The aim of this note is to prove the existence of invariant Ricci-flat Kähler metrics on complexifications of symmetric spaces of compact type. Before stating the result, let us fix the notation.

Let $(M, g)$ be a Riemannian symmetric space of compact type and $p$ a point in $M$. Let $G$ be the identity component of the isometry group of $(M, g)$ and let $K$ be the stabiliser of $p$ in $G$. Then $M \simeq G/K$ and the complexification of $M$ is $TM$ with the adapted complex structure [7] that can be identified with $G^C/K^C$. We are going to prove

**Theorem 1.** Let $(M, g)$ be an irreducible symmetric space of compact type. Let $G$ and $K$ be as above and suppose that $K$ is connected. Let $\rho$ be a real exact $G$-invariant $(1, 1)$-form on the complexification $TM \simeq G^C/K^C$. Then there exists a $G$-invariant Kähler metric on $TM$ whose Ricci form is $\rho$.

Remark. The Kähler form obtained in Theorem 1 is exact.

The above result has been proved in [9] for symmetric spaces of rank 1 and in [2] for compact groups, i.e. for the case when $G = K \times K$ and $K$ acts diagonally. For hermitian symmetric spaces and $\rho = 0$, Theorem 1 has also been known [4].

The proof given here is quite different from that given for group manifolds in [2]. We show that the complex Monge-Ampère equation on $G^C/K^C$ reduces, for $G$-invariant functions, to a real Monge-Ampère equation on the dual symmetric space $G^*/K$. We also show that the Monge-Ampère operator on non-compact symmetric spaces has a radial part, i.e. it is equal, for $K$-invariant functions, to another Monge-Ampère operator on the maximal abelian subspace of $G^*/K$. These facts, together with the theorem on $K$-invariant real Monge-Ampère equations proved in [3], yield Theorem 1.

1. Riemannian symmetric spaces of non-compact type

Here we recall some facts about the geometry of Riemannian symmetric spaces. The standard reference for this section is [6].

Let $M = G/K$ be a symmetric space of compact type with $K$ connected, and let $G^*/K$ be its dual. If $\mathfrak{g}$, $\mathfrak{g}^*$ and $\mathfrak{k}$ denote the Lie algebras of $G$, $G^*$ and

---

1 The complexification of a compact connected Lie group $G$ is the connected group $G^*$ whose Lie algebra is the complexification of the Lie algebra of $G$ which satisfies $\pi_1(G^*) = \pi_1(G)$. 

---

Received February 25, 2004.
K, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The restriction of the Killing form to $i\mathfrak{p}$ is positive definite and induces the Riemannian metric of $G^*/K$. Moreover, the Riemannian exponential mapping provides a diffeomorphism between $\mathfrak{p}$ and $G^*/K$. This can be viewed as the map:

$$p \mapsto e^{ip} K,$$

where $p \in \mathfrak{p}$ and $e$ is the group-theoretic exponential map for $G^*$. Thus we have two $K$-invariant Riemannian metrics on $\mathfrak{p} \simeq \mathbb{R}^n$: the Euclidean one given by the Killing form, and the negatively curved one given by the diffeomorphism (1.1).

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{l}$ its centraliser in $\mathfrak{k}$. Let $\Sigma$ be the set of restricted roots and $\Sigma^+$ the set of restricted positive roots. For each $\alpha \in \Sigma$, let $\mathfrak{p}_\alpha$ (resp. $\mathfrak{k}_\alpha$) denote the subspace of $\mathfrak{p}$ (resp. of $\mathfrak{k}$) where each $(\text{ad} H)_{\alpha}$, $H \in \mathfrak{a}$, acts with eigenvalue $\alpha(H)^2$. We have the direct decompositions

$$p = \mathfrak{a} + \sum_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha, \quad k = \mathfrak{l} + \sum_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha.$$  (1.2)

Let $\mathfrak{a}^+$ be an open Weyl chamber and let $\mathfrak{p}'$ be the union of $K$-orbits of points in $\mathfrak{a}^+$. Any $K$ orbit in $\mathfrak{p}'$ is isomorphic to $K/L$ where the Lie algebra of $L$ is $\mathfrak{l}$. Moreover, we have the diffeomorphism:

$$\mathfrak{a}^+ \times K/L \rightarrow \mathfrak{p}', \quad (h, k) \mapsto \text{Ad}(k)h.$$  (1.3)

