ADDENDUM TO THE PAPER
AFFINELY INFINITELY DIVISIBLE DISTRIBUTIONS AND
THE EMBEDDING PROBLEM

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Abstract. In our paper [5], in proving the general case of our theorem, a result from [3] on embedding of infinitely divisible measures on certain Lie groups with compact center was used. An error has been found in the proof in [3]. In this context we show in this note that the proof of the theorem in [5] can be completed without recourse to the result from [3].

1. Introduction

Let $A$ be a locally compact abelian group and let $P(A)$ denote the semigroup of probability measures on $A$, with the convolution product. Given $\mu \in P(A)$, a $\lambda \in P(A)$ is said to be an affine $k$-th root of $\mu$ (where $k$ is any natural number) if there exists a continuous automorphism $\rho$ of $A$ such that $\rho^k = I$ (the identity transformation) and $\lambda \ast \rho(\lambda) \ast \rho^2(\lambda) \ast \cdots \ast \rho^{k-1}(\lambda) = \mu$, and $\mu$ is said to be affinely infinitely divisible (on $A$) if it has affine $k$-th roots for all $k$. We recall also that $\mu \in P(A)$ is said to be infinitely divisible if, for every natural number $k$, $\mu$ admits a $k$-th (convolution) root. The following is the main theorem from [5]:

**Theorem 1.1.** Every affinely infinitely divisible probability measure on a connected abelian Lie group $A$ is infinitely divisible on $A$.

In [5], after various preparatory results, the theorem is first proved for $A = \mathbb{R}^n$ for any $n$, and then for a general $A$ as above, namely $A = \mathbb{T}^m \times \mathbb{R}^n$ for some $m$ and $n$. In the proof of the general case a theorem from [3] on the embeddability of infinitely divisible probability measures on a class of Lie groups with compact (nontrivial) center is used. It turns out that the proof in [3] has an error; see [4] for details. In this context we describe here a modified proof of Theorem 1.1 as above.

2. Proof of the theorem

As in [5] let $S$ be the maximal torus in $A$, $B$ the subgroup of $A$ containing $S$ and such that $B/S$ is the vector subspace of $A/S$ spanned by $(\text{supp}\mu)S/S$, and $V$ a vector subgroup of $B$ such that $B$ is the direct product of $S$ and $V$. 

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Let $\Gamma$ be the group of automorphisms of $B$ acting trivially on $S$, $\Theta$ the subgroup of $\Gamma$ consisting of automorphisms whose factor action on $B/S$ is trivial, and $\Delta$ the subgroup of $\Gamma$ consisting of automorphisms leaving $V$ invariant. Then the arguments in [5], until the penultimate paragraph of the proof show that there exists a compact subgroup $K$ of $\Delta$ such that $\mu$ is infinitely divisible on $B\Theta K$. From this point the next step is to prove that there exists a periodic one-parameter subgroup $\phi$ of $\Theta K$ such that $\mu$ is infinitely divisible on $B\phi$; this would enable, together with Corollary 4.2 of [5] to conclude that $\mu$ is infinitely divisible on $B$. To achieve this, in [5] we had appealed to a result from [3] on the embeddability of infinitely divisible measures on groups of the form $B\Theta K$ as above, but the proof of that result is found to have an error.

We shall therefore now proceed as follows. Let $M$ be a minimal closed subgroup of $\Theta K$ of the form $UC$ with $U$ a vector subspace of $\Theta$, and $C$ a compact subgroup of $\Theta K$ (not necessarily contained in $K$), such that $\mu$ is infinitely divisible on $BM = BUC$; such a subgroup exists, by considerations of dimension and the number of connected components. If $M^0$ is the connected component of the identity in $M$ then $BM^0$ is a subgroup of finite index in $BM$, and an argument as in the penultimate paragraph of [5] shows that $\mu$, which is infinitely divisible on $BM$, would also be infinitely divisible on $BM^0$. The minimality condition on $M$ therefore shows that $M^0 = M$, namely $M$ is connected. Hence $C$ is also connected.

Now let $H$ be the subgroup of $M$ consisting of all elements whose action on $B$ leaves $\mu$ invariant. Then $H$ is a closed subgroup, and by Lemma 2.2 of [5] $\mu$ is infinitely divisible on $BH$. A priori one does not know at this stage whether $H$ is a semidirect product of a subspace of $\Theta$ with a compact subgroup, so one can not conclude immediately that $H = M$. We shall however show that $H$ is compact, and hence $H = M = C$.

