EXISTENCE OF RELATIVE PERIODIC ORBITS NEAR RELATIVE EQUILIBRIA

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ABSTRACT. We show existence of relative periodic orbits (a.k.a. relative nonlinear normal modes) near relative equilibria of a symmetric Hamiltonian system under an appropriate assumption on the Hessian of the Hamiltonian. This gives a relative version of the Moser-Weinstein theorem.

1. Introduction

In this paper we discuss a generalization of the Weinstein-Moser theorem [Mo, W1, W2] on the existence of nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits (r.p.o.'s) near a relative equilibrium of a symmetric Hamiltonian system.

More specifically let $(M,\omega_M)$ be a symplectic manifold with a proper Hamiltonian action of a Lie group $G$ and a corresponding equivariant moment map $\Phi : M \to g^*$. Let $h \in C^\infty(M)^G$ be a $G$-invariant Hamiltonian. We will refer to the quadruple $(M,\omega_M,\Phi : M \to g^*, h \in C^\infty(M)^G)$ as a symmetric Hamiltonian system. The main result of the paper is the following theorem (the terms used in the statement are explained below):

**Theorem 1.** Let $(M,\omega_M,\Phi : M \to g^*, h \in C^\infty(M)^G)$ be a symmetric Hamiltonian system. Suppose $x \in M$ is a positive definite relative equilibrium of the system and let $\mu = \Phi(x)$. Then for every sufficiently small $E > 0$ the set \(\{h = E + h(x)\} \cap \Phi^{-1}(\mu)\) (if nonempty) contains a relative periodic orbit of $h$.

This theorem strengthens or complements numerous other results about the existence of relative periodic orbits; see, e.g., [LT, O1, O2].

We now define the relevant terms. Recall that for every invariant function $h \in C^\infty(M)^G$ the restriction $h|_{\Phi^{-1}(\mu)}$ descends to a continuous function $h_\mu$ on the symplectic quotient

$$M/\mu G := \Phi^{-1}(\mu)/G_\mu,$$

where $G_\mu$ denotes the stabilizer of $\mu \in g^*$ under the coadjoint action. If $G_\mu$ acts freely on $\Phi^{-1}(\mu)$ then the symplectic quotient $M/\mu G$ is a symplectic manifold.
and the flow (of the Hamiltonian vector field) of $h$ on $\Phi^{-1}(\mu)$ descends to the flow of $h_{\mu}$ on the quotient. More generally the quotient $M//\mu G$ is a symplectic stratified space: the quotient is a union of strata, which are symplectic manifolds and which fit together in a locally simple manner. In this case the flow of $h$ descends to a flow on $M//\mu G$, which is, on each stratum, the flow of the restriction of $h_{\mu}$ to the stratum.\footnote{The facts referred to above were first proved in [SL] in the special case of $G$ being compact and $\mu = 0$. The case where $G$ acts properly and the coadjoint orbit through $\mu$ is closed was dealt with in [BL]. The assumption on the coadjoint orbit was subsequently removed in [LW] following a suggestion in [O1].}

A point $x \in \Phi^{-1}(\mu)$ is a relative equilibrium of $h$ (strictly speaking of the symmetric Hamiltonian system $(M, \omega, \Phi : M \to g^*, h \in C^\infty(M)^G)$, if its image $[x] \in M//\mu G$ is stationary under the flow of $h_{\mu}$. If $x$ is a relative equilibrium then there is a vector $\xi \in g$ such that $dh(x) = d(\langle \Phi, \xi \rangle)(x)$. It is not hard to see that $\xi$ is in the Lie algebra $g_\mu$ of the stabilizer of $\mu$ and that $\xi$ is unique modulo $g_x$, the Lie algebra of the stabilizer of $x$. If we set

$$h^\xi := h - \langle \Phi, \xi \rangle,$$

then $dh^\xi(x) = 0$. Hence the Hessian $d^2(h^\xi)(x)$ is well-defined. The symplectic slice at $x \in M$ is, by definition, the vector space

$$V := T_x(G \cdot x)^\omega / (T_x(G \cdot x) \cap T_x(G \cdot x)^\omega),$$

where $\cdot^\omega$ denotes the symplectic perpendicular. Note that $V$ is naturally a symplectic representation of $G_x$, the stabilizer of $x$. Note also that $T_x(G \cdot x)^\omega = \ker d\Phi_x$, so $V$ is isomorphic to a maximal symplectic subspace of $\ker d\Phi_x$. In particular if $G_\mu$ acts freely at $x$ then $V$ models the symplectic quotient $M//\mu G$ near $[x]$. A computation shows that $T_x(G \cdot x) \cap T_x(G \cdot x)^\omega = T_x(G_\mu \cdot x)$ and that $h^\xi(g \cdot x) = h^\xi(x)$ for all $g \in G_\mu$. Hence the subspace $T_x(G_\mu \cdot x)$ lies in the kernel of the quadratic from $d^2(h^\xi)(x)$. It follows that the Hessian $d^2(h^\xi)(x)$ descends to a well-defined quadratic form $q$ on the symplectic slice $V$ (which depends on $\xi$). We say that the relative equilibrium $x$ is positive definite if $q$ is positive definite for some choice of $\xi$. When the action is free, $q$ can be thought of as the restriction of $d^2(h|_{\Phi^{-1}(\mu)})$ to the normal $V$ in $\Phi^{-1}(\mu)$ to the orbit $G \cdot x$. An integral curve $\gamma(t) \subset \Phi^{-1}(\mu)$ of $h$ is a relative periodic orbit if its projection $[\gamma(t)] \subset M//\mu G$ is periodic.

