DEHN SURGERY, THE FUNDAMENTAL GROUP AND $SU(2)$

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1. Introduction

The main result of this paper, which is a companion to [14], is the following theorem.

Theorem 1. Let $K$ be a non-trivial knot in $S^3$, and let $Y_r$ be the 3-manifold obtained by Dehn surgery on $K$ with surgery-coefficient $r \in \mathbb{Q}$. If $|r| \leq 2$, then $\pi_1(Y_r)$ is not cyclic. In fact, there is a homomorphism $\rho : \pi_1(Y_r) \to SU(2)$ with non-cyclic image.

The statement that $Y_r$ cannot have cyclic fundamental group was previously known for all cases except $r = \pm 2$. The case $r = 0$ is due to Gabai [12], the case $r = \pm 1$ is the main result of [14], and the case that $K$ is a torus knot is analysed for all $r$ in [16]. All remaining cases follow from the cyclic surgery theorem of Culler, Gordon, Luecke and Schalen [2]. It is proved in [15] that $Y_2$ cannot be homeomorphic to $\mathbb{RP}^3$. If one knew that $\mathbb{RP}^3$ was the only closed 3-manifold with fundamental group $\mathbb{Z}/2\mathbb{Z}$ (a statement that is contained in Thurston’s geometrization conjecture), then the first statement in the above theorem would be a consequence. The second statement in the theorem appears to sharpen the result slightly. In any event, we have:

Corollary 2. Dehn surgery on a non-trivial knot cannot yield a 3-manifold with the same homotopy type as $\mathbb{RP}^3$.

The proof of Theorem 1 provides a verification of the Property P conjecture that is independent of the results of the cyclic surgery theorem of [2]. Although the argument follows [14] very closely, we shall avoid making explicit use of instanton Floer homology and Floer’s exact triangle [11, 1]. Instead, we rely on the technique that forms just the first step of Floer’s proof from [11], namely the technique of “holonomy perturbations” for the instanton equations (see also the remark following Proposition 16 in [14]).
2. Holonomy perturbations

This section is a summary of material related to the "holonomy perturbations" which Floer used in the proof of his surgery exact triangle for instanton Floer homology [11]. Similar holonomy perturbations were introduced for the 4-dimensional anti-self-duality equations in [3]; see also [17]. Our exposition is taken largely from [1] with only small changes in notation. Some of our gauge-theory notation is taken from [13].

Let $Y$ be a compact, connected 3-manifold, possibly with boundary. Let $w$ be a unitary line bundle on $Y$, and let $E$ be a unitary rank 2 bundle equipped with an isomorphism

$$\psi : \text{det}(E) \to w.$$ 

Let $g_E$ denote the bundle whose sections are the traceless, skew-hermitian endomorphisms of $E$, and let $A$ be the affine space of $SO(3)$ connections in $g_E$. Let $G$ be the gauge group of unitary automorphisms of $E$ of determinant 1 (the automorphisms that respect $\psi$). We write $\mathcal{B}^w(Y)$ for the quotient space $A/G$. A connection $A$, or its gauge-equivalence class $[A] \in \mathcal{B}^w(Y)$, is irreducible if the stabilizer of $A$ is the group $\{\pm 1\} \subset G$, and is otherwise reducible. The reducible connections are the ones that preserve a decomposition of $g_E$ as $\mathbb{R} \oplus L$, where $L$ is an orientable 2-plane bundle; these connections have stabilizer either $S^1$ or (in the case of the product connection) the group $SU(2)$.

**Definition 3.** We write $\mathcal{R}^w(Y) \subset \mathcal{B}^w(Y)$ for the space of $G$-orbits of flat connections:

$$\mathcal{R}^w(Y) = \{ [A] \in \mathcal{B}^w(Y) \mid F_A = 0 \}.$$ 

This is the representation variety of flat connections with determinant $w$. □

We have the following straightforward fact:

**Lemma 4.** The representation variety $\mathcal{R}^w(Y)$ is non-empty if and only if $\pi_1(Y)$ admits a homomorphism $\rho : \pi_1(Y) \to SO(3)$ with $w_2(\rho) = c_1(w) \mod 2$. The representation variety contains an irreducible element if and only if there is such a $\rho$ whose image is not cyclic.

