Mathematical Research Letters **11**, 673–684 (2004)

BETWEEN MAHARAM'S AND VON NEUMANN'S PROBLEMS

Ilijas Farah and Jindřich Zapletal

Abstract. In the context of definable algebras Maharam's and von Neumann's problems essentially coincide. Consequently, random forcing is the only definable ccc forcing adding a single real that does not make the ground model reals null, and the only pairs of definable ccc σ-ideals with the Fubini property are ($\mathtt{meager}, \mathtt{meager}$) and ($\mathtt{null}, \mathtt{null}$).

In Scottish Book, von Neumann asked whether every ccc, weakly distributive complete Boolean algebra carries a strictly positive probability measure. Von Neumann's problem naturally splits into two: (a) whether all such algebras carry a strictly positive continuous submeasure, and (b) whether every algebra that carries a strictly positive continuous submeasure carries a strictly positive measure. The latter problem is known under the names of Maharam's Problem and Control Measure Problem (see [16], [9], [5, §393]). While von Neumann's problem has a consistently negative answer ([16]), Maharam's problem can be stated as a Σ^1_2 statement and is therefore, by Shoenfield's theorem, absolute between transitive models of set theory containing all countable ordinals.

Theorem 0.1. *Let I be a c.c.c. σ-ideal on Borel subsets of 2^ω that is analytic on G_δ. The following are equivalent:*

- *P_I is a weakly distributive notion of forcing*
- *there is a continuous submeasure on 2^ω such that I is the σ-ideal of its null sets.*

A suitable large cardinal assumption implies that the assumption 'I is analytic on G_δ' can be relaxed to 'I is definable.'

Here P_I is the partial ordering of I-positive Borel sets under inclusion, and its regular open algebra is isomorphic to the quotient Borel/I. Thus, the von Neumann's problem restricted to regular open algebras of definable partial orders of the form P_I coincides with the Control Measure Problem.

Our result was obtained in November 2003. In December 2003 we learned that Balcar, Jech and Pazák ([2]) and independently Velickovic ([26]) proved

Received January 15, 2004.

2000 *Mathematics Subject Classification.* 03E40, 28A05.

First author partially supported by NSERC..

Second author partially supported by grant GA ČR 201-03-0933, NSF grant DMS 0071437, and the first author's NSERC grant. Results of this paper were obtained in November and December 2003 while the second author was visiting York University.

that under the P-ideal dichotomy ([24]) every c.c.c. weakly distributive complete Boolean algebra carries a strictly positive continuous submeasure. Since the case of P-ideal dichotomy relevant to Boolean algebras of size continuum can always be forced without adding reals ([1]), Theorem 0.1 follows via an absoluteness argument. Quickert ([17]) used the P-ideal dichotomy earlier in a similar context.

In order to state some interesting consequences of our theorem succintly we quote the large cardinal version.

Corollary 0.2 (LC)**.** *Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset 2^\omega \times 2^\omega$ with all vertical sections in I and all horizontal sections of full Haar measure.*
2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the random forcing.*

In other words, if P_I does not force that the set of ground model reals is null, then below some condition P_I is forcing isomorphic to the random forcing. Modulo Theorem 0.1, this is a consequence of a result of Christensen ([4]). By an earlier result of Shelah a similar result holds on the meager side.

Fact 0.3 (LC)**.** *(*[20]*, see also* [27]*) Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset 2^\omega \times 2^\omega$ with all vertical sections in I and all horizontal sections comeager.*
2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the Cohen forcing.*

Another attractive corollary is that, up to the isomorphism, the only definable c.c.c. σ-ideals for which Fubini theorem holds are `meager` and `null` (Theorem 3.3). This shows that, under a large cardinal assumption, those two ideals are the only 'reasonable' ideals as introduced by Kunen in [14].

Terminology. Notation in this paper follows the set theoretic standard of [8]. For information on large cardinals and $L(\mathbb{R})$ see also [11]. An ideal I is analytic on G_δ if for every G_δ set $A \subseteq 2^\omega \times 2^\omega$ the set of all x such that the vertical section of A at x is in I is analytic. Both `meager` and `null` are analytic on G_δ (see [12]). Throughout the paper we will say that an ideal is *definable* if it belongs to the inner model $L(\mathbb{R})$. The suitable large cardinal assumption in Theorem 0.1 if I is in $L(\mathbb{R})$ is that there are ω Woodin cardinals with a measurable above them all. In all the subsequent results of this note no large cardinal assumptions are needed if I is assumed to be analytic on G_δ.