The motivation for Theorem 1 comes from a result of Weinstein [W1] generalizing a classical theorem of Liapunov which asserts that if $x$ is an equilibrium of a Hamiltonian $h$ on a symplectic manifold $(M, \omega)$ and if the Hessian $d^2 h(x)$ of $h$ at $x$ is positive definite, then for every $E > 0$ sufficiently small the energy surface

$$\{h = h(x) + E\}$$

carries at least $\frac{1}{2} \dim M$ periodic orbits. Now suppose $(M, \omega, \Phi : M \to g^*, h \in C^\infty(M)^G)$ is a symmetric Hamiltonian system, $\mu \in g^*$ is a point and the action

$$g \cdot x = \Phi(g \cdot x) = \Phi(g)x$$
of $G_\mu$ on $\Phi^{-1}(\mu)$ is free, so that the symplectic quotient $M//_\mu G$ is smooth. If a relative equilibrium $x \in \Phi^{-1}(\mu)$ is positive definite then the Hessian of $h_\mu$ at $[x]$ is positive definite. Hence by Weinstein’s theorem applied to $h_\mu$ at $[x]$, for every $E > 0$ sufficiently small the energy surface 
$$\{h_\mu = h_\mu([x]) + E\}$$
carries at least $\frac{1}{2} \dim M//_\mu G$ periodic orbits. In other words, under these natural assumptions the manifolds
$$\{h = h(x) + E\} \cap \Phi^{-1}(\mu)$$
carry relative periodic orbits of $h$. It is natural to ask what happens if $G_\mu$ does not act freely at or near the relative equilibrium $x$ of $h$. To address this issue let us recall where the strata of the symplectic quotients come from. For a subgroup $H$ of $G$ the set $M_{(H)}$ of points of orbit type $(H)$ is defined by
$$M_{(H)} := \{m \in M \mid \text{the stabilizer } G_m \text{ is conjugate to } H \text{ in } G\}.$$ Since the action of $G$ is proper, $M_{(H)}$ is a manifold. Moreover the set
$$(M//_\mu G)_{(H)} := (M_{(H)} \cap \Phi^{-1}(\mu)) / G_\mu$$
is naturally a symplectic manifold [BL, SL]. Now if $x \in \Phi^{-1}(\mu)$ is a relative equilibrium and $G_x$ is the stabilizer of $x$, then $(M//_\mu G)_{(G_x)}$ is the stratum of the quotient $M//_\mu G$ containing $[x]$. Hence by Weinstein’s theorem if $h_\mu|_{(M//_\mu G)_{(G_x)}}$ has a positive definite Hessian at $[x]$ then the energy surface
$$\{h_\mu = h_\mu([x]) + E\}$$contains at least $\frac{1}{2} \dim(M//_\mu G)_{(G_x)}$ periodic orbits of $h_\mu|_{(M//_\mu G)_{(G_x)}}$ for all $E > 0$ sufficiently small.

The trivial observation above raises a natural question. Suppose the stratum containing the point $[x]$ is not open in the quotient $M//_\mu G$. Under suitable assumptions on the second partials of $h$ at a relative equilibrium $x$, must the set
$$\{h_\mu = h_\mu([x]) + E\} \subset M//_\mu G$$contain more periodic orbits of the flow of $h_\mu$ than $\frac{1}{2}(\dim M//_\mu G)_{(H)}$? For example, are there periodic orbits of $h_\mu$ in nearby strata? Theorem 1 in effect affirmatively answer the question in a special case: the stratum of $M//_\mu G$ passing through $[x]$ is a single point $\{[x]\}$.

Keeping in mind that $V = T_{x}M$, when $x$ is a fixed point of the action, note that the following is a special case of Theorem 1 above:

**Theorem 2.** Let $K \to Sp(V, \omega)$ denote a symplectic representation of a compact Lie group $K$ on a symplectic vector space $(V, \omega)$, let $\Phi : V \to \mathfrak{t}^*$ denote the associated homogeneous moment map. Let $h \in C^\infty(V)^K$ be an invariant function with $dh(0) = 0$ and the quadratic form $q := d^2h(0)$ positive definite. Then for every $E > 0$ sufficiently small there is a relatively periodic orbit of $h$ on the set
$$\{h = h(0) + E\} \cap \{\Phi = 0\},$$
provided the set in question is non-empty.

We will show that in fact Theorem 2 implies Theorem 1.

**Theorem 3.** Let $x$ be a relative equilibrium of a symmetric Hamiltonian system $(M,\omega,\Phi : M \to g^*, h \in C^\infty(M)^G)$. Denote the stabilizer of $x$ by $G_x$, the stabilizer of $\mu = \Phi(x)$ by $G_\mu$, the symplectic slice at $x$ by $V$ and the moment map associated to the symplectic representation of $G_x$ on $V$ by $\Phi_V$.

Then there exists a Hamiltonian $h_V \in C^\infty(V)^{G_x}$ with $dh_V(0) = 0$ so that for any $E \in \mathbb{R}$ sufficiently small and for any $G_x$-relatively periodic orbit of $h_V$ in

$$\{h_V = E\} \cap \Phi_V^{-1}(0)$$

sufficiently close to 0 there is a $G$-relatively periodic orbit of $h$ in

$$\{h = h(x) + E\} \cap \Phi^{-1}(\mu).$$

Moreover, if $x$ is a positive definite relative equilibrium of $h$, then $h_V$ can be chosen so that the Hessian $d^2h_V(0)$ is positive definite.

Clearly Theorem 2 and Theorem 3 together imply Theorem 1. We will then reduce the proof of Theorem 2 to

**Theorem 4.** Let $Q$ be a compact manifold with a contact form $\alpha$ whose Reeb flow generates a torus action. Then for any contact form $\beta C^2$-close to $\alpha$, the Reeb flow of $\beta$ has at least one periodic orbit.

**A note on notation.** Throughout the paper the Lie algebra of a Lie group denoted by a capital Roman letter will be denoted by the same small letter in the fraktur font: thus $g$ denotes the Lie algebra of a Lie group $G$ etc. The identity element of a Lie group is denoted by 1. The natural pairing between $g$ and $g^*$ will be denoted by $\langle \cdot, \cdot \rangle$.

When a Lie group $G$ acts on a manifold $M$ we denote the action by an element $g \in G$ on a point $x \in G$ by $g \cdot x$; $G \cdot x$ denotes the $G$-orbit of $x$ and so on. The vector field induced on $M$ by an element $X$ of the Lie algebra $g$ of $G$ is denoted by $X_M$. The isotropy group of a point $x \in M$ is denoted by $G_x$; the Lie algebra of $G_x$ is denoted by $g_x$ and is referred to as the isotropy Lie algebra of $x$. We recall that $g_x = \{X \in g \mid X_M(x) = 0\}$. The image of a point $x \in M$ in $M/G$ under the orbit map is denoted by $[x]$.

If $P$ is a principal $G$-bundle then $[p,m]$ denotes the point in the associated bundle $P \times_G M = (P \times M)/G$ which is the orbit of $(p,m) \in P \times M$.

If $\omega$ is a differential form on a manifold $M$ and $Y$ is a vector field on $M$, the contraction of $\omega$ by $Y$ is denoted by $\iota(Y)\omega$.