If $c_1(w) = 0 \mod 2$, then $\mathcal{R}^w(Y)$ is isomorphic to the space of homomorphisms $\rho : \pi_1(Y) \to SU(2)$ modulo the action of conjugation. □

Suppose now that $Y$ is a closed oriented 3-manifold. The flat connections $A \in \mathcal{A}$ are the critical points of the Chern-Simons function

$$CS : \mathcal{A} \to \mathbb{R},$$

$$CS(A) = \frac{1}{4} \int_Y \text{tr}((A - A_0) \wedge (F_A + F_{A_0})), $$

where $A_0$ is a chosen reference point in $\mathcal{A}$, and tr denotes the trace on 3-by-3 matrices. We define a class of perturbations of the Chern-Simons functional, the holonomy perturbations.
Let $D$ be a compact 2-manifold with boundary, and let $\iota : S^1 \times D \rightarrow Y$. Choose a trivialization of $w$ over the image of $\iota$. With this choice, each connection $A \in \mathcal{A}$ gives rise to a unique connection $\tilde{A}$ in $E|_{\text{im}(\iota)}$ with the property that $\det(\tilde{A})$ is the product connection in the trivialized bundle $w|_{\text{im}(\iota)}$. Thus $\tilde{A}|_{\text{im}(\iota)}$ is an $SU(2)$ connection. Given a smooth 2-form $\mu$ with compact support in the interior of $D$ and integral 1, and given a smooth class-function $\phi : SU(2) \rightarrow \mathbb{R}$, we can construct a function

$$\Phi : \mathcal{A} \rightarrow \mathbb{R}$$

that is invariant under $\mathcal{G}$ as follows. For each $z \in D$, let $\gamma_z$ be the loop $t \mapsto \iota(t, z)$ in $Y$, and let $\text{Hol}_{\gamma_z}(\tilde{A})$ denote the holonomy of $\tilde{A}$ along $\gamma_z$, as an automorphism of the fiber $E$ at the point $y = \iota(0, z)$. The class-function $\phi$ determines also a function on the group of determinant-1 automorphisms of the fiber $E_y$, and we set

$$\Phi(A) = \int_D \phi(\text{Hol}_{\gamma_z}(\tilde{A}))\mu(z).$$

One can write down the equations for a critical point $A$ of the function $CS + \Phi$ on $\mathcal{A}$. They take the form

$$F_A = \phi'(H_A)\mu_Y,$$

where $H_A$ is the section of the bundle $\text{Aut}(E)$ over $\text{im}(\iota)$ obtained by taking holonomy around the circles, $\phi'$ is the derivative of $\phi$, regarded as a map from $\text{Aut}(E)$ to $\mathfrak{g}_E$, and $\mu_Y$ is the 2-form on $Y$ obtained by pulling back $\mu$ to $S^1 \times D$ and then pushing forward along $\iota$. (See [1].)

**Definition 5.** Given $\iota$ and $\phi$ as above, we write

$$\mathcal{R}^w_{\iota, \phi}(Y) = \{ [A] \in \mathcal{B}^w(Y) \mid F_A = \phi'(H_A)\mu_Y \}.$$ 

This is the *perturbed representation variety*. □

Now specialize to the case that $D$ is a disk, so $\iota$ is an embedding of a solid torus. Let

$$C = Y \setminus \text{im}(\iota)^\circ$$

be the complementary manifold with torus boundary. Let $z_0 \in \partial D$ be a basepoint, and let $a$ and $b$ be the oriented circles in $\partial C$ described by

$$a = \iota(S^1 \times \{z_0\}),$$

$$b = \iota(\{0\} \times \partial D).$$

(1)

These are the “longitude” and “meridian” of the solid torus. We continue to suppose that $w$ is trivialized on $\text{im}(\iota)$ and hence on $\partial C$. So the restriction of $E$ to $\partial C$ is given the structure of an $SU(2)$ bundle. Given a connection $A$ on $\mathfrak{g}_E$ that is flat on $\partial C$, let $\tilde{A}$ be the corresponding flat $SU(2)$ connection in $E|_{\partial C}$. One can choose a determinant-1 isomorphism between the fiber of $E$ at the basepoint
ι(0, z₀) so that the holonomies of ˜A around a and b become commuting elements of SU(2) given by

\[
\text{Hol}_a(\tilde{A}) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \quad \text{Hol}_b(\tilde{A}) = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix}.
\]

The pair \((\alpha(A), \beta(A)) \in \mathbb{R}^2\) is determined by \(A\) up to the ambiguities

1. adding integer multiples of \(2\pi\) to \(\alpha\) or \(\beta\);
2. replacing \((\alpha, \beta)\) by \((-\alpha, -\beta)\).

**Definition 6.** Let \(S \subset \mathbb{R}^2\) be a subset of the plane with the property that \(S + 2\pi \mathbb{Z}^2\) is invariant under \(s \mapsto -s\). Define the set

\[\mathcal{R}_w(C \mid S) \subset \mathcal{R}_w(C)\]

as

\[\mathcal{R}_w(C \mid S) = \{ [A] \in \mathcal{R}_w(C) \mid (\alpha(A), \beta(A)) \in S + 2\pi \mathbb{Z}^2 \} ,\]

where \((\alpha(A), \beta(A))\) are the longitudinal and meridional holonomy parameters, determined up to the ambiguities above.

One should remember that the choice of trivialization of \(w\) on im(ι) is used in this definition, and in general the set we have defined will depend on this choice.