A *continuous submeasure* (or a *Maharam submeasure*) on a complete Boolean algebra $\mathcal{B}$ is a function ϕ such that

1. $A \subseteq B$ implies $\phi(A) \leq \phi(B)$,
2. $\phi(A \cup B) \leq \phi(A) + \phi(B)$,
3. $\phi(0_{\mathcal{B}}) = 0$, and

4. if A_n is a decreasing sequence in $\mathcal{B}$ then $\phi(\bigcap_n A_n) = \lim_n \phi(A_n)$.

A complete Boolean algebra that carries a strictly positive continuous submeasure is called a *submeasure algebra*.

A forcing notion is *bounding* (or *weakly distributive*) if every element of ω^ω in the extension is dominated by a ground-model function in ω^ω. We use the words "bounding" and "weakly distributive" interchangeably. It is entirely irrelevant which uncountable Polish space the ideals in question measure; our choice is the Cantor space 2^ω for definiteness and ease of notation. To weed out trivial cases, we assume that ideals contain all singletons.

1. The proof of Theorem 0.1

If I is the ideal of null sets of some continuous submeasure then P_I is weakly distributive (see e.g., [5, 392I]). Suppose now that I is a definable, weakly distributive c.c.c. σ-ideal on Borel subsets of 2^ω. To find a continuous submeasure generating I we will use two ingredients. One is almost trivial:

Fact 1.1 ([28] Lemma 2.2.3.)**.** *Suppose I is a σ-ideal on 2^ω such that P_I is proper. The following are equivalent:*

- *P_I is weakly distributive*
- *compact sets are dense (every I-positive Borel set has an I-positive compact subset) and P_I allows continuous reading of names (for every I-positive Borel set B and a Borel function $f : B \to \omega^\omega$ there is an I-positive set $C \subset B$ such that $f \restriction C$ is continuous).*

This implies that I has a basis consisting of G_δ sets. For let $A \in I$ be a Borel set. The collection of compact I-positive sets disjoint from A is dense in P_I: for every I-positive Borel set B, the set $B \setminus A$ is still Borel and I-positive and therefore it has a compact I-positive subset. Choose then a maximal antichain X consisting of such compact sets. Since P_I is c.c.c. X is countable and $2^\omega \setminus \bigcup X$ is a G_δ set in I covering the set A.

The other ingredient is a result of Solecki. For an ideal I on 2^ω let $\hat{I}$ be the collection of subsets of $2^{<\omega}$ defined by putting $a \in \hat{I}$ if the set $B_a = \{r \in 2^\omega :$ for infinitely many $n, r \restriction n \in a\}$ is in I. It is immediate that $\hat{I}$ is an ideal, because $B_{a\cup b} = B_a \cup B_b$ and so if both $B_a, B_b \subset 2^\omega$ are in the σ-ideal I, so is $B_{a\cup b}$.

Fact 1.2. *Suppose I is a σ-ideal on Borel subsets of 2^ω that is analytic on G_δ. The following are equivalent:*

- *$\hat{I}$ is a P-ideal and I has a basis consisting of G_δ sets*
- *There is a continuous submeasure on 2^ω such that I is the collection of its null sets.*

Furthermore, large cardinals imply this equivalence for every definable I.

Proof. This was proved in [22, Theorem 5.2] in the case when I is analytic on G_δ. The definability assumption was used in this proof only to show that $\hat{I}$

is analytic. Assuming large cardinals, in [23, Theorem 4] it was proved that all definable P-ideals are analytic. □

Fact 1.2 clearly implies that we will be done once we prove $\hat{I}$ is a P-ideal. We fix a collection $\{a_n : n \in \omega\} \subset \hat{I}$ and aim to construct $b \in \hat{I}$ which includes each of them up to a finite set.

Claim 1.3. *The collection of compact I-positive sets C such that their associated tree on $2^{<\omega}$ has a finite intersection with each a_n ($n \in \omega$) is dense in P_I.*

Proof. Suppose $A \in P_I$ is a positive Borel set. Then $B = A \setminus \bigcup_n B_{a_n}$ is still an I-positive Borel set, and the function $f : B \to \omega^\omega$, $f(r)(n) = \max\{m \in \omega : r \restriction m \in a_n\}$, is Borel and well-defined on it. By Fact 1.1, there is an I-positive compact set $C \subset B$ such that $f \restriction C$ is continuous. By a compactness argument, for every n the set $\{f(r)(n) : r \in C\}$ is finite. The claim follows. □