We now wish to write the two $K$-invariant metrics on $\mathfrak{p}$ in coordinates given by this diffeomorphism. Let $\sum dr_i^2$ be the Killing metric on $\mathfrak{a}^+$ (the $r_i$ can be viewed as $K$-invariant functions on $\mathfrak{p}'$). For each $\mathfrak{k}_\alpha$, choose a basis $X_{\alpha,m}$ ($m$ runs from 1 to twice the multiplicity of $\alpha$) of vectors orthonormal for the Killing form and denote by $\theta_{\alpha,m}$ the corresponding basis of invariant 1-forms on $K/L$. We have

**Proposition 1.1.** Let $g_0$ be the Euclidean metric on $\mathfrak{p}$, given by the restriction of the Killing form, and let $g$ be the negatively curved symmetric metric on $\mathfrak{p}$ given by the diffeomorphism (1.1). Then, under the diffeomorphism (1.3) the metrics $g_0$ and $g$ can be written in the form

$$\sum_i dr_i^2 + \sum_{(\alpha,m)} F(\alpha(r))\theta^2(\alpha,m),$$  (1.4)

where $F(z) = z^2$ for $g_0$, and $F(z) = \sinh^2(z)$ for $g$.

**Proof.** Since all these metrics are $K$-invariant, it is enough to compute them at points of $\mathfrak{a}^+$. Let $H$ be such a point and let $(h, \rho)$, $h \in \mathfrak{a}$, $\rho \in T_{[1]} K/L$, be a tangent vector to $\mathfrak{a}^+ \times K/L$ at $(H, [1])$. The vector $\rho$ can be identified with an element of $\sum \mathfrak{k}_\alpha \subset \mathfrak{k}$. The corresponding (under (1.3)) tangent vector at $H \in \mathfrak{p}'$ is $h + [\rho, H]$. Computing the Killing form of this vector yields the formula (1.4) with $F(z) = z^2$ for $g_0$. The formula for $g$ follows from a similar computation, using the expression for the differential of the map (1.1) given in [6], Theorem IV.4.1. \qed
2. Monge-Ampère equation on symmetric spaces

Let \((M, g)\) be a Riemannian manifold and \(u: M \to \mathbb{R}\) a smooth function. Then the Hessian of \(u\) is the symmetric \((0,2)\)-tensor \(Ddu\) where \(D\) is the Levi-Civita connection of \(g\). In local coordinates \(x_i\), \(Ddu\) is represented by the matrix
\[
H_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial u}{\partial x_k}.
\]
(2.1)
We say that the function \(u\) is \(g\)-convex (resp. strictly \(g\)-convex), if \(Ddu\) is non-negative (resp. positive) definite. The Monge-Ampère equation on the manifold \((M, g)\) is then
\[
M_g(u) := (\det g)^{-1} \det Ddu = f
\]
(2.2)
where \(f\) is a given function.

Let \((G^* / K, g)\) be a symmetric space of non-compact type given by a Cartan decomposition \(g^* = \mathfrak{k} + i \mathfrak{p}\). As in the previous section, we identify \(M = G^* / K\) with \(\mathfrak{p}\) and denote by \(g_0\) the (flat) metric given by restricting the Killing form to \(\mathfrak{p}\). We have:

**Theorem 2.1.** Let \(M \simeq \mathfrak{p}\) be a symmetric space of noncompact type and let \(u\) be a \(K\)-invariant (smooth) function on \(M\). Then

1. \(u\) is \(g\)-convex if and only if \(u\) is \(g_0\)-convex (i.e. convex in the usual sense on \(\mathfrak{p}\)).
2. The following equality of Monge-Ampère operators holds:
\[
M_g(u) = F \cdot M_{g_0}(u),
\]
where \(F: M \to \mathbb{R}\) is a positive \(K\)-invariant smooth function depending only on \(M\).

We have proved in [3] a theorem on the existence and regularity of \(K\)-invariant solutions to Monge-Ampère equations on \(\mathbb{R}^n\). From this we immediately obtain

**Corollary 2.2.** Let \((G^* / K, g)\) be an irreducible symmetric space of noncompact type and let \(f\) be a positive smooth \(K\)-invariant function on \(G^* / K\). Then the Monge-Ampère equation (2.2) has a global smooth \(K\)-invariant strictly \(g\)-convex solution.

