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DEHN SURGERY, THE FUNDAMENTAL GROUP AND $SU(2)$

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1. Introduction

The main result of this paper, which is a companion to [14], is the following theorem.

Theorem 1. *Let K be a non-trivial knot in S^3, and let Y_r be the 3-manifold obtained by Dehn surgery on K with surgery-coefficient $r \in \mathbb{Q}$. If $|r| \leq 2$, then $\pi_1(Y_r)$ is not cyclic. In fact, there is a homomorphism $\rho : \pi_1(Y_r) \to SU(2)$ with non-cyclic image.*

The statement that Y_r cannot have cyclic fundamental group was previously known for all cases except $r = \pm 2$. The case $r = 0$ is due to Gabai [12], the case $r = \pm 1$ is the main result of [14], and the case that K is a torus knot is analysed for all r in [16]. All remaining cases follow from the cyclic surgery theorem of Culler, Gordon, Luecke and Schalen [2]. It is proved in [15] that Y_2 cannot be homeomorphic to $\mathbb{RP}^3$. If one knew that $\mathbb{RP}^3$ was the only closed 3-manifold with fundamental group $\mathbb{Z}/2\mathbb{Z}$ (a statement that is contained in Thurston's geometrization conjecture), then the first statement in the above theorem would be a consequence. The second statement in the theorem appears to sharpen the result slightly. In any event, we have:

Corollary 2. *Dehn surgery on a non-trivial knot cannot yield a 3-manifold with the same homotopy type as $\mathbb{RP}^3$.* □

The proof of Theorem 1 provides a verification of the Property P conjecture that is independent of the results of the cyclic surgery theorem of [2]. Although the argument follows [14] very closely, we shall avoid making explicit use of instanton Floer homology and Floer's exact triangle [11, 1]. Instead, we rely on the technique that forms just the first step of Floer's proof from [11], namely the technique of "holonomy perturbations" for the instanton equations (see also the remark following Proposition 16 in [14]).

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2. Holonomy perturbations

This section is a summary of material related to the "holonomy perturbations" which Floer used in the proof of his surgery exact triangle for instanton Floer homology [11]. Similar holonomy perturbations were introduced for the 4-dimensional anti-self-duality equations in [3]; see also [17]. Our exposition is taken largely from [1] with only small changes in notation. Some of our gauge-theory notation is taken from [13].

Let Y be a compact, connected 3-manifold, possibly with boundary. Let w be a unitary line bundle on Y, and let E be a unitary rank 2 bundle equipped with an isomorphism

$$\psi : \det(E) \to w.$$

Let $\mathfrak{g}_E$ denote the bundle whose sections are the traceless, skew-hermitian endomorphisms of E, and let $\mathcal{A}$ be the affine space of $SO(3)$ connections in $\mathfrak{g}_E$. Let $\mathcal{G}$ be the gauge group of unitary automorphisms of E of determinant 1 (the automorphisms that respect ψ). We write $\mathcal{B}^w(Y)$ for the quotient space $\mathcal{A}/\mathcal{G}$. A connection A, or its gauge-equivalence class $[A] \in \mathcal{B}^w(Y)$, is *irreducible* if the stabilizer of A is the group $\{\pm 1\} \subset \mathcal{G}$, and is otherwise *reducible.* The reducible connections are the ones that preserve a decomposition of $\mathfrak{g}_E$ as $\mathbb{R} \oplus L$, where L is an orientable 2-plane bundle; these connections have stabilizer either S^1 or (in the case of the product connection) the group $SU(2)$.

Definition 3. We write $\mathcal{R}^w(Y) \subset \mathcal{B}^w(Y)$ for the space of $\mathcal{G}$-orbits of flat connections:

$$\mathcal{R}^w(Y) = \{\, [A] \in \mathcal{B}^w(Y) \mid F_A = 0 \,\}.$$

This is the *representation variety* of flat connections with determinant w. □

We have the following straightforward fact:

Lemma 4. *The representation variety $\mathcal{R}^w(Y)$ is non-empty if and only if $\pi_1(Y)$ admits a homomorphism $\rho : \pi_1(Y) \to SO(3)$ with $w_2(\rho) = c_1(w)$ mod 2. The representation variety contains an irreducible element if and only if there is such a ρ whose image is not cyclic.*

If $c_1(w) = 0 \bmod 2$, then $\mathcal{R}^w(Y)$ is isomorphic to the space of homomorphisms $\rho : \pi_1(Y) \to SU(2)$ modulo the action of conjugation. □

Suppose now that Y is a closed oriented 3-manifold. The flat connections $A \in \mathcal{A}$ are the critical points of the Chern-Simons function

$$\mathrm{CS} : \mathcal{A} \to \mathbb{R},$$
$$\mathrm{CS}(A) = \frac{1}{4}\int_Y \mathrm{tr}\big((A - A_0) \wedge (F_A + F_{A_0})\big),$$

where A_0 is a chosen reference point in $\mathcal{A}$, and tr denotes the trace on 3-by-3 matrices. We define a class of perturbations of the Chern-Simons functional, the *holonomy* perturbations.