2. Reducing non-linear to linear: proof of Theorem 3

2.1. Facts about symplectic quotients. In this subsection we gather a few facts [SL, BL, LW] about symplectic quotients that we will need in the proof of Theorem 3. As we mentioned in the introduction, the symplectic quotient
at $\mu \in g^*$ for a proper Hamiltonian action of a Lie group $G$ on a symplectic manifold $(M, \omega)$ is the topological space

$$M//\mu G := \Phi^{-1}(\mu)/G\mu$$

where as before $\Phi : M \to g^*$ denotes the associated equivariant moment map. We define the set of smooth functions $C^\infty(M//\mu G)$ by

$$C^\infty(M//\mu G) = \{ f \in C^0(M//\mu G) \mid \pi^*_\mu f \in C^\infty(M)^G|_{\Phi^{-1}(\mu)} \} ,$$

where $\pi^*_\mu : \Phi^{-1}(\mu) \to M//\mu G$ denotes the orbit map. Note that since $\pi^*_\mu$ is surjective, given $h \in C^\infty(M)^G$ there is a unique $h_\mu \in C^\infty(M//\mu G)$ with

$$h|_{\Phi^{-1}(\mu)} = \pi^*_\mu h_\mu .$$

One often refers to $h_\mu$ as the reduction of the Hamiltonian $h$ at $\mu$.

**Theorem 2.1** (Arms-Cushman-Gotay [ACG]). The Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ induces a Poisson bracket $\{\cdot, \cdot\}_\mu$ on $C^\infty(M//\mu G)$ so that

$$\pi^*_\mu : C^\infty(M//\mu G) \to C^\infty(M)^G|_{\Phi^{-1}(\mu)}$$

is a Poisson map.

**Definition 2.2.** We say that a curve $\gamma : I \to M//\mu G$ is smooth ($C^\infty$) if $f \circ \gamma : I \to \mathbb{R}$ is smooth for any $f \in C^\infty(M//\mu G)$ ($I$, of course, is an interval). A curve $\gamma : I \to M//\mu G$ is an integral curve of a function $f \in C^\infty(M//\mu G)$ if for any $k \in C^\infty(M//\mu G)$ we have

$$\frac{d}{dt}(k \circ \gamma)(t) = \{f, k\}_\mu(\gamma(t)).$$

The following fact is an easy consequence of the well-known result that for proper group actions (the only kind of actions we consider) smooth invariant functions separate orbits:

**Proposition 2.3.** Integral curves of functions in $C^\infty(M//\mu G)$ are unique.

It is also not hard to see that Theorem 2.1 implies that if $\gamma$ is an integral curve of an invariant Hamiltonian $h \in C^\infty(M)^G$ lying in $\Phi^{-1}(\mu)$ then $\pi^*_\mu \circ \gamma$ is an integral curve of the corresponding reduced Hamiltonian $h_\mu \in C^\infty(M//\mu G)$ (as above $\pi^*_\mu h_\mu = h|_{\Phi^{-1}(\mu)}$). Combining this with Proposition 2.3 we get

**Lemma 2.4.** Let $h \in C^\infty(M)^G$ be an invariant Hamiltonian and $h_\mu \in C^\infty(M//\mu G)$ the corresponding reduced Hamiltonian at $\mu$. If $\gamma : I \to M//\mu G$ is an integral curve of $h_\mu$ there exists an integral curve $\tilde{\gamma} : I \to \Phi^{-1}(\mu)$ of $h$ so that

$$\pi^*_\mu \circ \tilde{\gamma} = \gamma .$$

Note that $\tilde{\gamma}$ is not unique: for any $a \in G\mu$ the curve $a \cdot \tilde{\gamma}$ is also an integral curve of $\gamma$ projecting down to $\gamma$. We will need one more fact about integral curves of functions in symplectic quotients, which is an easy consequence of Proposition 2.3.
Lemma 2.5. Suppose $\tau : M/\mu G \to M'/\mu' G'$ is a continuous map between two symplectic quotients such that the pull-back $\tau^*$ maps $C^\infty(M'/\mu' G')$ to $C^\infty(M/\mu G)$ preserving the Poisson brackets (i.e., $\tau$ is a morphism of symplectic quotients). Then for any $h' \in C^\infty(M'/\mu' G')$ if $\gamma$ is an integral curve of $\tau^*h'$ then $\tau \circ \gamma$ is an integral curve $h'$.

Proof. Since $\gamma$ is an integral curve of $\tau^*h'$
\[
\frac{d}{dt}((\tau^*f') \circ \gamma)(t) = \{\tau^*h', \tau^*f'\}_\mu(\gamma(t)).
\]
for any $f' \in C^\infty(M'/\mu' G')$. Hence
\[
\frac{d}{dt}f'(\tau \circ \gamma) = \{\tau^*h', \tau^*f'\}_\mu(\gamma) = \tau^*(\{h', f'\}_\mu)(\gamma) = \{h', f'\}_\mu'(\tau \circ \gamma).
\]

This ends our digression on the subject of symplectic quotients.

In proving Theorem 3 we will argue that there is a $G_x$-equivariant symplectic embedding $\sigma : V \to U$ of a $G_x$-invariant neighborhood $V$ of 0 in $V$ into a $G$-invariant neighborhood $U$ of $x$ in $M$ which induces a morphism
\[
\overline{\sigma} : V/\mu G_x \to U/\mu G
\]
of symplectic quotients so that $\overline{\sigma}$ embeds $V/\mu G_x$ as a connected component of $U/\mu G$. Note that for $\sigma$ to induce $\overline{\sigma}$ we would want $\sigma$ to map $\Phi_V^{-1}(0) \cap V$ into $\Phi^{-1}(\mu) \cap U$ in such a way that the diagram
\[
\begin{array}{ccc}
\Phi_V^{-1}(0) \cap V & \xrightarrow{\sigma} & \Phi^{-1}(\mu) \cap U \\
\pi_0 \downarrow & & \downarrow \pi_\mu \\
V/\mu G_x & \xrightarrow{\overline{\sigma}} & U/\mu G
\end{array}
\]
commutes, where $\pi_0$, $\pi_\mu$ are the respective orbit maps. As before given $f \in C^\infty(V)^{G_x}$ we denote by $f_0$ the unique function in $C^\infty(V/\mu G_x)$ with $f|_{\Phi_V^{-1}(0) \cap V} = \pi_0^*f_0$ and similarly $h_\mu \in C^\infty(U/\mu G)$ is determined by $\pi_\mu^*h_\mu = h|_{\Phi^{-1}(\mu)}$. Then the commutativity of (2.1) implies:
\[
(\sigma^*h)_0 = \overline{\sigma}^*h_\mu
\]
for any $h \in C^\infty(U)^G$.