We shall now prove Theorem 2.1. In fact we shall prove it in the following, more general situation. Suppose that we are given a \(K\)-invariant metric on \(\mathfrak{p}\) whose pullback under (1.3) can be written as (cf. (1.4)):
\[
\sum_i dr_i^2 + \sum_{(\alpha, m)} F(\alpha, m)(r) \theta^2(\alpha, m),
\]
(2.3)
where \(F(\alpha, m): \mathbb{R} \to \mathbb{R}\) are smooth functions vanishing at the origin such that \(z^{-1} \frac{dF(\alpha, m)}{dz}\) is smooth and positive everywhere. Proposition 1.1 implies that the symmetric metric on \(G^* / K\) is of this form. We claim that Theorem 2.1 holds for any metric \(g\) of the form (2.3).
In order to simplify the notation, let us write \( j \) for the index \((\alpha, m)\) and \( \alpha_j \) for \( \alpha \) if \( j = (\alpha, m) \). The metric \( g \) can be now written as
\[
\sum_i dr_i^2 + \sum_j F_j(\alpha_j(r))\theta_j^2.
\]
We recall the following formula:
\[
2Ddu = L\nabla u g,
\]
where \( L \) is the Lie derivative and \( \nabla u \) is the gradient of \( u \) with respect to the metric \( g \). On the other hand, for any \((0, 2)\)-tensor \( g \) and vector fields \( X, Y, Z \), we have:
\[
(L_X g)(Y, Z) = X.g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]).
\]
We now compute \( L\nabla u g \) on \( p' \) with respect to the basis vector fields \( \partial/\partial r_i, X_j \), where \( X_j \) are dual to \( \theta_j \). Here \( u \) is a \( K \)-invariant function. The gradient of \( u \) is just \( \sum \partial u/\partial r_i \partial/\partial r_i \), in particular it is independent of the functions \( F_j \). It follows immediately that \( (L\nabla u g)(\partial/\partial r_i, X_j) = 0 \) and that the matrix \( (L\nabla u g)(X_j, X_k) \) is equal to \( \nabla u.g(X_j, X_k) \) and hence it is diagonal with the \((jj)\)-entry equal to
\[
\nabla u(F_j(\alpha_j(r))) = \frac{dF_j}{dz}_{z=\alpha_j(r)} \alpha_j(\nabla_0 \bar{u}).
\]
Here \( \nabla_0 \bar{u} = \sum \partial u/\partial r_i \partial/\partial r_i \) is the gradient of \( u \) restricted to the Euclidean space \( \mathbb{R}^n \) in coordinates \( r_i \), and viewed as a map from \( \mathbb{R}^n \) to itself.

Theorem 2.1 with the more general metric (2.3) follows easily with the function \( F \) given explicitly by
\[
F = \prod \frac{\alpha_j(r)}{\alpha_j(r)} \prod \frac{1}{2} \frac{dF_j}{dz}_{z=\alpha_j(r)}.
\]
Observe that the assumptions on the \( F_j \) guarantee that \( F \) extends to a smooth positive function on \( p \).

3. Proof of the Main Theorem

Let \((M, g)\) be a Riemannian symmetric space of compact type, \( G \) its isometry group, and \( K \subset G \) the stabiliser group of a point. There is a canonical isomorphism between \( G^C/K^C \) and \( G \times_K p \) (i.e. the tangent bundle of \( G/K \)) given by the map:
\[
G \times p \rightarrow G^C \rightarrow G^C/K^C, \quad (g, p) \mapsto ge^{ip}.
\]
This isomorphism can be viewed in many ways: as an example of Mostow fibration [8], as given via Kähler reduction of \( G^C \simeq G \times \mathfrak{g} \) by the group \( K \) [5], or as given by the adapted complex structure construction [7] which provides a canonical diffeomorphism between the tangent bundle of \( G/K \) and a complexification of \( G/K \). In any case it provides a fibration
\[
\pi : G^C/K^C \rightarrow G/K.
\]
The fibers of this projection can be identified with \( \mathfrak{p} \) via the map (3.1). In particular, the fiber over \([1]\) is given by the \(K^C\)-orbits of elements \(e^{ip}, p \in \mathfrak{p}\). We shall relate \(G\)-invariant plurisubharmonic functions on \(G^C/K^C\) to convex functions on this fiber (see [1] for a different approach to this).