Let D be a compact 2-manifold with boundary, and let $\iota : S^1 \times D \hookrightarrow Y$. Choose a trivialization of w over the image of ι. With this choice, each connection $A \in \mathcal{A}$ gives rise to a unique connection $\tilde{A}$ in $E|_{\mathrm{im}(\iota)}$ with the property that $\det(\tilde{A})$ is the product connection in the trivialized bundle $w|_{\mathrm{im}(\iota)}$. Thus $\tilde{A}|_{\mathrm{im}(\iota)}$ is an $SU(2)$ connection. Given a smooth 2-form μ with compact support in the interior of D and integral 1, and given a smooth class-function

$$\phi : SU(2) \to \mathbb{R},$$

we can construct a function

$$\Phi : \mathcal{A} \to \mathbb{R}$$

that is invariant under $\mathcal{G}$ as follows. For each $z \in D$, let γ_z be the loop $t \mapsto \iota(t, z)$ in Y, and let $\mathrm{Hol}_{\gamma_z}(\tilde{A})$ denote the holonomy of $\tilde{A}$ along γ_z, as an automorphism of the fiber E at the point $y = \iota(0, z)$. The class-function ϕ determines also a function on the group of determinant-1 automorphisms of the fiber E_y, and we set

$$\Phi(A) = \int_D \phi(\mathrm{Hol}_{\gamma_z}(\tilde{A}))\mu(z).$$

One can write down the equations for a critical point A of the function $\mathrm{CS}+\Phi$ on $\mathcal{A}$. They take the form

$$F_A = \phi'(H_A)\mu_Y,$$

where H_A is the section of the bundle $\mathrm{Aut}(E)$ over $\mathrm{im}(\iota)$ obtained by taking holonomy around the circles, ϕ' is the derivative of ϕ, regarded as a map from $\mathrm{Aut}(E)$ to $\mathfrak{g}_E$, and μ_Y is the 2-form on Y obtained by pulling back μ to $S^1 \times D$ and then pushing forward along ι. (See [1].)

Definition 5. Given ι and ϕ as above, we write

$$\mathcal{R}^w_{\iota,\phi}(Y) = \{\, [A] \in \mathcal{B}^w(Y) \mid F_A = \phi'(H_A)\mu_Y \,\}.$$

This is the *perturbed representation variety.* □

Now specialize to the case that D is a disk, so ι is an embedding of a solid torus. Let

$$C = Y \setminus \mathrm{im}(\iota)^\circ$$

be the complementary manifold with torus boundary. Let $z_0 \in \partial D$ be a basepoint, and let a and b be the oriented circles in ∂C described by

$$\begin{aligned} a &= \iota\big(S^1 \times \{z_0\}\big) \\ b &= \iota\big(\{0\} \times \partial D\big). \end{aligned} \tag{1}$$

These are the "longitude" and "meridian" of the solid torus. We continue to suppose that w is trivialized on $\mathrm{im}(\iota)$ and hence on ∂C. So the restriction of E to ∂C is given the structure of an $SU(2)$ bundle. Given a connection A on $\mathfrak{g}_E$ that is flat on ∂C, let $\tilde{A}$ be the corresponding flat $SU(2)$ connection in $E|_{\partial C}$. One can choose a determinant-1 isomorphism between the fiber of E at the basepoint

$\iota(0, z_0)$ so that the holonomies of $\tilde{A}$ around a and b become commuting elements of $SU(2)$ given by

$$\mathrm{Hol}_a(\tilde{A}) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$$
$$\mathrm{Hol}_b(\tilde{A}) = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix}.$$

The pair $(\alpha(A), \beta(A)) \in \mathbb{R}^2$ is determined by A up to the ambiguities

1. adding integer multiples of 2π to α or β;
2. replacing (α, β) by $(-\alpha, -\beta)$.

Definition 6. Let $S \subset \mathbb{R}^2$ be a subset of the plane with the property that $S + 2\pi\mathbb{Z}^2$ is invariant under $s \mapsto -s$. Define the set

$$\mathcal{R}^w(C \mid S) \subset \mathcal{R}^w(C)$$

as

$$\mathcal{R}^w(C \mid S) = \{ [A] \in \mathcal{R}^w(C) \mid (\alpha(A), \beta(A)) \in S + 2\pi\,\mathbb{Z}^2 \},$$

where $(\alpha(A), \beta(A))$ are the longitudinal and meridional holonomy parameters, determined up to the ambiguities above. □

One should remember that the choice of trivialization of w on $\mathrm{im}(\iota)$ is used in this definition, and in general the set we have defined will depend on this choice.

A class-function ϕ on $SU(2)$ corresponds to a function $f : \mathbb{R} \to \mathbb{R}$ via

$$f(t) = \phi\left(\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right).$$

The function f satisfies $f(t) = f(t + 2\pi)$ and $f(-t) = f(t)$. The following observation of Floer's is proved as Lemma 5 in [1].