Suppose next that we have constructed $\sigma : V \to U$ with the desired properties. Given $h \in C^\infty(M)^G$ and any $\xi \in g_\mu$ let
\[
h_V = \sigma^*(h - \langle \Phi, \xi \rangle).
\]
Then $h_V \in C^\infty(V)^{G_x}$. Moreover, since (2.1) commutes,
\[
(h_V)_0 = (\sigma^*(h - \langle \Phi, \xi \rangle))_0 = \overline{\sigma}^*(h - \langle \Phi, \xi \rangle)_\mu = \overline{\sigma}^*h_\mu - \langle \mu, \xi \rangle.
\]

In other words,
\[
(h_V)_0 = \overline{\sigma}^*h_\mu + \text{constant}.
\]
Lemma 2.5 and equation (2.2) imply that if $\gamma_V$ is an integral curve of $h_V$ in $V \cap \Phi_V^{-1}(0) \cap \{h_V = h_V(0) + E\}$ then $\pi_0 \circ \gamma_V$ is an integral curve of $(h_V)_0$ in $\{(h_V)_0 = (h_V(\pi_0(0))) + E\}$. Hence $\tilde{\sigma} \circ \pi_0 \circ \gamma_V$ is an integral curve of $h_{\mu}$ in $\{h_{\mu} = h_{\mu}(\pi_{\mu}(x)) + E\}$. It follows that there is an integral curve $\gamma$ of $h$ in $\{h = h(x) + E\} \cap U \cap \Phi^{-1}(\mu)$ with

$$\pi_{\mu} \circ \gamma = \tilde{\sigma} \circ \pi_0 \circ \gamma_V.$$ 

If $\gamma_V$ is a $G_x$-relative periodic orbit of $h_V$ then $\pi_0 \circ \gamma_V$ is a periodic orbit of $(h_V)_0$. Consequently $\gamma$ is a $G$-relative periodic orbit of $h$.

Finally note that if additionally we can arrange for

$$d\sigma_0(T_{0V}) \subset T_x(G \cdot x)^{\omega},$$

then since $\sigma$ is symplectic

$$d\sigma_0(T_{0V}) \cap T_x(G_\mu \cdot x) = \{0\}.$$

Consequently, if $\xi \in g_\mu$ is such that $d(h - \langle \Phi, \xi \rangle)(x) = 0$ and $d^2(h - \langle \Phi, \xi \rangle)(x)|_{T_x(G \cdot x)^{\omega}}$ is positive semi-definite of maximal rank, then

$$d^2(h - \langle \Phi, \xi \rangle)(x)|_{d\sigma_0(T_{0V})}$$

is positive definite. Therefore the Hessian

$$d^2(\sigma^*(h - \langle \Phi, \xi \rangle))(0)$$

is positive definite as well. We conclude that in order to prove Theorem 3 it is enough to construct the embedding $\sigma$ with the desired properties. In other words it is enough to prove:

**Proposition 2.6.** Let $(M, \omega, \Phi : M \to g^*)$ be a symplectic manifold with a proper Hamiltonian action of a Lie group $G$. Fix a point $x$ in $M$. Let $\mu = \Phi(x)$, let $\Phi_V : V \to g_\mu^*$ be the homogeneous moment map associated with the symplectic slice representation $G_x \to \text{Sp}(V, \omega_V)$. There exists a $G_x$-invariant neighborhood $V$ of $0$ in $V$, a $G$-invariant neighborhood $U$ of $x$ in $M$ and a $G_x$-equivariant embedding $\sigma : V \to U$ with

$$\sigma(\Phi_V^{-1}(0) \cap V) \subset \Phi^{-1}(\mu) \cap U$$

such that the composition

$$\Phi_V^{-1}(0) \cap V \overset{\sigma}{\to} \Phi^{-1}(\mu) \cap U \overset{\pi_{\mu}}{\to} (\Phi^{-1}(\mu) \cap U)/G_\mu = U//G$$

drops down to

$$\tilde{\sigma} : V//_0G_x = (\Phi_V^{-1}(0) \cap V)/G_x \to U//_\mu G$$

making (2.1) commute. Moreover,

1. $\tilde{\sigma}(V//_0G_x)$ is a connected component of $U//_\mu G$ and $\tilde{\sigma}$ is a homeomorphism onto its image;
2. the pull-back $\tilde{\sigma}^*$ sends $C^\infty(U//_\mu G)$ isomorphically to $C^\infty(V//_0G_x)$ (as Poisson algebras).
Additionally
\[ d\sigma_0(T_0V) \subset T_x(G \cdot x)^\omega. \]

Proposition 2.6 will follow from

**Proposition 2.7.** As above let \((M, \omega, \Phi : M \rightarrow g^*)\) be a symplectic manifold with a proper Hamiltonian action of a Lie group \(G\). Fix a point \(x \in M\). Let \(\mu = \Phi(x)\), let \(\Phi_V : V \rightarrow g_x^*\) be the homogeneous moment map associated with the symplectic slice representation \(G_x \rightarrow \text{Sp}(V, \omega_V)\). There exists a slice \(\Sigma\) at \(x\) for the action of \(G\) on \(M\), a \(G_x\)-invariant neighborhood \(V\) of \(0\) in \(V\) and a \(G_x\)-equivariant embedding \(\sigma : V \rightarrow \Sigma\) so that

1. \(\sigma(V)\) is closed in \(\Sigma\);
2. \(\sigma(\Phi^{-1}_V(0) \cap V) = \Sigma \cap \Phi^{-1}(\mu)\);
3. \(\sigma^*\omega = \omega_V\) and \(\Phi \circ \sigma = i \circ \Phi_V + \mu\) where \(i : g_x^* \rightarrow g^*\) is a \(G_x\)-equivariant injection;
4. \(G_{\mu} \cdot (\Sigma \cap \Phi^{-1}(\mu))\) is a connected component of \(\Phi^{-1}(\mu) \cap U\), where \(U = G \cdot \Sigma\).

**Proof of Proposition 2.7.** Our proof of Proposition 2.7 uses the Bates-Lerman version [BL][pp. 212–215] of the local normal form theorem for moment maps.

