2001), 63–70.

[16] J. Lagarias, Y. Wang, *Spectral sets and factorizations of finite abelian groups*, J. Funct. Anal. **145** (1997), 73–98.

[17] ———, *Orthonormal bases of exponentials for the n-cube*, Duke Math. J. **103** (2000), 25–37.

[18] S. Pedersen, Y. Wang, *Universal spectra, universal tiling sets, and the spectral set conjecture*, Math. Scand. **88** (2001), 246–256.

[19] Y. Wang, *Wavelets, tiling and spectral sets*, Duke Math J. **114** (2002), 43–57.

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