Let X be a maximal antichain of I-positive compact sets from the claim. Since P_I is c.c.c., X is countable. Let $X = \{C_k : k \in \omega\}$ and let $T_k \subset 2^{<\omega}$ be the tree associated with the compact set C_k. Finally, let $b \subset 2^{<\omega}$ be the set $\bigcup_n (a_n \setminus \bigcup_{k<n} T_k)$. It is clear that b includes every a_n modulo finite. To show that $B_b \in I$ and $b \in \hat{I}$, note that for every $k \in \omega$ the intersection $T_k \cap b = \bigcup_{n \leq k} a_n$ is finite, and so the set B_b is disjoint from $\bigcup X$. However, the antichain $X \subset P_I$ was chosen to be maximal, and therefore the set $2^\omega \setminus \bigcup X$ is I-small and so is its subset B_b. The theorem follows.

2. Fubini failing

A submeasure ϕ is *pathological* if it does not dominate a positive nonzero finitely additive functional. A *control measure* for a continuous submeasure ϕ is a measure μ that has the same null sets as ϕ. A continuous submeasure is *Borel* if it is defined on the Borel algebra on 2^ω. A submeasure is *diffuse* if all countable sets are null. All results of this section are probably well-known.

Lemma 2.1. *The following are equivalent for a diffuse continuous Borel submeasure ϕ.*

1. *ϕ is not pathological.*
2. *There is a ϕ-positive set B such that the restriction of ϕ to B has a control measure.*
3. *There is a ϕ-positive set B such that $P_{\mathrm{Null}(\phi)}$ is forcing equivalent to random forcing below B.*

Proof. Let us write $I = \mathrm{Null}(\phi)$. Assume (1), so there is a nonzero finitely additive functional $\nu \leq \phi$ dominated by ϕ. There are two cases.

Assume there is a ϕ-positive set B such that $\nu(C) \neq 0$ for every I-positive set $C \subset B$. Then Borel$/I$ is weakly distributive (see e.g., [5, 392I]). By [5, 391D] there is a strictly positive measure on Borel$/I$, and therefore (2) holds.

Otherwise, every ϕ-positive set B contains a ϕ-positive set C such that $\nu(C) = 0$. In this case, choose a maximal antichain $\{C_n : n \in \omega\}$ of sets such that

$\phi(C_n) > 0$ and $\nu(C_n) = 0$, enumerated using the ccc of Borel/I. Consider the sets $D_m = \bigcup_{n>m} C_n$. By the finite additivity of the functional ν it is the case that $\nu(D_m) = \nu(2^\omega)$ for all $m \in \omega$. By the continuity of the submeasure ϕ, the numbers $\phi(D_m)$ converge to zero, since the D_ms form a decreasing collection of sets with empty intersection. This contradicts $\nu \leq \phi$.

Clause (2) implies (1) by [6, Theorem 2]. The equivalence of (2) and (3) follows by the separable case of Maharam's theorem. □

Lemma 2.2. *If ϕ is a continuous Borel submeasure then there is a Borel set A such that ϕ has a control measure on B and is pathological on B^{c}.*

Proof. Find a maximal family $\mathcal{F}$ of pairwise orthogonal measures dominated by ϕ, and let B be the union of their supports. By the ccc-ness of Borel/Null(ϕ), $\mathcal{F}$ is countable. If $\mathcal{F} = \{\mu_i | i \in \omega\}$ then $\sum_i 2^{-i}\mu_i$ is a control measure for ϕ on B. By Lemma 2.1, ϕ is pathological on the complement of B. □

Lemma 2.3 below was roughly proved by Christensen [4, Theorem 6]. We shall use his result. Let μ denote the Lebesgue measure on $[0, 1]$; the choice is immaterial as any other diffuse Borel probability measure would do.

Lemma 2.3. *Suppose I is the null ideal for some continuous Borel submeasure ϕ on 2^ω. Exactly one of the following holds:*

1. *There is a ϕ-positive Borel set B such that the restriction of ϕ to B has a control measure.*
2. *There is a Borel set $C \subseteq [0, 1] \times 2^\omega$ such that $\phi(C_x) = 0$ for all $x \in [0, 1]$ and $\mu([0, 1] \setminus C^y) = 0$ for every $y \in 2^\omega$.*

Proof. By Fubini's theorem, (1) excludes (2). Suppose now that (1) fails. By Lemma 2.1, ϕ is pathological. Christensen proved in [4, Theorem 6], Theorem 6 that if ϕ is pathological then (2) holds. □

A submeasure ϕ on 2^ω is *normalized* if $\phi(2^\omega) = 1$.