A class-function \(\phi\) on \(SU(2)\) corresponds to a function \(f : \mathbb{R} \to \mathbb{R}\) via

\[f(t) = \phi \left( \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right) .\]

The function \(f\) satisfies \(f(t) = f(t + 2\pi)\) and \(f(-t) = f(t)\). The following observation of Floer’s is proved as Lemma 5 in [1].

**Lemma 7.** Let \(f : \mathbb{R} \to \mathbb{R}\) correspond to \(\phi\) as above. Then restriction from \(Y\) to \(C\) gives rise to a bijection

\[\mathcal{R}_{\iota,\phi}^w(Y) \to \mathcal{R}_w^w(C \mid \beta = -f'(\alpha)) .\]

We also have the straightforward fact:

**Lemma 8.** If \(g : \mathbb{R} \to \mathbb{R}\) is a smooth odd function with period \(2\pi\), then there is a class-function \(\phi\) on \(SU(2)\) such that the corresponding function \(f\) satisfies \(f' = g\).
3. Removing flat connections by perturbation

Let us now take the case that \( Y \) is a homology \( S^1 \times S^2 \), and let \( w \to Y \) be a line-bundle with \( c_1(w) \) a generator for \( H^2(Y;\mathbb{Z}) = \mathbb{Z} \). Let \( N \to Y \) be an embedded solid torus whose core is a curve representing a generator of \( H_1(Y;\mathbb{Z}) \), and let \( C \) be the manifold with torus boundary

\[
C = Y \setminus N^0.
\]

By a “slope” we mean an isotopy class of essential closed curves on the torus \( \partial C \). For each slope \( s \), let \( Y_s \) denote the manifold obtained from \( C \) by Dehn filling with slope \( s \): that is, \( Y_s \) is obtained from \( C \) by attaching a solid torus in such a way that curves in the class \( s \) bound disks in the solid torus.

Parametrize \( N \) by a map \( \iota : S^1 \times D^2 \to N \). Let \( a \) and \( b \) be the curves (1) on \( \partial N \). The Dehn filling \( Y_b \) on the slope represented by \( b \) is just \( Y_b \). The manifold \( Y_a \) has \( H_1(Y_a;\mathbb{Z}) = 0 \). Let \( s \) be the slope

\[
s = [pa + qb],
\]

where \( p \) and \( q \) are coprime and both positive.

**Proposition 9.** Let \( s \) be as above, and suppose \( p/q \leq 2 \).

Suppose that neither \( \pi_1(Y_a) \) nor \( \pi_1(Y_s) \) admits a homomorphism to \( SU(2) \) with non-cyclic image. Then there is a holonomy-perturbation \( (\iota, \phi) \) for the manifold \( Y \) such that the perturbed representation variety \( R_{\iota,\phi}(Y) \) is empty.

**Proof.** Fix a trivialization \( \tau \) of \( w \) over \( N \). At this stage the choice is immaterial, because any two choices differ by an automorphism of \( w \) that extends over all of \( Y \). Write

\[
Y_a = C \cup N_a, \\
Y_s = C \cup N_s,
\]

where \( N_a \) and \( N_s \) are the solid tori from the Dehn surgery. The trivialization of \( w \) over \( \partial C \) allows us to extend \( w \) to a line-bundle \( w_a \to Y_a \) equipped with a trivialization \( \tau_a \) over \( N_a \), extending the given trivialization on \( \partial C \). Note that \( w_a \) is globally trivial on the homology 3-sphere \( Y_a \), but the global trivialization differs from \( \tau_a \) on the curve \( b \subset \partial C \) by a map \( b \to S^1 \) of degree 1. This is because there is a surface \( \Sigma \subset C \) with boundary \( b \), and the original trivialization \( \tau \) does not extend over \( \Sigma \). The same remarks apply to \( Y_s \).

On the manifold \( Y_s \), in addition to constructing \( w_s \) as above, we construct a different line bundle \( \tilde{w}_s \to Y_s \) as follows. Let \( \tilde{\tau} \) be the trivialization of \( w|_{\partial C} \) with the property that \( \tilde{\tau} \tilde{\tau}^{-1} \) is a map \( \partial C \to S^1 \) with degree \( q \) on \( b \) and degree 0 on \( a \). Let \( \tilde{w}_s \) be obtained by extending \( w \) as a trivial bundle over \( N_s \) extending the trivialization \( \tilde{\tau} \).

If \( p \) is odd, then \( Y_s \) has \( H^2(Y_s;\mathbb{Z}/2) = 0 \). When \( p \) is even, the construction of \( \tilde{w}_s \) makes \( c_1(\tilde{w}_s) \) divisible by 2. So in either case, elements of \( R_{\tau_s}(Y_s) \) correspond to homomorphisms \( \rho : \pi_1(Y_s) \to SU(2) \).
The following lemma is straightforward.