Lemma 7. *Let $f : \mathbb{R} \to \mathbb{R}$ correspond to ϕ as above. Then restriction from Y to C gives rise to a bijection*

$$\mathcal{R}^w_{\iota,\phi}(Y) \to \mathcal{R}^w\big(C \mid \beta = -f'(\alpha) \big).$$

□

We also have the straightforward fact:

Lemma 8. *If $g : \mathbb{R} \to \mathbb{R}$ is a smooth odd function with period 2π, then there is a class-function ϕ on $SU(2)$ such that the corresponding function f satisfies $f' = g$.* □

3. Removing flat connections by perturbation

Let us now take the case that Y is a homology $S^1 \times S^2$, and let $w \to Y$ be a line-bundle with $c_1(w)$ a generator for $H^2(Y;\mathbb{Z}) = \mathbb{Z}$. Let $N \hookrightarrow Y$ be an embedded solid torus whose core is a curve representing a generator of $H_1(Y;\mathbb{Z})$, and let C be the manifold with torus boundary

$$C = Y \setminus N^\circ.$$

By a "slope" we mean an isotopy class of essential closed curves on the torus ∂C. For each slope s, let Y_s denote the manifold obtained from C by Dehn filling with slope s: that is, Y_s is obtained from C by attaching a solid torus in such a way that curves in the class s bound disks in the solid torus.

Parametrize N by a map $\iota : S^1 \times D^2 \to N$. Let a and b be the curves (1) on ∂N. The Dehn filling Y_b on the slope represented by b is just Y. The manifold Y_a has $H_1(Y_a;\mathbb{Z}) = 0$. Let s be the slope

$$s = [pa + qb],$$

where p and q are coprime and both positive

Proposition 9. *Let s be as above, and suppose*

$$p/q \leq 2.$$

Suppose that neither $\pi_1(Y_a)$ nor $\pi_1(Y_s)$ admits a homomorphism to $SU(2)$ with non-cyclic image. Then there is a holonomy-perturbation (ι, ϕ) for the manifold Y such that the perturbed representation variety $\mathcal{R}^w_{\iota,\phi}(Y)$ is empty.

Proof. Fix a trivialization τ of w over N. At this stage the choice is immaterial, because any two choices differ by an automorphism of w that extends over all of Y. Write

$$\begin{aligned} Y_a &= C \cup N_a, \\ Y_s &= C \cup N_s, \end{aligned}$$

where N_a and N_s are the solid tori from the Dehn surgery. The trivialization of w over ∂C allows us to extend w to a line-bundle $w_a \to Y_a$ equipped with a trivialization τ_a over N_a, extending the given trivialization on ∂C. Note that w_a is globally trivial on the homology 3-sphere Y_a, but the global trivialization differs from τ_a on the curve $b \subset \partial C$ by a map $b \to S^1$ of degree 1. This is because there is a surface $\Sigma \subset C$ with boundary b, and the original trivialization τ does not extend over Σ. The same remarks apply to Y_s.

On the manifold Y_s, in addition to constructing w_s as above, we construct a different line bundle $\tilde{w}_s \to Y_s$ as follows. Let $\tilde{\tau}$ be the trivialization of $w|_{\partial C}$ with the property that $\tilde{\tau}\tau^{-1}$ is a map $\partial C \to S^1$ with degree q on b and degree 0 on a. Let $\tilde{w}_s$ be obtained by extending w as a trivial bundle over N_s extending the trivialization $\tilde{\tau}$.

If p is odd, then Y_s has $H^2(Y_s;\mathbb{Z}/2) = 0$. When p is even, the construction of $\tilde{w}_s$ makes $c_1(\tilde{w}_s)$ divisible by 2. So in either case, elements of $\mathcal{R}^{\tilde{w}_s}(Y_s)$ correspond to homomorphisms $\rho : \pi_1(Y_s) \to SU(2)$.

The following lemma is straightforward.

Lemma 10. *Restriction to C gives identifications*

$$\begin{aligned}\mathcal{R}^{w_a}(Y_a) &\to \mathcal{R}^{w}\big(\, C \mid \alpha = 0\,\big)\\ \mathcal{R}^{w_s}(Y_s) &\to \mathcal{R}^{w}\big(\, C \mid p\alpha + q\beta = 0\,\big)\\ \mathcal{R}^{\tilde{w}_s}(Y_s) &\to \mathcal{R}^{w}\big(\, C \mid p\alpha + q\beta = q\pi\,\big)\end{aligned}$$

□

The manifold C has $H_1(C;\mathbb{Z}) = \mathbb{Z}$, so the representation variety $\mathcal{R}^w(C)$ contains reducibles. The next lemma describes their α and β parameters.