Lemma 2.4. *Assume ψ is a normalized pathological Borel submeasure. Then for every $n \in \mathbb{N}$ there are pairwise disjoint sets A_i $(i < n)$ of submeasure at least $1/3$ each.*

Proof. This was proved by Kalton and Roberts ([10]) for an unspecified $\varepsilon > 0$ in place of $1/3$, and sharpened by Louveau ([15]) to the present form. □

Lemma 2.5. *Assume ϕ and ψ are normalized diffuse continuous Borel submeasures on 2^ω and ψ is pathological. Then there is a Borel set $C \subseteq 2^\omega \times 2^\omega$ such that $\psi(C_x) \geq 1/3$ for all $x \in 2^\omega$ and $\phi(C^y) = 0$ for all $y \in 2^\omega$.*

Proof. Since ϕ is diffuse and continuous, every set of submeasure δ has a subset of submeasure ϵ for every $\epsilon \in [0, \delta]$. For each n fix a maximal antichain of Borel sets such that the submeasure of each one is between 2^{-n-1} and 2^{-n}. Since ϕ is continuous, this antichain is finite and we can enumerate it as B^n_i $(i < k_n)$.

Using Lemma 2.4, fix a partition of 2^ω into Borel sets A^n_i $(i < k_n)$ such that $\psi(A^n_i) \geq 1/3$ for all n and i. Let

$$C(n) = \bigcup_{i=0}^{k_n-1} B^n_i \times A^n_i \qquad \text{and} \qquad C = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} C(n).$$

Note that $\psi(C(n)_x) \geq 1/3$ and that $\phi(C(n)^y) \leq 2^{-n}$ for all x, y in 2^ω. Therefore for all x, y we have $\psi(C_x) \geq 1/3$ and $\phi(C^y) \leq \sum_{n=m}^{\infty} 2^{-n} = 2^{-m+1}$ for all m, hence $\phi(C^y) = 0$. □

Lemma 2.6. *Assume ϕ and ψ are diffuse Borel continuous submeasures and ϕ does not have a control measure. Then there is a Borel set $A \subseteq 2^\omega \times 2^\omega$ such that $\psi(A_x) = 0$ for all x and $\inf_y \phi(A^y) > 0$.*

Proof. Let A be a Borel set such that the restriction of ϕ to D^{C} has a control measure while the restriction of ϕ to D is pathological, as given by Lemma 2.2. By our assumption, $\phi(D) > 0$. Again using Lemma 2.2, find a Borel partition $2^\omega = B \cup C$ so that ψ has a control measure on B and is pathological on C. By Lemma 2.3 there is Borel $E \subseteq D \times B$ such that $\phi(E^y) = \phi(D)$ for all $y \in B$ and $\psi(E_x) = 0$ for all x. By Lemma 2.5 there is a Borel $F \subseteq D \times C$ such that $\phi(F^y) \geq \frac{1}{3}\phi(D)$ for all $y \in C$ and $\psi(F_x) = 0$ for all x. Then $A = E \cup F$ is as required. □

3. Non-commutativity

Given σ-ideals I and J on the real line, let $I \perp J$ be the statement that there is a Borel subset B of the plane such that all of its vertical sections are in the ideal J and all of the horizontal sections of the complement are in the ideal I. Thus $I \perp J$ means that the Fubini theorem between I and J fails in a particularly violent manner. For example, if $I = \mathtt{meager} \cap \mathtt{null}$ then (I, I) does not have the Fubini property yet $I \perp I$ does not hold either. If the σ-ideals I and J are definable and c.c.c. then $I \perp J$ is easily seen to be equivalent to both $P_I \Vdash 2^\omega \cap V \in \dot{J}$ and $P_J \Vdash 2^\omega \cap V \in \dot{I}$ [28, 5.4.8].

Results of this section do not any require large cardinals if the ideals are assumed to be analytic on G_δ. This is because in this case both the compatibility and the incompatibility relations of P_I are analytic, and therefore the result of [21] applies. Let us recall and prove Corollary 0.2.