**Lemma 10.** Restriction to \( C \) gives identifications

\[
\begin{align*}
\mathcal{R}^w_s(Y_a) & \rightarrow \mathcal{R}^w(Y_a \mid \alpha = 0) \\
\mathcal{R}^w_s(Y_s) & \rightarrow \mathcal{R}^w(C \mid p\alpha + q\beta = 0) \\
\mathcal{R}^w_s(Y_s) & \rightarrow \mathcal{R}^w(C \mid p\alpha + q\beta = q\pi)
\end{align*}
\]

The manifold \( C \) has \( H_1(C; \mathbb{Z}) = \mathbb{Z} \), so the representation variety \( \mathcal{R}^w(C) \) contains reducibles. The next lemma describes their \( \alpha \) and \( \beta \) parameters.

**Lemma 11.** If \([A]\) is a reducible element of \( \mathcal{R}^w(C) \), then \((\alpha(A), \beta(A))\) lies on the line \( \beta = \pi \mod 2\pi \mathbb{Z} \).

**Proof.** If \([A]\) is a reducible element of \( \mathcal{R}^w(C) \), then \( A \) is a flat \( SO(3) \) connection on \( C \) with cyclic holonomy. The holonomy around \( b \) is the identity element of \( SO(3) \) because \( b \) bounds the surface \( \Sigma \) in \( C \). So the corresponding \( SU(2) \) connection \( \tilde{A} \) on \( E|b \) (regarding \( E|b \) as an \( SU(2) \) bundle using \( \tau \)) has holonomy \( \pm 1 \) in \( SU(2) \). It follows that \( \beta \) is 0 or \( \pi \mod 2\pi \). We can equip \( w \) on \( C \) with a connection \( \theta \) which respects the trivialization \( \tau \) on \( \partial C \) and whose curvature \( F_\theta \) integrates to \(-2\pi i \) on \( \Sigma \). The \( SU(2) \) connection \( \tilde{A} \) can be uniquely extended to a \( U(2) \) connection \( \tilde{A} \) on all of \( E|C \), in such a way that the associated \( SO(3) \) connection is \( A \) and such that the induced connection on \( \det(E) = w \) is \( \theta \). The connection reduces \( E \) to a sum of line bundles, both of which have curvature \( F_\theta/2 \). The holonomy of these line bundles on \( b \) is given by

\[
\exp \int_\Sigma (F_\theta/2) = -1.
\]

So \( \beta = \pi \mod 2\pi \) as claimed. This completes the proof of the lemma.

If we suppose that the homology-sphere \( Y_a \) has a fundamental group with no non-trivial homomorphisms to \( SU(2) \), then \( \mathcal{R}^w_s(Y_a) \) consists of a single reducible element. By the previous two lemmas, the \( \alpha \) and \( \beta \) parameters of this connection lie on the two line \( \alpha = 0 \) and \( \beta = \pi \). So it is the point

\[
v_a = (0, \pi)
\]

mod \( 2\pi \mathbb{Z}^2 \). Similarly the \( \alpha \) and \( \beta \) parameters of the reducible elements in \( \mathcal{R}^w_s(Y_s) \) lie on the line \( p\alpha + q\beta = \pi \mod 2\pi \) and the line \( \beta = \pi \). So they are represented by the points

\[
v_{k,k} = (2\pi k/p, \pi)
\]

mod \( 2\pi \mathbb{Z}^2 \). The next lemma is a standard result, from [11] of [1]. We supply the proof for completeness.

**Lemma 12.** Suppose \( \pi_1(Y_a) \) admits no non-trivial homomorphisms to \( SU(2) \). For any neighborhood \( W \) of \((0, \pi)\), let us write

\[
W^* = W \cap \{\beta \neq \pi\}.
\]
Then there exists a symmetric neighborhood $W$ of $(0, \pi)$ such that
\[
\mathcal{R}^w(C \mid W^*) = \emptyset.
\]

Proof. The space $\mathcal{R}^w(Y_a)$ consists of a single point, represented by the $SO(3)$ connection $A_a$ with trivial holonomy. By the one-to-one correspondence from Lemma 10, it follows that $\mathcal{R}^w(C \mid (0, \pi))$ consists of a single point $[A]$ represented by an $SO(3)$ connection which trivializes $g_E$. We need only show that a neighborhood of $[A]$ in $\mathcal{R}^w(C)$ consists entirely of reducibles. Equivalently, writing $\pi$ for $\pi_1(C)$, we can study a neighborhood of the trivial homomorphism $\rho_1: \pi \to SO(3)$ and show that it consists of reducible connections.

The deformations of $\rho_1$ are governed by $H^1(\pi; \mathbb{R}^3) = H^1(C) \otimes \mathbb{R}^3$, which is a copy of $\mathbb{R}^3$. It will be sufficient to exhibit a 1-parameter deformation of $\rho$ realizing any given vector in this $H^1$ as its tangent vector and consisting entirely of reducibles. This is straightforward. Given $\xi \in \mathfrak{so}(3)$, we can consider the 1-parameter family of connections in the trivial $SO(3)$ bundle given by the connection 1-forms $t\xi \eta$, where $\eta$ is a closed 1-form with period 1 on $C$ and $t \in \mathbb{R}$.

