ON THE REFINED CLASS NUMBER FORMULA FOR
GLOBAL FUNCTION FIELDS

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Abstract. We investigate a conjecture of Gross regarding a congruence relation
of the Stickelberger element. We consider the case when \( k \) is a global function field
of characteristic \( p \) and \( \text{Gal}(K/k) \) is an abelian \( l \)-group where \( l \) is a prime number
different from \( p \). Under the additional assumption that \( k \) does not contain a
primitive \( l \)-th root of unity and that the divisor class number of \( k \) is prime to \( l \),
we prove that the conjecture of Gross holds. This result generalizes the author’s
previous result on the elementary abelian case (cf. [6]).

1. Introduction

We describe the conjecture briefly, and refer the reader to [5] for details.

Let \( K/k \) be a finite abelian extension of global fields with Galois group \( G \).
Let \( S \) be a finite non-empty set of places of \( k \) which contains all archimedean
places and all places ramified in \( K \). Furthermore, let \( T \) be a finite non-empty
set of places of \( k \) which is disjoint from \( S \), such that the \((S,T)\)-unit group \( U_{S,T} \)
is torsion-free. Let \( n = |S| - 1 \) and let \( \hat{G} \) be the group of complex characters of
\( G \).

The Stickelberger element \( \theta_G \) is the unique element in \( \mathbb{Z}[G] \) which satisfies
\[
\chi(\theta_G) = L_{S,T}(\chi, 0)
\]
for all \( \chi \in \hat{G} \), where \( L_{S,T} \) is the \( S \)-truncated, \( T \)-modified Dirichlet \( L \)-function.
Gross has conjectured a congruence relation which bears striking resemblance to
the analytic class number formula. In order to describe the conjecture we need
to introduce some further notation.

Choose an ordered basis \( \{u_1, \ldots, u_n\} \) of \( U_{S,T} \). Pick a place \( v_0 \in S \), and for
each \( v_i \in S \setminus \{v_0\} \), we let \( f_i : k^\times \to G \) denote the homomorphism induced from
local Artin map for \( v_i \). We set
\[
\det_G \lambda := \det_{1 \leq i, j \leq n}(f_i(u_j) - 1).
\]
The conjecture of Gross states that

\[(1) \quad \theta_G \equiv m \cdot \det_G \lambda \pmod{I^{n+1}}.\]

Here \(I\) is the augmentation ideal of \(Z[G]\) and the integer \(m = \pm h_{S,T}\) is the \(T\)-modified class number of the \(S\)-integers of \(k\) whose sign is determined by the \((S,T)\)-version of the analytic class number formula.

Let \(\text{Gr}(K/k, S, T)\) denote the congruence relation (1). For the reader’s convenience, we list (without proofs) some of the basic facts regarding this conjecture. Consult [1] or [8] for details.

**Proposition 1.** (a) If \(v \notin S \cup T\) and \(S' = S \cup \{v\}\), then \(\text{Gr}(K/k, S, T)\) implies \(\text{Gr}(K/k, S', T)\).

(b) Suppose \(H\) is a subgroup of \(G\). The natural map \(Z[G] \to Z[G/H]\) maps \(\theta_G\) and \(\det_G \lambda\) to \(\theta_{G/H}\) and \(\det_{G/H} \lambda\) respectively, and \(\text{Gr}(K/k, S, T)\) implies \(\text{Gr}(K^H/k, S, T)\).

(c) If \(n = 0\) then \(\text{Gr}(K/k, S, T)\) holds, being equivalent to the analytic class number formula.

We also note that the conjecture has been verified for numerous cases [1, 3, 4, 6, 7].

2. The Main Result

Let \(G = G_0 \times G_1 \times \cdots \times G_m\), and set \(X = \{0, \ldots, m\}\). For each \(i \in X\), we have \(Z[G_i] \cong Z \oplus I_i\) as a direct sum of abelian groups. Here \(I_i\) is the augmentation ideal of \(Z[G_i]\).

As \(G_i\) is a subgroup of \(G\), \(Z[G_i]\) is naturally embedded in \(Z[G]\), and so is \(I_i\). For each non-empty subset \(A\) of \(X\), we define \(I_A := \prod_{i \in A} I_i \subset Z[G]\), and we define \(I_\emptyset := Z\). Then we have

\[Z[G] \cong \bigotimes_{i \in X} Z[G_i] \cong \bigotimes_{i \in X} (Z \oplus I_i) \cong \bigoplus_{A \subset X} I_A.\]

We observe that \(I_A \cdot I_B \subset I_{A \cup B}\). Therefore \(Z[G]\) is a graded ring with respect to the monoid of subsets of \(X\) with union as monoid operation. Also, we have

\[I = \bigoplus_{\emptyset \neq A \subset X} I_A,\]

therefore \(I\) is a homogeneous ideal of \(Z[G]\) and so is \(I^n\) for \(n \geq 1\).