Lemma 11. *If $[A]$ is a reducible element of $\mathcal{R}^w(C)$, then $(\alpha(A), \beta(A))$ lies on the line $\beta = \pi \bmod 2\pi\mathbb{Z}$.*

Proof. If $[A]$ is a reducible element of $\mathcal{R}^w(C)$, then A is a flat $SO(3)$ connection on C with cyclic holonomy. The holonomy around b is the identity element of $SO(3)$ because b bounds the surface Σ in C. So the corresponding $SU(2)$ connection $\tilde{A}$ on $E|_b$ (regarding $E|_b$ as an $SU(2)$ bundle using τ) has holonomy ± 1 in $SU(2)$. It follows that β is 0 or π mod 2π. We can equip w on C with a connection θ which respects the trivialization τ on ∂C and whose curvature F_θ integrates to $-2\pi i$ on Σ. The $SU(2)$ connection $\tilde{A}$ can be uniquely extended to a $U(2)$ connection $\tilde{A}$ on all of $E|_C$, in such a way that the associated $SO(3)$ connection is A and such that the induced connection on $\det(E) = w$ is θ. The connection reduces E to a sum of line bundles, both of which have curvature $F_\theta/2$. The holonomy of these line bundles on b is given by

$$\exp \int_\Sigma (F_\theta/2) = -1.$$

So $\beta = \pi$ mod 2π as claimed. This completes the proof of the lemma. □

If we suppose that the homology-sphere Y_a has a fundamental group with no non-trivial homomorphisms to $SU(2)$, then $\mathcal{R}^{w_a}(Y_a)$ consists of a single reducible element. By the previous two lemmas, the α and β parameters of this connection lie on the two line $\alpha = 0$ and $\beta = \pi$. So it is the point

$$v_a = (0, \pi)$$

mod $2\pi\mathbb{Z}^2$. Similarly the α and β parameters of the reducible elements in $\mathcal{R}^{\tilde{w}_s}(Y_s)$ lie on the line $p\alpha + q\beta = \pi$ mod 2π and the line $\beta = \pi$. So they are represented by the points

$$v_{s,k} = (2\pi k/p,\ \pi)$$

mod $2\pi\mathbb{Z}^2$. The next lemma is a standard result, from [11] of [1]. We supply the proof for completeness.

Lemma 12. *Suppose $\pi_1(Y_a)$ admits no non-trivial homomorphisms to $SU(2)$. For any neighborhood W of $(0, \pi)$, let us write*

$$W^* = W \cap \{\beta \neq \pi\}.$$

Then there exists a symmetric neighborhood W *of* $(0,\pi)$ *such that*

$$\mathcal{R}^w(C \mid W^*) = \emptyset.$$

Proof. The space $\mathcal{R}^{w_a}(Y_a)$ consists of a single point, represented by the $SO(3)$ connection A_a with trivial holonomy. By the one-to-one correspondence from Lemma 10, it follows that $\mathcal{R}^w(C \mid (0,\pi))$ consists of a single point $[A]$ represented by an $SO(3)$ connection which trivializes $\mathfrak{g}_E$. We need only show that a neighborhood of $[A]$ in $\mathcal{R}^w(C)$ consists entirely of reducibles. Equivalently, writing π for $\pi_1(C)$, we can study a neighborhood of the trivial homomorphism $\rho_1 : \pi \to SO(3)$ and show that it consists of reducible connections.

The deformations of ρ_1 are governed by $H^1(\pi;\mathbb{R}^3) = H^1(C)\otimes\mathbb{R}^3$, which is a copy of $\mathbb{R}^3$. It will be sufficient to exhibit a 1-parameter deformation of ρ realizing any given vector in this H^1 as its tangent vector and consisting entirely of reducibles. This is straightforward. Given $\xi \in \mathfrak{so}(3)$, we can consider the 1-parameter family of connections in the trivial $SO(3)$ bundle given by the connection 1-forms $t\xi\eta$, where η is a closed 1-form with period 1 on C and $t \in \mathbb{R}$. □

We need one more lemma before completing the proof of Proposition 9.

Lemma 13. *For any* S*, there is a one-to-one correspondence between* $\mathcal{R}^w(C \mid S)$ *and* $\mathcal{R}^w(C \mid S')$*, where* S' *is the translate* $S + (\pi, 0)$*.*

Proof. Let ϵ be an automorphism of the $U(2)$ bundle $E \to C$ whose determinant is a function $C \to S^1$ which has degree 1 on the curve a. (The automorphism ϵ does not belong to the gauge group $\mathcal{G}$, because elements of $\mathcal{G}$ have determinant 1.) The element ϵ acts on the space of flat connections A in $\mathcal{A}(C)$, and gives rise to a bijective self-map of the space $\mathcal{R}^w(C)$:

$$\bar{\epsilon} : \mathcal{R}^w(C) \to \mathcal{R}^w(C).$$

This map restricts to a bijection $\bar{\epsilon} : \mathcal{R}^w(C \mid S) \to \mathcal{R}^w(C \mid S')$. □