We need one more lemma before completing the proof of Proposition 9.

Lemma 13. For any $S$, there is a one-to-one correspondence between $\mathcal{R}^w(C \mid S)$ and $\mathcal{R}^w(C \mid S')$, where $S'$ is the translate $S + (\pi, 0)$.

Proof. Let $\epsilon$ be an automorphism of the $U(2)$ bundle $E \to C$ whose determinant is a function $C \to S^1$ which has degree 1 on the curve $a$. (The automorphism $\epsilon$ does not belong to the gauge group $G$, because elements of $G$ have determinant 1.) The element $\epsilon$ acts on the space of flat connections $A$ in $A(C)$, and gives rise to a bijective self-map of the space $\mathcal{R}^w(C)$:
\[
\bar{\epsilon}: \mathcal{R}^w(C) \to \mathcal{R}^w(C).
\]

This map restricts to a bijection $\bar{\epsilon}: \mathcal{R}^w(C \mid S) \to \mathcal{R}^w(C \mid S')$.

We can now conclude the proof of the proposition. Suppose that $\pi_1(Y_a)$ admits only the trivial homomorphism to $SU(2)$, and that the only homomorphisms $\rho: \pi_1(Y_a) \to SU(2)$ are those with cyclic image. Let $L \subset \mathbb{R}^2$ be the closed line segment
\[
L = \{ (\alpha, \beta) \mid \alpha = 0, -\pi \leq \beta \leq \pi \}
\]
and let $L^*$ be the open line-segment obtained by removing the endpoints. Let $L^*_\pi$ and $L^*_{-\pi}$ be the translates of this line segment by the vectors $(\pi, 0)$ and $(-\pi, 0)$. By Lemmas 10 and 11, the hypothesis on $\pi_1(Y_a)$ means that
\[
\mathcal{R}^w(C \mid L^*) = \emptyset.
\]

By Lemma 13, we therefore have
\[
\mathcal{R}^w(C \mid L^*_\pm) = \emptyset.
\]
Let $P_1$ be the line
\[ P = \{ p\alpha + q\beta = q\pi \} \]
and let $P_2 = P_1 - (0, 2\pi)$. The hypothesis on $\pi_1(Y_s)$ means that $R^w(C \mid P_i)$ consists only of reducibles, lying over the points on $P_i$ where $\beta = \pi \mod 2\pi$. Let $S \subset \mathbb{R}^2$ be the piecewise-linear arc with vertices at the points
\[
\begin{align*}
  z_1 &= (-\pi, 0) \\
  z_2 &= (-\pi, -(1 - p/q)\pi) \\
  z_3 &= (0, -\pi) \\
  z_4 &= (0, \pi) \\
  z_5 &= (\pi, (1 - p/q)\pi) \\
  z_6 &= (\pi, 0).
\end{align*}
\]

Figure 1 shows the set $S$ in the case $p/q = 5/3$. Because $p/q \leq 2$, the set is contained in the region $-\pi \leq \beta \leq \pi$. If $p/q = 2$, then $S$ has four points on the lines $\beta = \pm\pi$; otherwise it has just two.

Let $S^*$ be the complement in $S$ in of the points whose $\beta$ coordinates are $\pm\pi$.

Given any symmetric neighborhood $U$ of $S$, let $U^*$ similarly stand for
\[ U^* = U \setminus \{ \beta = \pm\pi \}. \]

We know that $R^w(C \mid S^*) = \emptyset$, because $S$ is entirely contained in the union of $L$, $L_{\pm\pi}$ and the two lines $P_1$, $P_2$. From Lemma 12 and the compactness of $R^w(C)$, it follows that there is a symmetric neighborhood $U$ of $S$ such that
\[ R^w(C \mid U^*) = \emptyset. \]

We now observe that, given any neighborhood $U$ of $S$, we can find a smooth odd function $g$ with period $2\pi$ such that the graph of $-g$ on the interval $[-\pi, \pi]$. 

\[ \text{Figure 1. The set } S, \text{ for } p/q = 5/3. \text{ The } (\alpha, \beta) \text{ parameters of reducible elements of } R^w(C) \text{ lie on the dashed lines.} \]
is entirely contained in $U^*$. By Lemma 7 and Lemma 8, there exists a $\phi$ such that

$$R^w_{i,\phi}(Y) = R^w(C \mid \beta = -g(\alpha)).$$

The right hand side is empty because it is contained in the empty set $(3)$. This finishes the proof of the proposition.

We can reformulate the result of Proposition 9 in the special case that $Y_{w}$ is $S^3$ as follows.

**Corollary 14.** Let $K$ be a knot in $S^3$ and let $Y_r$ be the manifold obtained by Dehn surgery with coefficient $r \in \mathbb{Q}$. Let $Y_0$ be the manifold obtained by $0$-surgery, and let $w \to Y_0$ be a line bundle whose first Chern class is a generator of $H^2(Y_0; \mathbb{Z})$. Suppose $\pi_1(Y_r)$ admits no homomorphism $\rho$ to $SU(2)$ with non-cyclic image. Then, if $0 < r < 2$, the manifold $Y_0$ admits a holonomy deformation $(\iota, \phi)$ so that $R^w_{i,\phi}(Y_0)$ is empty.