**Theorem 2.8 ([BL]).** Let \((M, \omega)\) be a symplectic manifold with a proper Hamiltonian action of a Lie group \(G\) and a corresponding equivariant moment map \(\Phi : M \rightarrow g^*\). Fix \(x \in M\), let \(\mu = \Phi(x)\), \((V, \omega_V)\) the symplectic slice at \(x\), \(\Phi_V : V \rightarrow g_x^*\) the associated homogeneous moment map. Choose a \(G_x\)-equivariant splitting
\[ g^* = g_x^* \oplus (g_\mu / g_x)^* \oplus g_\mu^0 \]
\((g_\mu^0\) denotes the annihilator of \(g_\mu\) in \(g^*)\) and thereby \(G_x\)-equivariant injections
\[ i : g_x^* \rightarrow g^*, ~ j : (g_\mu / g_x)^* \rightarrow g^*. \]

Let
\[ Y = G \times_{G_x} ((g_\mu / g_x)^* \times V); \]

it is a homogeneous vector bundle over \(G/G_x\). There exists a closed 2-form \(\omega_Y\) on \(Y\) which is non-degenerate in a neighborhood of the zero section \(G \times_{G_x} \emptyset\) \(\{0,0\}\) \((0\) denotes the identity in \(G\)) such that

1. a \(G\)-invariant neighborhood of \(G \cdot x\) in \((M, \omega)\) is \(G\)-equivariantly symplectomorphic to a neighborhood of \(G \cdot [1,0,0]\) in \((Y, \omega_Y)\);
2. the moment map \(\Phi_Y\) for the action of \(G\) on \((Y, \omega_Y)\) is given by
\[ \Phi_Y([g, \eta, v]) = g \cdot (\mu + j(\eta) + i(\Phi_V(x))) \]
for all \((g, \eta, v) \in G \times (g_\mu / g_x)^* \times V; \]
3. the embedding \(\iota : V \rightarrow Y, \iota(v) = [1,0,v]\) is symplectic: \(\iota^*\omega_Y = \omega_V\).
Hence, we can assume without loss of generality that \((M, \omega, \Phi) = (Y, \omega_Y, \Phi_Y)\) and \(x = [1, 0, 0]\). Note that the embedding \(\kappa: (\mathfrak{g}_\mu / \mathfrak{g}_x)^* \times V \rightarrow Y, \kappa(\eta, v) = [1, \eta, v]\) is a slice at \(x\) for the action of \(G\) on \(Y\).

We now argue that for a small enough \(G_x\)-invariant neighborhood \(V\) of 0 in \(V\) and a small enough \(G_x\)-invariant neighborhood \(W\) of 0 in \((\mathfrak{g}_\mu / \mathfrak{g}_x)^*\)

\[ \Sigma := \kappa(W \times V) \subset Y \]

is the desired slice and

\[ \sigma = \kappa|_{\{0\} \times V}: V \rightarrow \Sigma, \quad \sigma(v) = [1, 0, v] \]

is the desired embedding.

Note that no matter how \(V\) and \(W\) are chosen we automatically have that \(\sigma(V)\) is closed in \(\Sigma\) and \(\sigma^* \omega_Y = \omega_Y\). Hence \(\sigma(V)\) is closed in \(U := G \cdot \Sigma\), which is a \(G\)-invariant neighborhood of \(x\). Note also that

\[ \Phi_Y \circ \sigma(v) = \Phi_Y([1, 0, v]) = \mu + i(\Phi_V(v)). \]

Next we make our choice of \(V\) and \(W\) and prove that the resulting embedding \(\sigma\) has all the desired properties. For this purpose, factor \(\Phi_Y : Y \rightarrow \mathfrak{g}^*\) as a sequence of maps (we identify \(\mathfrak{g}_\mu^*\) with \(j((\mathfrak{g}_\mu / \mathfrak{g}_x)^*) \oplus i(\mathfrak{g}_x^*) \subset \mathfrak{g}^*\):

\[ G \times G_x ((\mathfrak{g}_\mu / \mathfrak{g}_x)^* \times V) \xrightarrow{F_1} G \times G_x ((\mathfrak{g}_\mu / \mathfrak{g}_x)^* \times \mathfrak{g}_x^*) \xrightarrow{F_2} G \times \mathfrak{g}_\mu \xrightarrow{\mathcal{E}} \mathfrak{g}^*, \]

where

\[ F_1([g, \eta, v]) = [g, \eta, \Phi_V(v)], \]

\[ F_2([g, \eta, \nu]) = [g, j(\eta) + i(\nu)] \]

and

\[ \mathcal{E}(g, \partial) = g \cdot (\mu + \partial). \]

Since the tangent space \(T_\mu(G \cdot \mu)\) is canonically isomorphic to the annihilator \(\mathfrak{g}_\mu^0\)

and since \(\mathfrak{g}^* = \mathfrak{g}_\mu^* \oplus \mathfrak{g}_x^*\), the vector bundle \(G \times \mathfrak{g}_\mu^*\) is the normal bundle for the embedding \(G \cdot \mu \hookrightarrow \mathfrak{g}^*\) and \(\mathcal{E}: G \times \mathfrak{g}_\mu^* \rightarrow \mathfrak{g}^*\) is the exponential map for a flat \(G_x\)-invariant metric on \(\mathfrak{g}_x^*\). Therefore \(\mathcal{E}\) is a local diffeomorphism near the zero section. In particular there is a \(G_x\)-invariant neighborhood \(\mathcal{O}\) of \([1, 0] \in G \times \mathfrak{g}_\mu^*\) so that \(\mathcal{E}|_\mathcal{O}\) is a diffeomorphism onto its image. Let \(\mathcal{O}' = (F_2 \circ F_1)^{-1}(\mathcal{O})\). Then

\[ \mathcal{O}' \cap \Phi_Y^{-1}(\mu) = \mathcal{O}' \cap (F_2 \circ F_1)^{-1}((\mathcal{E}|_\mathcal{O})^{-1}(\mu)) = \mathcal{O}' \cap F_1^{-1}(F_2^{-1}([1, 0])) = \mathcal{O}' \cap F_1^{-1}(G_\mu \times G_x \{0, 0\}) = \mathcal{O}' \cap G_\mu \times G_x (\{0\} \times \Phi_V^{-1}(0)). \]

We may take \(\mathcal{O}\) to be of the form \(A \times G_x (W \times V')\) where \(A \subset G\) is a \(G_x \times G_\mu\)-invariant neighborhood of 1, \(W \subset (\mathfrak{g}_\mu / \mathfrak{g}_x)^*\) is a \(G_x\)-invariant neighborhood of 0 and \(V' \subset \mathfrak{g}_x^*\) is a convex \(G_x\)-invariant neighborhood of 0. We take \(V = \Phi_V^{-1}(V')\).