We can now conclude the proof of the proposition. Suppose that $\pi_1(Y_a)$ admits only the trivial homomorphism to $SU(2)$, and that the only homomorphisms $\rho : \pi_1(Y_s) \to SU(2)$ are those with cyclic image. Let $L \subset \mathbb{R}^2$ be the closed line segment

$$L = \{\, (\alpha,\beta) \mid \alpha = 0, -\pi \le \beta \le \pi \,\}$$

and let L^* be the open line-segment obtained by removing the endpoints. Let L^*_π and $L^*_{-\pi}$ be the translates of this line segment by the vectors $(\pi,0)$ and $(-\pi,0)$. By Lemmas 10 and 11, the hypothesis on $\pi_1(Y_a)$ means that

$$\mathcal{R}^w(C \mid L^*) = \emptyset.$$

By Lemma 13, we therefore have

$$\mathcal{R}^w(C \mid L^*_{\pm\pi}) = \emptyset.$$

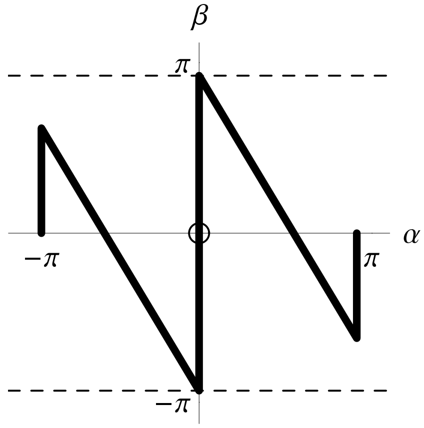

FIGURE 1. The set S, for $p/q = 5/3$. The (α, β) parameters of reducible elements of $\mathcal{R}^w(C)$ lie on the dashed lines.

Let P_1 be the line

$$P = \{\, p\alpha + q\beta = q\pi \,\}$$

and let $P_2 = P_1 - (0, 2\pi)$. The hypothesis on $\pi_1(Y_s)$ means that $\mathcal{R}^w(C \mid P_i)$ consists only of reducibles, lying over the points on P_i where $\beta = \pi \bmod 2\pi$. Let $S \subset \mathbb{R}^2$ be the piecewise-linear arc with vertices at the points

$$\begin{aligned} z_1 &= (-\pi, 0) \\ z_2 &= (-\pi,\ -(1 - p/q)\pi) \\ z_3 &= (0, -\pi) \\ z_4 &= (0, \pi) \\ z_5 &= (\pi,\ (1 - p/q)\pi) \\ z_6 &= (\pi, 0). \end{aligned}$$

Figure 1 shows the set S in the case $p/q = 5/3$. Because $p/q \le 2$, the set is contained in the region $-\pi \le \beta \le \pi$. If $p/q = 2$, then S has four points on the lines $\beta = \pm\pi$; otherwise it has just two,

Let S^* be the complement in S in of the points whose β coordinates are $\pm\pi$. Given any symmetric neighborhood U of S, let U^* similarly stand for

$$U^* = U \setminus \{\, \beta = \pm\pi \,\}. \tag{2}$$

We know that $\mathcal{R}^w(C \mid S^*) = \emptyset$, because S is entirely contained in the union of L, $L_{\pm\pi}$ and the two lines P_1, P_2. From Lemma 12 and the compactness of $\mathcal{R}^w(C)$, it follows that there is a symmetric neighborhood U of S such that

$$\mathcal{R}^w(C \mid U^*) = \emptyset. \tag{3}$$

We now observe that, given any neighborhood U of S, we can find a smooth odd function g with period 2π such that the graph of $-g$ on the interval $[-\pi, \pi]$

is entirely contained in U^*. By Lemma 7 and Lemma 8, there exists a ϕ such that

$$\mathcal{R}^w_{\iota,\phi}(Y) = \mathcal{R}^w\big(C \mid \beta = -g(\alpha) \big).$$

The right hand side is empty because it is contained in the empty set (3). This finishes the proof of the proposition. □

We can reformulate the result of Proposition 9 in the special case that Y_a is S^3 as follows.

Corollary 14. *Let K be a knot in S^3 and let Y_r be the manifold obtained by Dehn surgery with coefficient $r \in \mathbb{Q}$. Let Y_0 be the manifold obtained by 0-surgery, and let $w \to Y_0$ be a line bundle whose first Chern class is a generator of $H^2(Y_0;\mathbb{Z})$. Suppose $\pi_1(Y_r)$ admits no homomorphism ρ to $SU(2)$ with non-cyclic image. Then, if $0 < r < 2$, the manifold Y_0 admits a holonomy deformation (ι, ϕ) so that $\mathcal{R}^w_{\iota,\phi}(Y_0)$ is empty.* □

4. Proof of the theorem

4.1. A stretching argument. Let X be a closed, oriented 4-manifold containing a connected, separating 3-manifold Y. Let g_1 be metric on X that is cylindrical on a collar region $[-1,1] \times Y$ containing Y in X. For $L > 0$, let $X_L \cong X$ be the manifold obtained from X by removing the piece $[-1,1] \times Y$ and replacing it with $[-L, L] \times Y$. There is a metric g_L on X_L that contains a cylindrical region of length of $2L$ and agrees with the original metric on the complement of the cylindrical piece.

Let $v \to X$ be a line bundle, let $E \to X$ be a unitary rank-2 bundle with $\det(E) = v$, and form the configuration space $\mathcal{B}^v(X,E)$ of connections in $\mathfrak{g}_E$ modulo determinant-1 gauge transformations of E, as we did in the 3-dimensional case. In dimension 4, the bundle E is not determined up to isomorphism by v alone, so we include it in our notation. Inside $\mathcal{B}^v(X,E)$ is the moduli space of anti-self-dual connections,

$$M^v(X,E) = \{\, [A] \in \mathcal{B}^v(X,E) \mid F^+_A = 0 \,\}.$$

For each $L > 0$, we also have a moduli space

$$M^v(X_L, E) \subset \mathcal{B}^v(X_L, E).$$

(We do not take the trouble to introduce the additional notation v_L and E_L for the corresponding bundles on X_L.)