4. Proof of the theorem

4.1. A stretching argument. Let $X$ be a closed, oriented 4-manifold containing a connected, separating 3-manifold $Y$. Let $g_1$ be metric on $X$ that is cylindrical on a collar region $[-1,1] \times Y$ containing $Y$ in $X$. For $L > 0$, let $X_L \cong X$ be the manifold obtained from $X$ by removing the piece $[-1,1] \times Y$ and replacing it with $[-L,L] \times Y$. There is a metric $g_L$ on $X_L$ that contains a cylindrical region of length of $2L$ and agrees with the original metric on the complement of the cylindrical piece.

Let $v \to X$ be a line bundle, let $E \to X$ be a unitary rank-2 bundle with $\det(E) = v$, and form the configuration space $B^v(X,E)$ of connections in $g_E$ modulo determinant-1 gauge transformations of $E$, as we did in the 3-dimensional case. In dimension 4, the bundle $E$ is not determined up to isomorphism by $v$ alone, so we include it in our notation. Inside $B^v(X,E)$ is the moduli space of anti-self-dual connections,

$$M^v(X,E) = \{ [A] \in B^v(X,E) \mid F_A^+ = 0 \}.$$

For each $L > 0$, we also have a moduli space

$$M^v(X_L,E) \subset B^v(X_L,E).$$

(We do not take the trouble to introduce the additional notation $v_L$ and $E_L$ for the corresponding bundles on $X_L$.)

Let $(\iota, \phi)$ be data for a holonomy perturbation for the bundle $E|_Y$. Following [10, 11, 4], we shall use $\phi$ also to perturb the anti-self-duality equations on $X_L$. We use $\iota$ to embed $[-L,L] \times S^1 \times D$ into $X_L$, and let $\mu_X$ be the 2-form on the cylindrical part $[-L,L] \times Y$ obtained by pulling back $\mu$ from $D$ and pushing forward using this embedding. We choose a trivialization of $v = \det(E)$ on the image of the embedding so that each $SO(3)$ connection in $g_E$ determines uniquely an $SU(2)$ connection. For each $A$, the holonomy around the circles
defines, as before, a section $H_A$ over $[-L, L] \times \text{im}(\iota)$ of the bundle $\text{Aut}(E)$, and we obtain

$$\phi'(H_A) \in C^\infty([-L, L] \times \text{im}(\iota); g_E).$$

For $L > 1$, let $\beta : X_L \to [0, 1]$ be a smooth cut-off function, supported in $[-L, L] \times Y$ and equal to 1 on $[-L + 1, L - 1] \times Y$. On $X_L$, the perturbed anti-self-duality equation is the equation

$$F_A^+ + \beta \phi'(H_A) \mu^+ = 0. \tag{4}$$

We define the corresponding moduli space:

$$M^v_\phi(X_L, E) = \{ [A] \in B^v(X_L, E) \mid \text{equation (4) holds} \}. \tag{5}$$

**Proposition 15.** Let $w = v|_Y$. Suppose that there is a holonomy perturbation on $Y$ such that the perturbed representation variety $R^v_{i, \phi}(Y)$ is empty. Then for each $E$ with determinant $v$ on $X$, there exists an $L_0$ such that $M^v_\phi(X_L, E)$ is also empty, for all $L \geq L_0$.

**Proof.** The proof is some subset of a standard discussion of holonomy perturbations and compactness in Floer homology theory (see [11, 1, 4]). Suppose on the contrary that we can find $[A_i]$ in $M^v_\phi(X_L, E)$ for an increasing, unbounded sequence of lengths $L_i$. We start as usual with the fact that the quantity

$$E(A_i) = \int_{X_{L_i}} \text{tr}(F_{A_i} \wedge F_{A_i})$$

$$= \|F_{A_i}^-\|^2 - \|F_{A_i}^+\|^2$$

is independent of $i$ and depends only on the Chern numbers of the bundle $E$. (The norms are $L^2$ norms.) We write this quantity as the sum of three terms:

$$E(A_i) = E(A_i | X^1) + E(A_i | X^2) + E(A_i | X^3_i),$$

where

$$X^1 = X_{L_i} \setminus ([L_i - 1, L_i] \times Y)$$

$$X^2 = ([L_i - 1, L_i] \times Y) \cup ([L_i - 1, L_i] \times Y)$$

$$X^3_i = [-L_i + 1, L_i - 1] \times Y.$$

Only the third piece has a geometry which depends on $i$. From the equation (4), we have

$$E(A_i | X^1) \geq 0$$

because $\beta$ is zero on $X^1$. The second term in equation (4) is pointwise uniformly bounded, so

$$E(A_i | X^2) \geq -C_2$$

where $C_2$ is independent of $i$. Because the sum of the three terms is constant, we deduce that

$$E(A_i | X^3_i) \leq K,$$

where $K$ is independent of $i$. 