For a function \(w\) on a complex manifold one defines its Levy form \(Lw\) to be the Hermitian \((0,2)\) tensor given in local coordinates as

\[
\frac{\partial^2 w}{\partial z_k \partial \bar{z}_l} dz_k \otimes d\bar{z}_l.
\]

This form does not depend on the choice of local coordinates. We shall compute this form for a \(G\)-invariant function \(w\) on \(G^C/K^C\). It is enough to compute it at points \(e^{ip}, p \in \mathfrak{p}\). First of all, we choose local holomorphic coordinates at such a point:

**Lemma 3.1.** In a neighbourhood of a point \(e^{ip}, p \in \mathfrak{p}\), complex coordinates are provided by the map \(\mathfrak{p}^C \to G^C \to G^C/K^C, (a + ib) \mapsto e^{a+ib} e^{ip}\).

**Proof.** We have to show that the map \((a + ib) \mapsto e^{a+ib} e^{ip} K^C\) has a non-singular differential at 0. This is equivalent to \(\text{ad} e^{-ip}\) \(u \not\in \mathfrak{k}^C\) for \(u \in \mathfrak{p}^C\). We have

\[
(\text{ad} e^{-ip}) u = e^{\text{ad}(-ip)} u = \cosh(\text{ad}(-ip)) u + \sinh(\text{ad}(-ip)) u,
\]

where the first term of the sum lies in \(\mathfrak{p}^C\) and the second one in \(\mathfrak{k}^C\). To show that the first term does not vanish recall that \((\text{ad}(-ip))^2\) has all eigenvalues nonnegative. \(\square\)

We now have:

**Lemma 3.2.** In the complex coordinates \(z = a + ib\) given by the previous lemma, the Levy form (3.3) of a \(G\)-invariant function \(w\) satisfies the equation:

\[
\frac{\partial^2 w}{\partial z_k \partial \bar{z}_l} \bigg|_{a=0, b=0} = \frac{1}{4} \frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib} e^{ip})_{b=0}.
\]

**Proof.** The polar decomposition of \(G^C\) implies that \(e^{a+ib}\) can be uniquely written as \(ge^{iy}\), where \(g \in G\) and \(y \in \mathfrak{g}\). Any \(G\)-invariant function on \(G^C/K^C\) in a neighbourhood of \(e^{ip}\) is a function of \(y\) only. On the other hand, as \(e^{2iy} = (e^{x} e^{iy})(e^{x} e^{iy}) = e^{-a+ib} e^{a+ib}\), it follows from the Campbell-Hausdorff formula that \(y = b + a/2 + \text{higher order terms}\). Hence the matrix of second derivatives in (3.3) at \(e^{ip}\) (i.e. at \(a = 0, b = 0\)) is the same as the matrix of second derivatives of

\[
(a, b) \mapsto e^{(ib + \frac{a}{2}) (b, a)} e^{ip}
\]

at \(a = 0, b = 0\). We shall now show that for a \(G\)-invariant function \(w\) on \(G^C/K^C\), this matrix of second derivatives is equal to the right-hand side of (3.5).

The Campbell-Hausdorff formula implies that up to order 2 in \(a, b\), we have \(e^{(ib + \frac{a}{2}) (b, a)} = e^{ib} e^{\frac{a}{2} (b, a)}\). Set \(c = \frac{b}{2}\), which is a point in \(\mathfrak{k}\). We are going to show that modulo terms of order 2 in \(c\) (hence of order 4 in \(a, b\)), \(e^{ic} e^{ip}\) is equal
to \( e^{\rho}e^{ip}e^{iq} \), where \( \rho \in \mathfrak{g} \) and \( q \in \mathfrak{k} \) are both linearly dependent on \( c \). We note that this proves the lemma, as

\[
e^{ib}e^{\rho}e^{ip}e^{iq} = e^{ib+O(3)}e^{ip}e^{iq} = e^{ib+O(3)}e^{ip}
\]

in \( G^C/K^C \), where \( O(3) \) denotes terms of order 3 and higher in \( a, b \).