Then \(V \cap \Phi_V^{-1}(0)\) is connected. With the choices above, \(\mathcal{O}' = A \times G_x (W \times V)\) and

\[ \mathcal{O}' \cap \Phi_Y^{-1}(\mu) = (A \times G_x (W \times V)) \cap (G_\mu \times G_x (\{0\} \times (\Phi_V^{-1}(0) \cap V))). \]
Since $G_\mu \times G_x (\{0\} \times (\Phi^{-1}_V(0) \cap V))$ is closed in $G \times G_x (\mathcal{W} \times \mathcal{V}) = G \cdot \Sigma = \mathcal{U}$ and
since $\Phi^{-1}_V(0) \cap V$ is connected, $G_\mu \cdot \sigma(\Phi^{-1}_V(0) \cap V) = G_\mu \times G_x (\{0\} \times (\Phi^{-1}_V(0) \cap V))$
is a connected component of $\Phi^{-1}_V(0) \cap \mathcal{U}$. This proves property (4).

Note that $\Sigma = \{[1, \eta, v] \mid \eta \in \mathcal{W}, v \in \mathcal{V}\} \subset \mathbb{A} \times G_x (\mathcal{W} \times \mathcal{V})$. Hence $\Phi^{-1}_V(0) \cap \Sigma = (\Phi^{-1}_V(\mu) \cap \mathcal{M}) \cap \Sigma = \{[1, 0, v] \mid v \in \Phi^{-1}_V(0) \cap \mathcal{V}\}$, i.e.,

$$\sigma(\Phi^{-1}_V(0) \cap \mathcal{V}) = \Phi^{-1}_V(\mu) \cap \Sigma,$$

which proves property (2) and thereby finishes the proof of Proposition 2.7. $\square$

**Proof of Proposition 2.6.** We continue to use the notation above. Since $\sigma(\Phi^{-1}_V(0) \cap \mathcal{V}) = \Phi^{-1}_V(\mu) \cap \Sigma$ and since $\sigma : \mathcal{V} \to \Sigma$ is a closed embedding, the restriction $\sigma|_{\Phi^{-1}_V(0) \cap \mathcal{V}} : \Phi^{-1}_V(0) \cap \mathcal{V} \to \Phi^{-1}_V(\mu) \cap \Sigma$ is a $G_\mu$-equivariant homeomorphism. Hence

$$\tilde{\sigma} : (\Phi^{-1}_V(0) \cap \mathcal{V})/G_x \to (\Phi^{-1}_V(\mu) \cap \Sigma)/G_x$$
is a homeomorphism as well. Since $\Sigma$ is a slice at $x$ for the action of $G$ on $M$, it is also a slice for the action of $G_\mu$. Consequently

$$(G_\mu \cdot (\Phi^{-1}_V(\mu) \cap \Sigma))/G_\mu \cong (\Phi^{-1}_V(\mu) \cap \Sigma)/G_x.$$

Since $G_\mu \cdot (\Phi^{-1}_V(\mu) \cap \Sigma)$ is a component of $\Phi^{-1}_V(\mu) \cap \mathcal{U}$, $\tilde{\sigma} : \mathcal{V}/G_x \to \mathcal{U}/G_\mu$ is a homeomorphism onto its image. Moreover, the diagram (2.1) commutes.

We now argue that $\tilde{\sigma}$ pulls back the smooth functions in $C^\infty(\mathcal{V}/G_x)$ to smooth functions in $C^\infty(\mathcal{U}/G_\mu)$ and that the pull-back is an isomorphism of Poisson algebras. Since $\Sigma$ is a slice and $\mathcal{U} = G \cdot \Sigma$, the restriction

$$C^\infty(\mathcal{U})^G \to C^\infty(\Sigma)^{G_x}, \ f \mapsto f|_\Sigma$$
is a bijection. Since $\sigma(\Phi^{-1}_V(0) \cap \mathcal{V}) \subset \Phi^{-1}_V(\mu) \cap \mathcal{U}$ and since

$$C^\infty(\mathcal{U})^G \quad \xrightarrow{\sigma^*} \quad C^\infty(\mathcal{V})^{G_x}$$

commutes, where the vertical arrows are restrictions, and since the top arrow is surjective, the bottom arrow is surjective as well. Since $\Sigma$ is a slice and $\mathcal{U} = G \cdot \Sigma$, any function $f \in C^\infty(\mathcal{U})^G|_{\Phi^{-1}_V(\mu) \cap \mathcal{U}}$ is uniquely defined by its values on $\Phi^{-1}_V(\mu) \cap \Sigma = \sigma(\Phi^{-1}_V(0) \cap \mathcal{V})$. Hence the bottom arrow $\sigma^*$ is also

injective. Since $\sigma : \mathcal{V} \to \mathcal{U}$ is symplectic, $\sigma^* : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{V})$ is Poisson. Hence $\sigma^* : C^\infty(\mathcal{U})^G \to C^\infty(\mathcal{V})^{G_x}$ is also Poisson. Consequently $\sigma^* : C^\infty(\mathcal{U})^G|_{\Phi^{-1}_V(\mu) \cap \mathcal{U}} \to C^\infty(\mathcal{V})^{G_x}|_{\Phi^{-1}_V(0) \cap \mathcal{V}}$ is Poisson as well. We conclude that

$$\sigma^* : C^\infty(\mathcal{U}/G_\mu) \to C^\infty(\mathcal{V}/G_x)$$
is an isomorphism of Poisson algebras.

Finally since $\Phi \circ \sigma = i \circ \Phi_V + \mu$,

$$d\Phi_x \circ d\sigma_0 = i \circ d(\Phi_V)_0.$$
Since $\Phi_V$ is quadratic homogeneous, $d(\Phi_V)_0 = 0$. Hence
\[ d\sigma_0(T_0 V) \subset \ker d\Phi_x = T_x(G \cdot x)^w. \]

This concludes our proof of Theorem 3 as well.

3. From invariant Hamiltonians on vector spaces to Reeb flows:

Theorem 4 implies Theorem 2

**Remark 3.1.** Theorem 2 is easily seen to be true in a special case: the set $V^K$ of $K$-fixed vectors is a subspace of $V$ of positive dimension. Indeed, since $h$ is $K$-invariant its Hamiltonian flow preserves the symplectic subspace $V^K$, which is contained in the zero level set $\Phi^{-1}(0)$ of the moment map. Moreover the flow of $h$ in $V^K$ is the Hamiltonian flow of the restriction $h|_{V^K}$. Hence Weinstein’s theorem applied to the Hamiltonian system $(V^K, \omega|_{V^K}, h|_{V^K})$ guarantees that for any $E > 0$ sufficiently small there are at least $\frac{1}{2} \dim V^K$ periodic orbits of $h|_{V^K}$ in the surface
\[ \{ h|_{V^K} = E \} = \{ h = E \} \cap V^K \subset \{ h = E \} \cap \{ \Phi = 0 \}. \]

To show that Theorem 4 implies Theorem 2 we first need to digress on the subject of contact quotients.