To understand the term $\mathcal{E}(A_i \mid X^3_i)$ better, one must reinterpret (4). On $X^3_i$, the function $\beta$ is 1. Identify $E$ on this cylinder with the pull-back of a bundle $E_Y \to Y$, and choose a gauge representative $A_i$ for $[A_i]$ in temporal gauge. Write

$$A_i(t) = A_i|_{\{t\} \times Y}, \quad (-L_i + 1 \leq t \leq L_i - 1).$$

Thus $A_i(t)$ becomes a path in the space of connections $\mathcal{A}(Y; E_Y)$. The equation (4) is equivalent on $X^3_i$ to the condition that $A_i(t)$ solves the downward gradient flow equation for the perturbed Chern-Simons functional on $\mathcal{A}(Y; E_Y)$:

$$\frac{d}{dt} A_i(t) = -\text{grad}(CS + \Phi).$$

In particular, $CS + \Phi$ is monotone decreasing along the path (or constant). The function $|\Phi|$ is a bounded function on $\mathcal{A}(Y; E_Y)$: we can write $|\Phi| \leq K'$. The change in $CS$ is equal to the quantity $-\mathcal{E}$; that is,

$$CS(A_i(-L_i + 1)) - CS(A_i(L_i - 1)) = \mathcal{E}(A_i \mid X^3_i) \leq K.$$

So from the bound on $|\Phi|$ we obtain

$$(CS + \Phi)(A_i(-L_i + 1)) - (CS + \Phi)(A_i(L_i - 1)) \leq K + 2K'.$$

Now let $\delta > 0$ be given. Because $CS + \Phi$ is decreasing and the total drop is bounded by $K + 2K'$, we can find intervals

$$(a_i, b_i) \subset [-L_i + 1, L_i + 1]$$

of length $\delta$, so that the drop in $CS + \Phi$ along $(a_i, b_i)$ tends to zero as $i$ goes to infinity. Because the equation is a gradient-flow equation, this means

$$\lim_{i \to \infty} \int_{a_i}^{b_i} \|\text{grad}(CS + \Phi)(A_i(t))\|_{L^2(Y)}^2 dt = 0.$$

We have an expression for $\text{grad}\Phi$ as a uniformly bounded form, so

$$\limsup_{i \to \infty} \int_{a_i}^{b_i} \|F_{A_i(t)}\|_{L^2(Y)}^2 dt \leq \delta J$$

for some constant $J$ depending on $\phi$. So given any $\epsilon > 0$, we can find a $\delta > 0$ and a sequence of intervals $(a_i, b_i)$ of length $\delta$ so that

$$\int_{(a_i, b_i) \times Y} |F_{A_i}|^2 dvol \leq \epsilon$$

for all $i \geq i_0$. We now regard the $A_i$ as connections on the fixed cylinder $(0, \delta) \times Y$. At this point, if $\epsilon$ is smaller than the threshold for Uhlenbeck’s gauge fixing theorem on the 4-ball, we can find 4-dimensional gauge transformations on the cylinder so that, after applying these gauge transformations and passing to a subsequence, the connections converge in $C^\infty$ on compact subsets. (See for example [4, section 5.5].)
If \( A \) is the limiting connection on \((0, \delta) \times Y\), in temporal gauge, then the function \( \text{CS} + \Phi \) is constant along the path \( A(t) \). It follows that \( A(t) \) is constant and is a critical point of \( \text{CS} + \Phi \). This tells us that \([A(t)]\) belongs to the perturbed representation variety \( R_{\iota, \phi}^w(Y) \), which we were supposing to be empty.

The proposition above has the following corollary for the Donaldson polynomial invariants. (Our notation and conventions for these invariants is taken from [13].)

**Corollary 16.** Let \( X \) be an admissible 4-manifold in the sense of [13], so that its Donaldson polynomial invariants \( D^v_X \) are defined. (For example, suppose \( H_1(X; \mathbb{Z}) \) is zero and \( b^+(X) \) is greater than 1.) Then, under the assumptions of the previous proposition, the polynomial invariants are identically zero, regarded as a map

\[
D^v_X : \mathbb{A}(X) \to \mathbb{Z}.
\]

**Proof.** The definition of \( D^v_X \) involves first choosing a Riemannian metric on \( X \) so that the moduli spaces \( M^v(X, E) \) are smooth submanifolds of \( B^v(X, E) \), containing no reducibles and cut out transversely by the equations. If \( X \) is admissible, then this can always be done, by changing the metric inside a ball in \( X \). The value of the invariant is then defined as a signed count of the intersection points between \( M^v(X, E) \) and some specially-constructed finite-codimension submanifolds of \( B^v(X, E) \). This part of the construction of \( D^v_X \) involves only transversality arguments, which can be carried out equally with \( M^v_{\phi}(X_L, E) \) in place of \( M^v(X, E) \), for any fixed \( L \). That the signed count is independent of the choices made, in the unperturbed setting, is a consequence of the compactness theorem for the moduli space. The Uhlenbeck compactification works the same way for \( M^v_{\phi}(X_L, E) \) as it does for the unperturbed anti-self-duality equations (see [4] for example); so the Donaldson invariants can be defined using the perturbed moduli spaces. Each moduli space is empty once \( L \) is large enough, so the invariants are zero.