Corollary 3.1 (LC)**.** *Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset 2^\omega \times 2^\omega$ with all vertical sections in I and all horizontal sections of full Haar measure.*
2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the random forcing.*

Proof. By Fubini's theorem, two clauses exclude each other. Assume that P_I is not isomorphic to the random algebra below any positive set B. By Theorem 0.1

and Lemma 2.3, we may assume P_I is not bounding, so by a result of Shelah ([21]) it adds a Cohen real. Let $f\colon 2^\omega \to 2^\omega$ be a Borel function such that the preimages of meager sets are in I. Fix a Borel set $B \subseteq 2^\omega \times 2^\omega$ whose all vertical sections are null and whose complements of horizontal sections are meager. Then the set $D = \{(x, y) | (f(x), y) \in A\}$ witnesses that $I \perp \texttt{null}$. $\square$

We do not know whether $\mathrm{Null}(\phi) \perp \mathrm{Null}(\psi)$ whenever ϕ and ψ are continuous submeasures at least one of which is pathological.

Shelah [20] defined the notion of commutation for definable c.c.c. σ-ideals I, J: they commute if for all reals r, s in all generic extensions of V, the statement "r is $V[s]$-generic for P_I and s is V-generic for P_J" is equivalent to "s is $V[r]$-generic for P_J and r is V-generic for P_I." (Note that in this situation r is automatically V-generic for P_I, since it avoids all sets in I coded in V.) Shelah proved that the only ideal commuting with `meager` is `meager` itself. Corollary 3.1 can be formulated by saying that the only ideal commuting with `null` is `null` itself. In [20], Problem 11.5, Shelah asked whether there are ccc Suslin forcings other than Cohen and random that commute with themselves. (A forcing notion $\mathbb{P}$ is *suslin* if its underlying set is $\mathbb{R}$ and both $\leq_\mathbb{P}$ and $\perp_\mathbb{P}$ are analytic subsets of the plane. If I is analytic on G_δ, then P_I is easily Suslin.) By Theorem 3.3, the answer to this question restricted to definable forcings of the form P_I is negative.

Recław and Zakrzewski ([18]) say that a pair of ideals I, J has the *Fubini property* if for every Borel $B \subseteq 2^\omega \times 2^\omega$ such that $\{x | B_x \notin J\} = \emptyset$ we have $\{y | B^y \notin I\} \in J$. They have proved that in a certain restricted class of ccc σ-ideals of Borel sets (`meager`, `meager`) and (`null`, `null`) are the only pairs that have the Fubini property. They have also found a consistent example (using a large cardinal assumption) of a ccc σ-ideal I such that both (I, `null`) and (I, `meager`) have the Fubini property, and asked whether there are other 'natural' examples of pairs of ccc ideals with Fubini property. Theorem 3.3 gives a negative answer to their question restricted to the class of definable ideals.

In order to give unified treatment of ideals `meager` and `null` and the corresponding forcing notions Cohen and random, in [14, Definition 1.26] Kunen introduced the class of 'reasonable' ideals. Among other properties, every reasonable ideal is a *Fubini ideal* ([14, Definition 1.3]) and this implies that (I, I) has the Fubini property. Therefore by Theorem 3.3, `meager` and `null` are the only reasonable ideals that are analytic on G_δ. The definition of reasonable also involves being absolute ([14, Definition 1.20]) and under large cardinals every absolute set of reals belongs to $L(\mathbb{R})$ by [7, Theorem 3.2]. Therefore large cardinals imply that `meager` and `null` are the only reasonable ideals.

If the assumption that I is a Fubini ideal is dropped from the definition of a reasonable ideal then there are many ideals satisfying the weaker notion ([19]).

Lemma 3.2. *Suppose I and J are definable c.c.c. σ-ideals on 2^ω. Then the following are equivalent.*

1. *P_I and P_J commute.*
2. *If $B \subseteq 2^\omega \times 2^\omega$ is Borel then $\{x | B_x \notin J\} \notin I$ implies $\{y | B^y \notin I\} \neq \emptyset$.*

3. *Pair J, I has the Fubini property.*

Proof. Assume (2) fails and fix a Borel B such that $\{x|B_x \notin J\} \notin I$ and $C = \{y|B^y \notin I\} \in J$. Let $A = B \setminus 2^\omega \times C$, and note that $\{x|A_x \notin J\} \notin I$ and $\{y|A^y \notin I\} = \emptyset$. Let x be V-generic for P_I so that that $A_x \notin J$ and let $y \in A_x$ be $V[x]$-generic for P_J. Since $A^y \in I$ and $x \in A^y$, x is not P_I-generic over $V[y]$.