3.1. Facts about contact quotients. Suppose that a Lie group $G$ acts properly on a manifold $\Sigma$, preserving a contact form $\beta$. The associated moment map $\Psi: \Sigma \to \mathfrak{g}^*$ is defined by
\[ \langle \Psi(x), \xi \rangle = \beta_x(\xi_M(x)) \]
for all $x \in \Sigma$, all $\xi \in \mathfrak{g}^*$. The map $\Psi$ is $G$-equivariant. The contact quotient at zero $\Sigma//G$ is, by definition, the set
\[ \Sigma//G := \Psi^{-1}(0)/G. \]
Just as in the case of symplectic quotients the contact quotients are stratified spaces [LW] with the stratification induced by the orbit type decomposition:
\[ \Sigma//G := \bigsqcup_{H < G} (\Psi^{-1}(0) \cap \Sigma(H))/G, \]
where the disjoint union is taken over conjugacy classes of subgroups of $G$. Additionally each stratum
\[ (\Sigma//G)(H) := (\Psi^{-1}(0) \cap \Sigma(H))/G \]
is a contact manifold and the contact form $\beta(H)$ on each stratum is induced by the contact form $\beta$ on $\Sigma$ [Wi, Theorem 3, p. 4256]. More precisely for each subgroup $H$ of $G$ the set $\Psi^{-1}(0) \cap \Sigma(H)$ is a manifold and
\[ \pi^*_H(\beta(H)) = \beta|_{\Psi^{-1}(0) \cap \Sigma(H)}, \]
where $\pi_H: \Psi^{-1}(0) \cap \Sigma(H) \to (\Psi^{-1}(0) \cap \Sigma(H))/G = (\Sigma//G)(H)$ is the orbit map. It is not hard to see that the flow of the Reeb vector field $X$ of $\beta$ preserves
the moment map and the orbit type decomposition, hence descends to a strata-preserving flow on the quotient $\Sigma//G$. Also, on each stratum the induced flow is the Reeb flow of the induced contact form $\beta_H$.

We're now ready to prove that Theorem 4 implies Theorem 2. It is no loss of generality to assume that $h(0) = 0$. Since the quadratic form $q = d^2h(0)$ is positive definite, the energy surface $\{q = E\}$, $E > 0$, is a $K$-invariant hypersurface star-shaped about 0. Hence

$$\alpha_E := \iota(R)\omega|_{\{q = E\}}$$

is a $K$-invariant contact form, where $R(v) = v$ denotes the radial vector field on $V$. For $E > 0$ sufficiently small, the $K$-invariant set

$$\{h = E\}$$

is a hypersurface which is $C^2$-close to $\{q = E\}$. Hence

$$\beta_E := \iota(R)\omega|_{\{h = E\}}$$

is also a $K$-invariant contact form. By the implicit function theorem, for $E > 0$ sufficiently small, there is a function $f : \{q = E\} \rightarrow (0, \infty)$, which is $C^2$ close to 1, so that

$$\phi : \{q = E\} \rightarrow \{h = E\} \quad \phi(x) = f(x)x$$

is a $K$-equivariant diffeomorphism. Since $\phi^*\beta_E = f^2\alpha_E$, the manifolds $\{q = E\}$ and $\{h = E\}$ are $K$-equivariantly contactomorphic. Moreover, under the identification $\phi$ the two contact forms $\alpha_E$ and $\beta_E$ are $C^2$-close (again, provided $E$ is small). Note that the two associated contact moment maps are $\Phi|_{\{q = E\}}$ and $\Phi|_{\{h = E\}}$ respectively.

Up to re-parameterization the integral curves of the Hamiltonian vector field of $h$ in $\{h = E\}$ are the integral curves of the Reeb vector field $X_E$ of $\beta_E$. Similarly the integral curves of the Hamiltonian vector field of $q$ on $\{q = E\}$ are the integral curves of the Reeb vector field $Y_E$ of $\alpha_E$. In particular the relatively periodic orbits of $h$ on $\{h = E\}$ are relatively periodic orbits of $X_E$. Since the hypersurface $\{h = E\}$ is compact and since the orbit type decomposition of the contact quotient $\{h = E\}//G$ is a stratification, the minimal strata of the quotient are compact. Let $Q = (\{h = E\}//G)|_{(L)}$ be one such stratum. Then the relatively periodic orbits of $h$ in $\{h = E\} \cap \Phi^{-1}(0) \cap V_L$ descend to periodic orbits of the Reeb vector field $X$ of the contact form $\beta_L$ on $Q = (\{h = E\} \cap \Phi^{-1}(0) \cap V_L)//G$. Therefore to prove Theorem 2 it is enough to establish the existence of periodic orbits of $X$. For this, according to Theorem 4, it suffices to establish the existence of a contact form $\alpha$ on $Q$ whose Reeb vector field $Y$ generates a torus action and such that $\beta_L$ is $C^2$ close to $\alpha$ when $E > 0$ is small enough.
The form \( \alpha \), of course, is the form induced by \( \alpha_E \). Let us prove that it does have the desired properties. Since \( \phi : \{ q = E \} \to \{ h = E \} \) is an equivariant contactomorphism it induces an identification of the contact manifold \( Q \) with \( \{ q = E \} \cap \Phi^{-1}(0) \cap V(L) \). Moreover, since \( \alpha_E \) and \( \beta_E \) are \( C^2 \)-close, the induced forms \( \beta(L) \) and \( \alpha = \alpha(L) \) are \( C^2 \)-close as well. Since \( q \) is definite, its Hamiltonian flow generates a linear symplectic action of a torus \( T \) on \( V \). The restriction of this action to \( \{ q = E \} \) is also generated by the Reeb vector field \( Y_E \) of \( \alpha_E \). Since \( q \) is \( K \)-invariant, the action of \( T \) commutes with the action of \( K \) and preserves the moment map \( \Phi \). Hence it descends to an action of \( T \) on \( Q \).

4. Perturbations of Reeb flows: proof of Theorem 4

In the proof of Theorem 4 we will need the following elementary result.

**Lemma 4.1.** Let \( \phi_t \) be a dense one-parameter subgroup in a torus \( T \) and let \( H \) be a subgroup of \( T \) topologically generated by an element \( \phi^\tau \), \( \tau > 0 \). Then either \( H \) has codimension one in \( T \) or \( H = T \).