4.2. Concluding the proof. The rest of the argument is essentially the same as the proof of the main theorem in [14]. Let \( K \) be a knot in \( S^3 \) that is a counterexample to Theorem 1. We will obtain a contradiction.

The manifold \( Y_0 \) obtained by zero-surgery admits a taut foliation and is not \( S^1 \times S^2 \), by the results of [12]. The following proposition is proved in [14] using the results of [6] and [7]:

**Proposition 17.** Let \( Y \) be a closed orientable 3-manifold admitting an oriented taut foliation. Suppose \( Y \) is not \( S^1 \times S^2 \). Then \( Y \) can be embedded as a separating hypersurface in a closed symplectic 4-manifold \((X, \Omega)\). Moreover, we can arrange that \( X \) satisfies the following additional conditions.

1. The first homology \( H_1(X; \mathbb{Z}) \) vanishes.
2. The euler number and signature of \( X \) are the same as those of some smooth hypersurface in \( \mathbb{C}P^3 \), whose degree is even and not less than 6.
3. The restriction map $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is surjective.
4. The manifold $X$ contains a tight surface of positive self-intersection number, and a sphere of self-intersection $-1$. 

We apply this proposition to the manifold $Y_0$, to obtain an $X$ with all of the above properties. Using the results of [8], it was shown in [14] that a 4-manifold satisfying these conditions satisfies Witten’s conjecture relating the Seiberg-Witten and Donaldson invariants. (See [14, Conjecture 5 and Corollary 7] for an appropriate statement of Witten’s conjecture in this context.) Because $X$ is symplectic, its Seiberg-Witten invariants are non-trivial by [18]. For the same reason, $X$ has Seiberg-Witten simple type. From Witten’s conjecture, it follows that the Donaldson invariants $D^v_X$ are non-trivial, for all $v$ on $X$.

By the penultimate condition on $X$ in Proposition 17, we can choose $v \to X$ so that $c_1(v)$ restricts to a generator of $H^2(Y_0;\mathbb{Z})$. Write $w = v|_{Y_0}$. If $K$ is a counterexample to Theorem 1, then Corollary 14 tells us there is a holonomy perturbation $\phi$ such that

$$\mathcal{R}_{w,\phi}(Y_0) = \emptyset.$$ 

Corollary 16 then tells us that $D^v_X$ is zero. This is the contradiction.

4.3. Further remarks. An analysis of the proof of Theorem 1 reveals that it proves a slightly stronger result (stronger, that is, if one is granted the results of [12]). For example, we can state:

**Theorem 18.** Let $N$ be an embedded solid torus in an irreducible closed 3-manifold $Y$ with $H_1(Y) = \mathbb{Z}$. Let $C = Y \setminus N^\circ$ be the complementary manifold with torus boundary.

Then there is at most one Dehn filling of $C$ which yields a homotopy sphere.

Indeed, for all but one slope, the fundamental group of the manifold obtained by Dehn filling admits a non-trivial homomorphism to $SU(2)$.

The point here is that the original hypothesis need not be that $K$ is a non-trivial knot in $S^3$. What one wants is that zero-surgery on $K$ should be an irreducible homology $S^1 \times S^2$; and if we make this our hypothesis, then we can also consider the case that $K$ is a knot in (for example) a homotopy sphere.

One can also ask whether there is a non-trivial extension of Theorem 1 to other integer surgeries. The results of [15] show that surgery with coefficient 3 or 4 on a non-trivial knot cannot be a lens space. It would be interesting to know whether the fundamental groups of $Y_3$ and $Y_4$ must admit homomorphisms to $SU(2)$ with non-abelian image when $K$ is non-trivial. Surgery with coefficient $+5$ on the right-handed trefoil produces a lens space, so one does not expect to extend Theorem 1 further in the direction of integer surgeries without additional hypotheses. Dunfield [5] has provided an example of a non-trivial knot in $S^3$ for which the Dehn filling $Y_{37/2}$ has a fundamental group which is not cyclic but admits no homomorphism to $SU(2)$ (or even $SO(3)$) with non-abelian image. (The knot is the $(-2,3,7)$ pretzel knot, for which $Y_{18}$ and $Y_{19}$ are both lens
spaces [9].) This example shows that the property of having cyclic fundamental group and the property of admitting no cyclic homomorphic image in $SU(2)$ are in general different for 3-manifolds obtained by Dehn surgery.

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