Let (ι, ϕ) be data for a holonomy perturbation for the bundle $E|_Y$. Following [10, 11, 4], we shall use ϕ also to perturb the anti-self-duality equations on X_L. We use ι to embed $[-L, L] \times S^1 \times D$ into X_L, and let μ_X be the 2-form on the cylindrical part $[-L, L] \times Y$ obtained by pulling back μ from D and pushing forward using this embedding. We choose a trivialization of $v = \det(E)$ on the image of the embedding so that each $SO(3)$ connection in $\mathfrak{g}_E$ determines uniquely an $SU(2)$ connection. For each A, the holonomy around the circles defines, as before, a section H_A over $[-L,L]\times \operatorname{im}(\iota)$ of the bundle $\operatorname{Aut}(E)$, and we obtain

$$\phi'(H_A)\in C^\infty([-L,L]\times \operatorname{im}(\iota);\mathfrak{g}_E).$$

For $L>1$, let $\beta : X_L \to [0,1]$ be a smooth cut-off function, supported in $[-L,L]\times Y$ and equal to 1 on $[-L+1,L-1]\times Y$. On X_L, the *perturbed anti-self-duality equation* is the equation

$$F_A^+ + \beta\phi'(H_A)\mu^+ = 0. \tag{4}$$

We define the corresponding moduli space:

$$M^v_\phi(X_L,E) = \{\,[A]\in \mathcal{B}^v(X_L,E) \mid \text{equation (4) holds}\,\}. \tag{5}$$

Proposition 15. *Let $w=v|_Y$. Suppose that there is a holonomy perturbation on Y such that the perturbed representation variety $\mathcal{R}^w_{\iota,\phi}(Y)$ is empty. Then for each E with determinant v on X, there exists an L_0 such that $M^v_\phi(X_L,E)$ is also empty, for all $L\geq L_0$.*

Proof. The proof is some subset of a standard discussion of holonomy perturbations and compactness in Floer homology theory (see [11, 1, 4]). Suppose on the contrary that we can find $[A_i]$ in $M^v_\phi(X_{L_i},E)$ for an increasing, unbounded sequence of lengths L_i. We start as usual with the fact that the quantity

$$\begin{aligned}\mathcal{E}(A_i) &= \int_{X_{L_i}} \operatorname{tr}(F_{A_i}\wedge F_{A_i})\\ &= \|F^-_{A_i}\|^2 - \|F^+_{A_i}\|^2\end{aligned}$$

is independent of i and depends only on the Chern numbers of the bundle E. (The norms are L^2 norms.) We write this quantity as the sum of three terms:

$$\mathcal{E}(A_i) = \mathcal{E}(A_i \mid X^1) + \mathcal{E}(A_i \mid X^2) + \mathcal{E}(A_i \mid X^3_i),$$

where

$$\begin{aligned}X^1 &= X_{L_i}\setminus\big([-L_i,L_i]\times Y\big)\\ X^2 &= \big([-L_i,-L_i+1]\times Y\big)\cup\big([L_i-1,L_i]\times Y\big)\\ X^3_i &= [-L_i+1,L_i-1]\times Y.\end{aligned}$$

Only the third piece has a geometry which depends on i. From the equation (4), we have

$$\mathcal{E}(A_i \mid X^1)\geq 0$$

because β is zero on X^1. The second term in equation (4) is pointwise uniformly bounded, so

$$\mathcal{E}(A_i \mid X^2)\geq -C_2$$

where C_2 is independent of i. Because the sum of the three terms is constant, we deduce that

$$\mathcal{E}(A_i \mid X^3_i)\leq K,$$

where K is independent of i.

To understand the term $\mathcal{E}(A_i \mid X_i^3)$ better, one must reinterpret (4). On X_i^3, the function β is 1. Identify E on this cylinder with the pull-back of a bundle $E_Y \to Y$, and choose a gauge representative A_i for $[A_i]$ in temporal gauge. Write

$$A_i(t) = A_i|_{\{t\}\times Y}, \quad (-L_i + 1 \le t \le L_i - 1).$$

Thus $A_i(t)$ becomes a path in the space of connections $\mathcal{A}(Y; E_Y)$. The equation (4) is equivalent on X_i^3 to the condition that $A_i(t)$ solves the downward gradient flow equation for the perturbed Chern-Simons functional on $\mathcal{A}(Y; E_Y)$:

$$\frac{d}{dt} A_i(t) = -\mathrm{grad}(\mathrm{CS} + \Phi).$$

In particular, $\mathrm{CS} + \Phi$ is monotone decreasing along the path (or constant). The function $|\Phi|$ is a bounded function on $\mathcal{A}(Y; E_Y)$: we can write

$$|\Phi| \le K'.$$

The change in CS is equal to the quantity $-\mathcal{E}$: that is,

$$\begin{aligned}\mathrm{CS}\big(A_i(-L_i+1)\big) - \mathrm{CS}\big(A_i(L_i-1)\big) &= \mathcal{E}(A_i \mid X_i^3)\\ &\le K\end{aligned}$$