**Proof of Lemma 4.1.** It suffices to show that the map

\[
[0, \tau] \times H \to T, \quad F(t, h) = \phi^t \cdot h
\]

is onto \( T \). Pick \( g \in T \). Assume first that \( g \) is in the one-parameter subgroup, i.e., \( g = \phi^t \) for some \( t \). Then we have \( t = k\tau + t' \) with \( 0 \leq t' < \tau \) and, clearly,

\[
g = \phi^{t' \cdot [(\phi^\tau)^k]} = F(t', (\phi^\tau)^k).
\]

Hence \( g \) is in the image of \( F \).

Let now \( g \) be in \( T \), but not in the one-parameter subgroup \( \phi^t \). Then there exists a sequence \( t_r \to \pm \infty \) such that \( \phi^{t_r} \to g \). (This sequence must go to positive or negative infinity, for otherwise \( g \) would be in the one-parameter subgroup.) Assume that \( t_r \to \infty \); the case of negative infinity can be dealt with in a similar fashion. As above, we write

\[
t_r = k_r \tau + t'_r,
\]

where \( k_r \to \infty \) as \( r \to \infty \) and \( 0 < t'_r < \tau \).

The elements \( (\phi^\tau)^{k_r} \) are in \( H \) and, since \( H \) is compact, we may assume that \( (\phi^\tau)^{k_r} \to h \in H \) by passing if necessary to a subsequence. Furthermore, by passing if necessary to a subsequence again, we may assume that \( t'_r \to t' \in [0, \tau] \).

We claim now that \( g = F(t', h) \). To see this note that as above

\[
\phi^{t_r} = \phi^{k_r \cdot \tau + t'_r} = \phi^{t'_r \cdot [(\phi^\tau)^{k_r}]}.
\]

As \( r \) goes to infinity, \( \phi^{t_r} \to \phi^{t'} \) and the second term goes to \( h \). Hence,

\[
g = \phi^{t'} \cdot h = F(t', h).
\]

This completes the proof of the lemma.
Proof of Theorem 4. First, let us set notation. We denote by $X$ the Reeb vector field of $\alpha$ and by $\phi$ its Reeb flow. By the hypotheses of the theorem, the flow $\phi$ generates an action of a torus $T$ on $Q$. We will view $\phi$ as a dense one-parameter subgroup of $T$. The points on periodic orbits of $X$ will be referred to as periodic points. We break up the proof of the theorem into four steps. Steps 1–3 concern exclusively properties of the Reeb flow of $\alpha$. The perturbed form $\beta$ enters the proof only at the last step.

1. We claim that the periodic points of $X$ are exactly the points $x \in Q$ whose stabilizers $T_x$ have codimension one in $T$.

   Indeed, let $x \in Q$ be a periodic point, i.e., $\phi^T(x) = x$ for some $T > 0$. Since $\phi$ is dense in $T$, the Reeb orbit through $x$ is dense in the $T$-orbit through $x$. Since $x$ is a periodic point, the Reeb orbit is closed and thus equal to the $T$-orbit. Hence, $T/T_x$ is a circle and thus $T_x$ has codimension one. Conversely, if $T_x$ has codimension one, the Reeb orbit through $x$ must be dense in the $T$-orbit and hence equal to the $T$-orbit because the latter is a circle.

2. Let now $N$ be a minimal stratum of the $T$-action, which is comprised entirely of periodic points. We claim that such a stratum exists, is a smooth submanifold, and all points of $N$ have the same period, i.e., the $T$-action on $N$ factors through a free circle action.

   Since the $T$-action has no fixed points, periodic points lie in minimal strata of the action. Furthermore, the Reeb flow of $\alpha$ has at least one periodic orbit (in fact, at least two unless $Q$ is a circle); this follows, for example, from a theorem of Banyaga and Rukimbira, [BR]. Now it suffices to take as $N$ a minimal stratum containing a periodic point. The fact that $N$ is smooth is a general result about compact group actions. Finally, all points in $N$ have the same stabilizer $T_x$ and the action of the circle $T/T_x$ on $N$ is free because $N$ is minimal. The period $T$ of $x \in N$ is the first $T > 0$ such that $\phi^T \in T_x$.

3. We claim that $N$ is a non-degenerate invariant submanifold for the Reeb flow of $\alpha$.

   Let $x \in N$. We need to show that the linearization $d\phi^T$ on the normal space $\nu_x$ to $N$ at $x$ does not have unit as an eigenvalue. By definition, this linearization is just the linearized action of $\phi^T \in T_x$ on $\nu_x$. As is well known, the isotropy representation of $T_x$ on $\nu_x$ contains no trivial representations in its decomposition into the sum of irreducible representations. Hence, it suffices to show that the subgroup generated by $\phi^T$ is dense in $T_x$.

   Let $m$ be the first positive integer such that $(\phi^T)^m$ is in $T^0_x$, the connected component of identity in $T_x$. (Such an integer $m$ exists because $T_x/T^0_x$ is a finite subgroup of the circle $T/T^0_x$.) Since $T_x/T^0_x$ is finite cyclic, it suffices to show that the subgroup $H$ topologically generated by $\phi^m$ is equal to $T^0_x$. This follows immediately from Lemma 4.1. Indeed, by the lemma, the group $H$ is either equal to $T$ or has codimension one in $T$. Since $H \subset T^0_x$ and $T^0_x$ has codimension one, the group $H$ must have codimension one. Thus $H$ is a closed subgroup of $T^0_x$ of the same dimension as $T^0_x$ and hence $H = T^0_x$. 

4. Now we invoke the following theorem due to Kerman [K] (p. 967). Let \( \text{Crit}(P) \) be the minimal possible number of critical points of a smooth function on a compact manifold \( P \).

**Theorem 4.2** (Kerman, [K]). Let \( Q \) be a compact odd-dimensional manifold, \( X \) a non-vanishing vector field on \( Q \), and \( N \) a non-degenerate periodic submanifold of \( X \). Let \( \Omega \) be a closed maximally non-degenerate two-form on \( Q \) whose kernel is \( C^2 \)-close to \( X \) and such that the class \([\Omega|_N]\) is in the image of the pull-back from \( H^2(N/S^1) \) to \( H^2(N) \). Then \( \Omega \) has at least \( \text{Crit}(N/S^1) \) closed characteristics near \( N \).

Applying this theorem to \( Q, N \) and \( X \) as above, and \( \Omega = d\beta \) we obtain the required result.

**Remark 4.3.** In fact, our proof of Theorem 4 establishes the existence of two distinct periodic orbits when \( Q \) is not a circle. As a consequence, in the setting of Theorems 1 and 2 there exist at least two distinct relative periodic orbits unless \( Q \) is a circle.

**References**


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