**Lemma 2.** For each \(i \in X\), let \(H_i = G_0 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_m\) and let \(\phi_i : Z[G] \to Z[H_i]\) be the map induced by natural projection. Pick an integer \(r\) with \(0 \leq r \leq m\). If \(\alpha\) is an element of \(Z[G]\) with \(\phi_i(\alpha) \in I_{H_i}^{r+1}\) for \(i = 0, \ldots, r\), then \(\alpha \in I^{r+1}\).

**Proof.** Write \(\alpha = \sum_{A \subset X} \alpha_A\). We need to show that \(\alpha_B \in I^{r+1}\) for all \(B \subset X\).
If \{0, \ldots, r\} \subset B, then by the definition of \( I_B \) it follows that \( \alpha_B \in I^{r+1} \).

Suppose \( i \not\in B \) for some \( 0 \leq i \leq r \). It is straightforward to verify that

\[
\phi_i : \bigoplus_{A \subset X} I_A \rightarrow \bigoplus_{A \subset X \setminus \{i\}} I_A
\]

is the projection onto the \( A \)-components with \( i \not\in A \). As \( i \not\in B \), \( \alpha_B = \phi_i(\alpha_B) \) is the \( B \)-component of \( \phi_i(\alpha) \). Since \( \phi_i(\alpha) \in I^{r+1}_i \) by assumption and \( I^{r+1}_i \) is a homogeneous ideal of \( \mathbb{Z}[H_i] \), \( \alpha_B \in I^{r+1} \) as well. If we view \( H_i \) as a subgroup of \( G \), we have \( I_{H_i} \subset I \) and hence \( I^{r+1}_{H_i} \subset I^{r+1} \). Therefore \( \alpha_B \in I^{r+1} \).

\[ \square \]

**Theorem 3.** Let \( K/k \) be a finite abelian extension with Galois group \( G = G_0 \times G_1 \times \cdots \times G_m \) and let \( S = \{v_0, \ldots, v_n\} \). Suppose that for each \( 0 \leq i \leq n \), its inertia group \( I_{v_i} \) of \( v_i \) is contained in \( G_i \). Then \( \text{Gr}(K/k, S, T) \) holds.

**Proof.** We prove the theorem by induction on \( n \). When \( n = 0 \), \( \text{Gr}(K/k, S, T) \) holds as noted in Proposition 1(c).

In the general case, we apply Lemma 2 to \((\theta_G - m \cdot \det_G \lambda)\). As \( v_i \) is unramified in the subextension \( K^{G_i}/k \), the induction hypothesis together with Proposition 1(a) implies that the hypothesis of Lemma 2 is satisfied. Hence we conclude that \( \text{Gr}(K/k, S, T) \) holds.

\[ \square \]

**Corollary 4.** Fix a prime number \( l \). For a global function field \( k \), let \( p \) be its characteristic, \( h \) be its divisor class number and \( w \) be the number of roots of unity in \( k \). If \( l \) does not divide \( phw \), then \( \text{Gr}(K/k, S, T) \) holds whenever \( \text{Gal}(K/k) \) is an abelian \( l \)-group.

**Proof.** For each positive integer \( e \geq 1 \), let \( k_{S,e} \) be the maximal abelian extension of \( k \) unramified outside of \( S \) whose Galois group has exponent \( l^e \). Thanks to Proposition 1(b), we may assume \( K = k_{S,e} \). Theorem 5 of the next section ensures that the hypothesis of Theorem 3 is satisfied in this case.

\[ \square \]

### 3. Some Class Field Theory

In this section we use the results from class field theory, and study the structure of \( G \) using ideles. The reader may consult [2] for example.

We keep the assumptions of Corollary 4. Let \( \mathbb{F}_q \) be the exact field of constants of \( k \), and for each place \( v \) of \( k \) let \( \mathbb{F}_v \) be its residue field. For each finite nonempty set \( S \) of places of \( k \) and for each integer \( e \geq 1 \), let \( G_{S,e} := \text{Gal}(k_{S,e}/k) \).