Now assume (1) fails, and fix a countable transitive model M of a large enough fragment of ZFC containing definitions of I and J. Since $\{x|x$ is M-generic for $P_I\}$ is equal to the complement of the union of all Borel sets coded in M that belong to I, it is Borel. Similarly, the set

$$A_{IJ} = \{(x,y)|x \text{ is } M\text{-generic for } P_I \text{ and } y \text{ is } M[x]\text{-generic for } P_J\}$$

is Borel, and $B = A_{IJ} \setminus \{(x,y)|(y,x) \in A_{JI}\}$ is a Borel set consisting of all pairs (x,y) that fail the commutativity condition. This set is nonempty by our assumption, and it satisfies (2).

To see that (2) and (3) are equivalent, take the contrapositive of (2). □

Theorem 3.3 (LC)**.** *Suppose I, J are definable c.c.c. σ-ideals on 2^ω. Then one of the following holds:*

1. *Both P_I and P_J are isomorphic to the Cohen algebra.*
2. *Both P_I and P_J are isomorphic to the Lebesgue measure algebra.*
3. *P_I and P_J do not commute.*

In particular, if P_I of this kind commutes with itself, then it is isomorphic to either Cohen or random.

By Fubini's and Kuratowski–Ulam theorems at most one of three statements holds. The rest of the proof of Theorem 3.3 breaks into several cases according to whether the posets P_I, P_J are bounding or not, with wildly different arguments in each case.

Lemma 3.4 (LC)**.** *Suppose I, J are definable c.c.c. σ-ideals on 2^ω such that both forcings P_I and P_J add an unbounded real. Exactly one of the following holds:*

- *there are Borel I-positive set B and a Borel J-positive set C such that both P_I below B and P_J below C are isomorphic to the Cohen algebra*
- *$I \perp J$*

Proof. There is nothing really new here. By the Kuratowski–Ulam theorem the first item implies the failure of $I \perp J$. On the other hand, suppose that the first item fails. Then one of the partial orders, P_I say, is not isomorphic to the Cohen algebra below any condition. By [20] 9.16 or [27] 6.6, $P_I \Vdash 2^\omega \cap V$ is meager, so $I \perp \texttt{meager}$ and there is a Borel set $E \subset 2^\omega \times 2^\omega$ such that its vertical sections are meager and the horizontal sections of its complement are I-small. By [21], 1.14, P_J adds a Cohen real over V and so there is a Borel function $f : 2^\omega \to 2^\omega$ such that preimages of meager sets are J-small. It is not difficult to verify that the Borel set $D \subset 2^\omega \times 2^\omega$ defined by $\langle x, y\rangle \in D$ if and only if $\langle x, f(y)\rangle \in E$ witnesses $I \perp J$. The lemma follows. □

Lemma 3.5 (LC)**.** *Suppose I, J are definable c.c.c. σ-ideals such that both forcings P_I and P_J are bounding. If P_I is not equivalent to random, then there is a Borel $B \subseteq 2^\omega \times 2^\omega$ such that $B_x \in J$ for all x and $B^y \notin I$ for all y.*

Proof. By Theorem 0.1, both I and J are null ideals for some continuous submeasures ϕ and ψ, respectively. By Lemma 2.1, ϕ does not have a control measure. Therefore we are in the situation of Lemma 2.6. □

Lemma 3.6 (LC)**.** *Suppose that I, J are definable c.c.c. σ-ideals on 2^ω such that P_I is bounding while P_J adds an unbounded real. Then $I \perp J$.*

Proof. By Theorem 0.1, there is a continuous submeasure ϕ such that I is the null ideal for ϕ. We will first prove that $I \perp \texttt{meager}$. For $s \in 2^n$ let $[s] = \{x \in 2^\omega | x \restriction n = s\}$.

Claim 3.7. *If ϕ is a continuous submeasure on the Borel algebra of 2^ω, then for every $\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $\phi([s]) \leq \varepsilon$ for every $s \in 2^{m_\varepsilon}$.*

Proof. Assume not, and find $s_m \in 2^m$ such that $\phi([s_m]) \geq \varepsilon$ for all m. Assume for a moment there is an infinite set $B \subseteq \omega$ such that $[s_m]$ $(m \in B)$ are pairwise disjoint. In this case the open sets $U_n = \bigcup\{[s_m] | m \geq n, n \in B\}$ have all submeasure at least ε and they are decreasing with empty intersection. Since ϕ is a continuous submeasure, this is impossible.