We find \( q \) from the equation \( \cosh(\text{ad}(ip))(q) = c \), which can be solved uniquely as \( \cosh(\text{ad}(ip)) \) is symmetric and positive-definite on \( \mathfrak{f} \subset \mathfrak{g} \). We then put \( \rho = -i \sinh(\text{ad}(ip))(q) \). We observe that \( \rho \in \mathfrak{g} \) and \( e^{\rho-ic} = e^{ip}e^{-iq}e^{-ip} \), thanks to (3.4). Moreover, modulo terms quadratic in \( c \), \( e^{ib} = e^{ic}e^{\rho-ic} \) and, consequently:

\[
e^{ib}e^{ip}e^{iq} = e^{ic}e^{\rho-ic}e^{ip}e^{iq} = e^{ic}(e^{ip}e^{-iq}e^{-ip})e^{ip}e^{iq} = e^{ic}e^{ip},
\]

again modulo terms quadratic in \( c \). This finishes the proof of the lemma.

According to this lemma, we have to compute \( \frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib}e^{ip})_{b=0} \). Now, since \( e^{ib}e^{ip} \in G^* \), \( e^{ib}e^{ip} = ke^{iz} \), where \( z = z(b) \in \mathfrak{p} \) and \( k \in K \). As \( w \) is \( G \)-invariant, \( w(e^{ib}e^{ip}) = w(e^{iz}) \) and therefore

\[
\frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib}e^{ip})_{b=0} = \frac{\partial^2}{\partial b_k \partial b_l} w(e^{iz(b)})_{b=0}.
\]

Thus we compute the matrix of second derivatives of a function defined on \( \exp(\mathfrak{p}) \) in the coordinates given by \( b \mapsto e^{ib}e^{ip} \mapsto e^{iz(b)} \). These, however, are the geodesic coordinates at the point \( e^{ip} \) in the symmetric space dual to \( M \) (being translations of geodesics at \([1]\)) and hence the matrix of second derivatives in these coordinates is equal to the Riemannian Hessian (2.1) for the symmetric metric on the dual space. If we assume that \( K \) is connected, then this dual space is \( G^*/K \), and we obtain

**Theorem 3.3.** Suppose that \( K \) is connected. Let \( w \) be a smooth \( G \)-invariant function on \( X = G^C/K^C \) and let \( \bar{w} \) be its restriction to the fiber \( S = \exp(\mathfrak{p}) \) of (3.2) over \([1]\). Let \( g \) denote the symmetric metric on \( S \simeq G^*/K \). Then \( w \) is (strictly) plurisubharmonic if and only if \( \bar{w} \) is (strictly) \( g \)-convex. Moreover, the following equality holds:

\[
\partial \bar{\partial} \log \det \bar{L} = \partial \bar{\partial} \log M_g(\bar{w}),
\]

where \( \bar{u} : X \to \mathbb{R} \) is a \( G \)-invariant function such that \( \bar{u} \) is a given \( K \)-invariant function \( u \) on \( S \).

We are now ready to prove Theorem 1. Recall that \( X = G^C/K^C \) is a Stein manifold and so if \( \rho \) is an exact \((1,1)\) form on \( X \), then \( \rho = -i \partial \bar{\partial} h \) for some function \( h \). If \( \rho \) is \( G \)-invariant, then we can assume that \( h \) is \( G \)-invariant. We can restrict \( h \) to the fiber \( S \) defined in the last theorem and thanks to Corollary 2.2 we can find a strictly \( g \)-convex \( K \)-invariant smooth solution \( \bar{u} \) to the equation (2.2) with \( f = h \), where the metric \( g \) is the symmetric metric on \( S \simeq G^*/K \). We can extend this solution via \( G \)-action to a \( G \)-invariant function \( u \) on \( X \). Theorem 3.3 implies now that \( u \) is strictly plurisubharmonic and that the Ricci form of the Kähler metric with potential \( u \) is \( \rho \).
Acknowledgement

This work has been supported by an Advanced Research Fellowship from the Engineering and Physical Sciences Research Council of Great Britain. I thank Hassan Azad and the anonymous referee for helpful comments which resulted in an improved presentation.

References


Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK
E-mail address: R.Bielawski@maths.gla.ac.uk