So from the bound on $|\Phi|$ we obtain

$$(\mathrm{CS} + \Phi)\big(A_i(-L_i+1)\big) - (\mathrm{CS} + \Phi)\big(A_i(L_i-1)\big) \le K + 2K'.$$

Now let $\delta > 0$ be given. Because $\mathrm{CS} + \Phi$ is decreasing and the total drop is bounded by $K + 2K'$, we can find intervals

$$(a_i, b_i) \subset [-L_i + 1, L_i + 1]$$

of length δ, so that the drop in $\mathrm{CS} + \Phi$ along (a_i, b_i) tends to zero as i goes to infinity. Because the equation is a gradient-flow equation, this means

$$\lim_{i\to\infty} \int_{a_i}^{b_i} \|\mathrm{grad}(\mathrm{CS} + \Phi)(A_i(t))\|^2_{L^2(Y)}\, dt = 0.$$

We have an expression for $\mathrm{grad}\Phi$ as a uniformly bounded form, so

$$\limsup_{i\to\infty} \int_{a_i}^{b_i} \|F_{A_i(t)}\|^2_{L^2(Y)}\, dt \le \delta J$$

for some constant J depending on ϕ. So given any $\epsilon > 0$, we can find a $\delta > 0$ and a sequence of intervals (a_i, b_i) of length δ so that

$$\int_{(a_i,b_i)\times Y} |F_{A_i}|^2 \, d\mathrm{vol} \le \epsilon$$

for all $i \ge i_0$. We now regard the A_i as connections on the fixed cylinder $(0, \delta) \times Y$. At this point, if ϵ is smaller than the threshold for Uhlenbeck's gauge fixing theorem on the 4-ball, we can find 4-dimensional gauge transformations on the cylinder so that, after applying these gauge transformations and passing to a subsequence, the connections converge in C^∞ on compact subsets. (See for example [4, section 5.5].)

If A is the limiting connection on $(0,\delta)\times Y$, in temporal gauge, then the function $\mathrm{CS}+\Phi$ is constant along the path $A(t)$. It follows that $A(t)$ is constant and is a critical point of $\mathrm{CS}+\Phi$. This tells us that $[A(t)]$ belongs to the perturbed representation variety $\mathcal{R}^w_{\iota,\phi}(Y)$, which we were supposing to be empty. □

The proposition above has the following corollary for the Donaldson polynomial invariants. (Our notation and conventions for these invariants is taken from [13].)

Corollary 16. *Let X be an admissible 4-manifold in the sense of [13], so that its Donaldson polynomial invariants D^v_X are defined. (For example, suppose $H_1(X;\mathbb{Z})$ is zero and $b^+(X)$ is greater than 1.) Then, under the assumptions of the previous proposition, the polynomial invariants are identically zero, regarded as a map*

$$D^v_X : \mathbb{A}(X) \to \mathbb{Z}.$$

Proof. The definition of D^v_X involves first choosing a Riemannian metric on X so that the moduli spaces $M^v(X,E)$ are smooth submanifolds of $\mathcal{B}^v(X,E)$, containing no reducibles and cut out transversely by the equations. If X is admissible, then this can always be done, by changing the metric inside a ball in X. The value of the invariant is then defined as a signed count of the intersection points between $M^v(X,E)$ and some specially-constructed finite-codimension submanifolds of $\mathcal{B}^v(X,E)$. This part of the construction of D^v_X involves only transversality arguments, which can be carried out equally with $M^v_\phi(X_L,E)$ in place of $M^v(X,E)$, for any fixed L. That the signed count is independent of the choices made, in the unperturbed setting, is a consequence of the compactness theorem for the moduli space. The Uhlenbeck compactification works the same way for $M^v_\phi(X_L,E)$ as it does for the unperturbed anti-self-duality equations (see [4] for example); so the Donaldson invariants can be defined using the perturbed moduli spaces. Each moduli space is empty once L is large enough, so the invariants are zero. □

4.2. Concluding the proof. The rest of the argument is essentially the same as the proof of the main theorem in [14]. Let K be a knot in S^3 that is a counterexample to Theorem 1. We will obtain a contradiction.