If there is no such B, by Ramsey's theorem there is an infinite set D such that $[s_m]$ $(m \in D)$ form a decreasing chain. The intersection $\bigcap_{m \in D} [s_m]$ is a singleton, $\{x\}$, and again by the continuity of the submeasure, $\phi(\{x\}) \geq \varepsilon$. Thus $\{x\} \notin I$, contradiction. □

Let $f(n) = m_{2^{-n}}$ as given by Claim 3.7. Interpret the Cohen forcing as adding a function $g \in \prod_n 2^{f(n)}$ with finite conditions. Let $D_m = \bigcup_{n>m}[g(n)]$. It is not difficult to see that $V \cap 2^\omega \subset D_m$ for every number $m \in \omega$ and the submeasures $\phi(D_m)$ converge to zero. Therefore $\bigcap_m D_m$ is a submeasure zero set containing all the ground model reals.

To show $I \perp J$ note that by a result of Shelah [21] the poset P_J adds a Cohen real. The argument is concluded in a manner similar to Lemma 3.4. □

Proof of Theorem 3.3. Let I, J be definable c.c.c. σ-ideals, and suppose that the first two alternatives in the Theorem fail. Use the c.c.c. to find partitions $2^\omega = B_0 \cup B_1$ and $2^\omega = C_0 \cup C_1$ into Borel sets such that P_I below B_0 and P_J below C_0 are bounding forcings while the posets P_I below B_1 and P_J below C_1 add an unbounded real. Pick i, j such that $B_i \notin I$ and $C_j \notin J$. If $i = j$ we may assure that if P_I is `meager` (`null`, respectively) below B_i then P_J is not `meager` (`null`, respectively) below C_j. In either case, by one of lemmas 3.4, 3.5 or 3.6 we are in the situation of Lemma 3.2. □

4. Concluding remarks

Another corollary of Theorem 0.1 precisely determines the extent of ccc-ness of a weakly distributive definable forcing P_I. Recall that a subset F of a poset $\mathbb{P}$ is n-linked if every n-element subset of F has a lower bound, and that $\mathbb{P}$ is *σ-n-linked* if it can be covered by countably many n-linked sets. An $F \subseteq \mathbb{P}$ is centered if every finite subset of F has a lower bound, and $\mathbb{P}$ is σ-centered if it can be covered by countably many centered subsets. It is well-known that all these chain conditions are different. Also, by a result of Todorcevic ([25], see also [3, 3.6.C]), there is a Borel ccc poset that is not σ-2-linked.

Corollary 4.1 (LC)**.** *If I is a σ-ideal of Borel sets and P_I is weakly distributive, then the following hold.*

1. *P_I is not σ-centered.*
2. *If I is moreover definable, then P_I is ccc if and only if it is σ-n-linked for all n.*

Proof. Assume $\mathcal{B}$ is σ-centered and fix centered sets X_n maximal under the inclusion whose union covers $\mathcal{B}$. Since by Fact 1.1 every positive set has a compact subset the intersection of each X_n is a singleton. This implies that a co-countable set belongs to I, a contradiction.

For 2, by Theorem 0.1 it suffices to prove a well-known fact that if ϕ is a continuous submeasure on Borel algebra of 2^ω and $\mathrm{Null}(\phi)$ contains all countable sets, then the quotient algebra is σ-n-linked for all n (this is [5, Exercise 393Y(a)]). Recall first that it is completely generated by its countable subalgebra $\mathcal{B}_0$ given by the name for the P_I-generic real. Now consider the metric on $\mathcal{B}$ defined by $\rho(A,B) = \phi(A\Delta B)$. It is not difficult to check that $(\mathcal{B},\rho)$ is a complete metric space, and as an easy consequence of [5, 393B (c)], it is isomorphic to the completion of $(\mathcal{B}_0,\rho)$, and in particular separable. For $A \in \mathcal{B}_0$ the set

$$F_A = \{C | \rho(A,C) < \rho(0_{\mathcal{B}}, A)/n\}.$$

is n-linked, and $\bigcup_{A\in\mathcal{B}_0} F_A$ covers $\mathcal{B}$. □

A poset $\mathbb{P}$ is poly-linked if for some sequence $\{k_n\}$ of natural numbers such that $\lim_n k_n = \infty$ it can be covered by sets F_n such that each F_n is k_n-linked. David Fremlin has recently proved that an infinite measure algebra cannot be poly-linked. We do not know whether 1. of Corollary 4.1 can be improved into 'P_I is not poly-linked.'

We conclude with a question asked by Solecki (personal communication).

Question 4.2. Are the following equivalent for every c.c.c. σ-ideal I on Borel subsets of 2^ω that is analytic on G_δ?