**Theorem 5.** \( G_{S,e} \cong \prod_{v \in S} I_v \times \mathbb{Z}/l^e \mathbb{Z} \).

**Proof.** For each place \( v \) of \( k \), let \( k_v \) be the completion of \( k \) at \( v \), \( U_v \) the set of local units in \( k_v \), and \( U^1_v \subset U_v \) the local units which are congruent to 1 \( \pmod{v} \). Also let \( U := \prod_v U_v \) and let \( U_S := \prod_{v \in S} U_v \cdot \prod_{v \in S} U^1_v \).

There is an exact sequence

\[
0 \rightarrow U/\mathbb{F}_q^* \cdot U_S \rightarrow J/k^* \cdot U_S \rightarrow J/k^* \cdot U \rightarrow 0.
\]
We note that the profinite completion of $J/k^* \cdot U_S$ is $\text{Gal}(k_S/k)$ where $k_S$ is the maximal abelian extension of $k$ unramified outside of $S$ and tamely ramified in $S$. Similarly, the profinite completion of $J/k^* \cdot U$ is $\text{Gal}(k_{unr}/k)$ where $k_{unr}$ is the maximal unramified abelian extension of $k$.

Let $J_0$ be the group of ideles of $k$ of degree 0. Then $J$ is isomorphic to $J_0 \times \langle c \rangle$, where $c$ is an idele of degree 1. Therefore we may rewrite the above sequence as

\[
0 \to \left( \prod_{v \in S} \mathbb{F}_v^*/ \mathbb{F}_v^{*l_v} \right)/\mathbb{F}_q^* \to (J_0/k^* \cdot U_S) \times \mathbb{Z} \to (J_0/k^* \cdot U) \times \mathbb{Z} \to 0.
\]

Note that for each $v \in S$, the inertia group of $v$ is the image of $\mathbb{F}_v^*$ in the first term of the sequence (2).

As we assume that the order of $J_0/k^* \cdot U$ (which is canonically isomorphic to the divisor class group of $k$) is not divisible by $l$, we have $(J_0/k^* \cdot U \times \mathbb{Z}) \otimes \mathbb{Z}/l^e \mathbb{Z} = \mathbb{Z}/l^e \mathbb{Z}$ and $\text{Tor}(J_0/k^* \cdot U \times \mathbb{Z}, \mathbb{Z}/l^e \mathbb{Z}) = 0$. Hence tensoring the exact sequence with $\mathbb{Z}/l^e \mathbb{Z}$ preserves the exactness;

\[
0 \to \left( \prod_{v \in S} \mathbb{F}_v^*/ \mathbb{F}_v^{*l_v} \right)/\mathbb{F}_q^* \to (J_0/k^* \cdot U_S) \times \mathbb{Z}/l^e \mathbb{Z} \to (J_0/k^* \cdot U) \times \mathbb{Z}/l^e \mathbb{Z} \to 0,
\]

where $\mathbb{F}_q^*$ is the image of $\mathbb{F}_q^*$ in $\prod_{v \in S} \mathbb{F}_v^*/ \mathbb{F}_v^{*l_v}$. Therefore $G_{S,e}$, the middle term of the above exact sequence, is isomorphic to $\left( \prod_{v \in S} \mathbb{F}_v^*/ \mathbb{F}_v^{*l_v} \right)/\mathbb{F}_q^* \cong \mathbb{Z}/l^e \mathbb{Z}$.

As we assume that $k$ does not contain a primitive $l$-th root of unity, $\mathbb{F}_q^* \cong \{1\}$, and hence $G_{S,e} \cong \prod_{v \in S} \mathbb{F}_v^*/ \mathbb{F}_v^{*l_v} \times \mathbb{Z}/l^e \mathbb{Z} \cong \prod_{v \in S} I_v \times \mathbb{Z}/l^e \mathbb{Z}$.

\[\square\]

Remark. If we assume that $k$ contains an $l$-th root of unity, one can prove that $G_{S,e}$ is isomorphic to $\prod_{v \in S'} I_v \times H'$ where $S' = S \setminus \{v_0\}$ for a suitable choice of $v_0 \in S$. Therefore, one may apply Lemma 2 to $\theta_G$ to conclude that $\theta_G \in I^n$.

Acknowledgements

This paper grew out of some stimulating discussions with Noboru Aoki, and I would like to thank him for his generosity. I also thank Ki-Seng Tan and John Tate for many suggestions for improvement.

References


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