The manifold Y_0 obtained by zero-surgery admits a taut foliation and is not $S^1\times S^2$, by the results of [12]. The following proposition is proved in [14] using the results of [6] and [7]:

Proposition 17. *Let Y be a closed orientable 3-manifold admitting an oriented taut foliation. Suppose Y is not $S^1\times S^2$. Then Y can be embedded as a separating hypersurface in a closed symplectic 4-manifold (X,Ω). Moreover, we can arrange that X satisfies the following additional conditions.*

1. *The first homology $H_1(X;\mathbb{Z})$ vanishes.*
2. *The euler number and signature of X are the same as those of some smooth hypersurface in $\mathbb{CP}^3$, whose degree is even and not less than 6.*

3. *The restriction map* $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ *is surjective.*
4. *The manifold* X *contains a tight surface of positive self-intersection number, and a sphere of self-intersection* -1. □

We apply this proposition to the manifold Y_0, to obtain an X with all of the above properties. Using the results of [8], it was shown in [14] that a 4-manifold satisfying these conditions satisfies Witten's conjecture relating the Seiberg-Witten and Donaldson invariants. (See [14, Conjecture 5 and Corollary 7] for an appropriate statement of Witten's conjecture in this context.) Because X is symplectic, its Seiberg-Witten invariants are non-trivial by [18]. For the same reason, X has Seiberg-Witten simple type. From Witten's conjecture, it follows that the Donaldson invariants D^v_X are non-trivial, for all v on X.

By the penultimate condition on X in Proposition 17, we can choose $v \to X$ so that $c_1(v)$ restricts to a generator of $H^2(Y_0;\mathbb{Z})$. Write $w = v|_{Y_0}$. If K is a counterexample to Theorem 1, then Corollary 14 tells us there is a holonomy perturbation ϕ such that

$$\mathcal{R}^w_{\iota,\phi}(Y_0) = \emptyset.$$

Corollary 16 then tells us that D^v_X is zero. This is the contradiction. □

4.3. Further remarks. An analysis of the proof of Theorem 1 reveals that it proves a slightly stronger result (stronger, that is, if one is granted the results of [12]). For example, we can state:

Theorem 18. *Let* N *be an embedded solid torus in an irreducible closed* 3-*manifold* Y *with* $H_1(Y) = \mathbb{Z}$. *Let* $C = Y \setminus N^\circ$ *be the complementary manifold with torus boundary.*

Then there is at most one Dehn filling of C *which yields a homotopy sphere. Indeed, for all but one slope, the fundamental group of the manifold obtained by Dehn filling admits a non-trivial homomorphism to* $SU(2)$. □

The point here is that the original hypothesis need not be that K is a non-trivial knot in S^3. What one wants is that zero-surgery on K should be an irreducible homology $S^1 \times S^2$; and if we make this our hypothesis, then we can also consider the case that K is a knot in (for example) a homotopy sphere.

One can also ask whether there is a non-trivial extension of Theorem 1 to other integer surgeries. The results of [15] show that surgery with coefficient 3 or 4 on a non-trivial knot cannot be a lens space. It would be interesting to know whether the fundamental groups of Y_3 and Y_4 must admit homomorphisms to $SU(2)$ with non-abelian image when K is non-trivial. Surgery with coefficient $+5$ on the right-handed trefoil produces a lens space, so one does not expect to extend Theorem 1 further in the direction of integer surgeries without additional hypotheses. Dunfield [5] has provided an example of a non-trivial knot in S^3 for which the Dehn filling $Y_{37/2}$ has a fundamental group which is not cyclic but admits no homomorphism to $SU(2)$ (or even $SO(3)$) with non-abelian image. (The knot is the $(-2,3,7)$ pretzel knot, for which Y_{18} and Y_{19} are both lens spaces [9].) This example shows that the property of having cyclic fundamental group and the property of admiting no cyclic homomorphic image in $SU(2)$ are in general different for 3-manifolds obtained by Dehn surgery.

References

1. P. J. Braam and S. K. Donaldson, *Floer's work on instanton homology, knots and surgery*, The Floer memorial volume, Progr. Math., **133** (1995), 95–256.
2. M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), 237–300.
3. S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987), 397–428.
4. ———, *Floer homology groups in Yang-Mills theory*, Cambridge Tracts in Mathematics, **147**, Cambridge University Press, Cambridge, 2002, With the assistance of M. Furuta and D. Kotschick.
5. N. M. Dunfield, Private communication.
6. Y. M. Eliashberg and W. P. Thurston, *Confoliations*, University Lecture Series, **13**, American Mathematical Society, 1998.
7. Yakov Eliashberg, *A few remarks about symplectic filling*, Geom. Topol. **8** (2004), 277–293.
8. P. M. N. Feehan and T. G. Leness, *A general SO(3)-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, arXiv:math.DG/0203047, 2003.
9. R. Fintushel and R. J. Stern, *Constructing lens spaces by surgery on knots*, Math. Z. **175** (1980), 33–51.
10. A. Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), 215–240.
11. ———, *Instanton homology and Dehn surgery*, The Floer memorial volume, Progr. Math., **133** (1995), 77–97.
12. D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. **26** (1987), 479–536.
13. P. B. Kronheimer and T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), 573–734.
14. P B Kronheimer and T S Mrowka, *Witten's conjecture and Property P*, Geom. Topol. **8** (2004), 295–310.
15. P. B. Kronheimer, T. S. Mrowka, P. S. Ozsvath, and Z. Szabo, *Monopoles and lens space surgeries*, arXiv:math.GT/0310164.
16. L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.
17. C. H. Taubes, *Casson's invariant and gauge theory*, J. Differential Geom. **31** (1990), 547–599.
18. ———, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822.

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