1. Compact sets are dense in P_I and I is ccc.
2. I is the null ideal of some continuous submeasure.

If the answer is positive, this would strengthen Theorem 0.1 and nicely complement a result of [13] where it was proved that every ccc σ-ideal σ-generated by compact sets is Borel-isomorphic to `meager`.

References

[1] U. Abraham and S. Todorčević, *Partition properties of ω_1 compatible with CH*, Fund. Math., **152** (1997), 165–181.

[2] B. Balcar., T. Jech, and T. Pazák, *Complete ccc boolean algebras, the order sequential topology, and a problem of von Neumann.* preprint, 2003.

[3] T. Bartoszynski and H. Judah, *Set theory: on the structure of the real line.* A.K. Peters, 1995.

[4] J.P.R. Christensen, *Some results with relation to the control measure problem*, Vector space measures and applications II, **645**, *Lecture Notes in Mathematics*, pages 27–34. Springer, 1978.

[5] D.H. Fremlin, *Measure Theory*, **3**. Torres–Fremlin, 2002.

[6] W. Herer and J.P.R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Mathematische Annalen, **213** (1975), 203–210.

[7] I. Neeman and J. Zapletal, *Proper forcing and absoluteness in l(r)*, Commentationes Mathematicae Universitatis Carolinae, **39** (1998), 281–301.

[8] T. Jech, *Set Theory*, Academic Press, San Diego, 1978.

[9] N.J. Kalton, *The Maharam problem*, Séminaire Initiation à Í Analyse, 28e Anńee, **18** (1988), 1–13.

[10] N.J. Kalton and J.W. Roberts, *Uniformly exhaustive submeasures and nearly additive set functions*, Transactions of the American Mathematical Society, **278** (1983), 803–816.

[11] A. Kanamori, *The higher infinite: large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer–Verlag, Berlin–Heidelberg–New York, 1995.

[12] A.S. Kechris, *Classical descriptive set theory*, **156**, Graduate texts in mathematics. Springer, 1995.

[13] A.S. Kechris and S. Solecki, *Approximating analytic by Borel sets and definable chain conditions*, Israel J. Math., **89** (1995), 343–356.

[14] K. Kunen, *Random and cohen reals*, Handbook of Set-Theoretic Topology. North-Holland, 1984.

[15] A. Louveau, *Progres recents sur le probleme de Maharam d'apres N.J. Kalton et J.W. Roberts*, Séminaire Initiation à Í Analyse, 28e Anńee, **20** (1983), 1–8.

[16] D. Maharam, *An algebraic characterization of measure algebras*, Annals of mathematics, **48** (1947), 154–167.

[17] S. Quickert, *CH and the Sacks property*, Fund. Math., **171** (2002), 93–100.

[18] I. Recław and P. Zakrzewski, *Fubini properties of ideals*, Real Anal. Exchange, **25** (1999), 565–578.

[19] A. Roslanowski and S. Shelah, *Norms on possibilities ii: more ccc ideals on* 2^ω, Journal of Applied Analysis, **3** (1997), 103–127.

[20] S. Shelah, *Properness without elementaricity*, Journal of Applied Analysis, accepted.

[21] ———, *How special are cohen and random forcings i.e. boolean algebras of the family of subsets of reals modulo meagre or null*, Israel Journal of Mathematics, **88** (1994), 159–174.

[22] S. Solecki, *Analytic ideals and their applications*, Annals of Pure and Applied Logic, **99** (1999), 51–72.

[23] S. Todorcevic, *Definable ideals and gaps in their quotients*, Set Theory: Techniques and Applications, 213–226. Kluwer Academic Press, 1997.

[24] ———, *A dichotomy for P-ideals of countable sets*, Fundamenta Mathematicae, **166** (2000), 251–267.

[25] ———, *Two examples of Borel partially ordered sets with the countable chain condition*, Proc. Amer. Math. Soc., **112** (1991), 1125–1128.

[26] B. Velickovic, *CCC forcings and splitting reals*, preprint, 2003.

[27] J. Zapletal, *Proper forcing and rectangular Ramsey theorems*, preprint.

[28] ———, *Descriptive Set Theory and Definable Forcing.* Memoirs of American Mathematical Society. AMS, Providence, 2004.

Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON, Canada M6G 1N5

Matematicki Institut, Kneza Mihaila 35, Belgrade, Serbia and Montenegro
E-mail address: farah@mathstat.yorku.ca

Department of Mathematics and Statistics, University of Florida, Gainesville, FL 32611-8105
E-mail address: zapletal@math.ufl.edu