We note firstly that if $Q$ is a closed subgroup of $M$ such that every element $x$ of $M$ which is of finite order can be expressed as $hyh^{-1}$ for some $h \in H$ and $y \in Q$, then $\mu$ is infinitely divisible on $BQ$. This may be seen as follows: Let $k$ be any natural number, and $\nu$ be a $k$-th root of $\mu$ on $BM$. Then it has the form $\nu = \lambda x$ where $\lambda$ is a probability measure on $B$ and $x \in M$ is such that $x^k = e$, the identity element; see [5]. Now let $x$ be expressed as $hyh^{-1}$, with $h \in H$ and $y \in Q$ as above. Then $(h^{-1}\nu h)^k = h^{-1}\mu h = \mu$, since $\mu$ is $h$-invariant. Thus $h^{-1}\nu h$ is a $k$-th root of $\mu$. On the other hand, $h^{-1}\nu h = (h^{-1}\lambda h)(h^{-1}xh) = (h^{-1}\lambda h)y$, so its support is contained in $BQ$. This shows that $\mu$ is infinitely divisible on $BQ$, as claimed.

We now return to the subgroup $M = UC$ as above. The vector subspace $U$ can be decomposed under the conjugation action of $C$ as $U_0 \oplus U_1$ such that $U_0$ is pointwise fixed and $U_1$ contains no nonzero fixed points. Then $M$ is a direct product of $M_1 = U_1 C$ and $U_0$. Since supp$\mu \subset B \subset BM_1$ and $BM_1/BM_1$ is a vector group, it follows that for every root $\lambda$ of $\mu$ on $BM$, supp$\lambda$ is contained in $BM_1$, and hence that $\mu$ is infinitely divisible on $BM_1$. By the minimality of $M$
we get therefore that \( M_1 = M \); thus \( U_0 \) is trivial and the action of \( C \) on \( U \) has no nonzero fixed point.

Consider now the subgroup \( \overline{UH} \) (the closure of \( UH \)). It is of the form \( UC' \) for some compact subgroup \( C' \) of \( C \), and so by the minimality condition on \( M \) we get that \( M = \overline{UH} \). We note that \( H \cap U \) is normalised by \( H \) and \( U \), and hence the preceding conclusion implies that it is a normal subgroup of \( M \). Let \( W = H \cap U \). Then \( W \) can be expressed as a direct product of its identity component \( W^0 \) with a discrete subgroup \( D \) which is invariant under the action of \( C \). Since \( C \) is connected and its action on \( U \) has no nontrivial fixed point, it follows that \( D \) is trivial, and hence \( W \) is a vector subspace of \( U \). We can now express \( U \) as \( U = W \oplus W' \) where \( W' \) is a \( C \)-invariant subspace of \( U \). It can be verified, using elementary linear algebra, that if \( \tau \) is an affine automorphism of \( W' \) of the form \( w \mapsto \sigma(w) + w_0 \) for all \( w \in W' \), where \( \sigma \) is an automorphism of \( W \) and \( w_0 \in W \), and if \( \tau \) is of finite order then \( \tau \) and \( \sigma \) are conjugate as affine automorphisms, by a translation from \( W \). Using this we see that every element \( x \) of \( UC \) which has finite order can be expressed as \( h y h^{-1} \), with \( h \in W \subset H \) and \( y \in W'C \). Therefore by the remark above \( \mu \) is infinitely divisible on \( BW'C \), and hence by the minimality condition on \( M \) we have \( M = W'C \). Thus, in the notation as above, \( H \cap U \) is trivial.

Let \( R \) be the (solvable) radical of (the connected Lie group) \( M \) and \( H^0 \) be the connected component of the identity in \( H \). Since \( R \) contains \( U \), \( H^0 R \) is normalised by \( U \). It is also normalised by \( H \), and since \( UH \) is dense in \( M \) it follows that \( H^0 R \) is a normal Lie subgroup of \( M \). Since \( M/R \) is a semisimple Lie group this implies that \( H^0 R/R \) is closed, and furthermore \( M/R \) can be expressed as \( M_1(H^0 R/R) \), where \( M_1 \) is a compact connected normal subgroup of \( M/R \) such that \( M_1 \cap (H^0 R/R) \) is finite. Let \( T \) be a maximal torus in the compact group \( H^0 R/R \) and let \( M' \) be the closed subgroup of \( M \) containing \( R \) and such that \( M'/R = M_1T \). By the conjugacy of maximal tori (see [6], Chapter 5, Theorem 15) in \( H^0 R/R \) we get that every \( x \) in \( M \) can be expressed as \( h y h^{-1} \) for some \( h \in H^0 \), and \( y \in M' \). Therefore, by our observation above, \( \mu \) is infinitely divisible on \( BM' \), and hence by the minimality condition on \( M \) we have \( M' = M \). Thus \( M/R = M'/R = M_1T \), and since \( M/R \) is semisimple we see that \( T \) must be trivial. Therefore \( H^0 \) is a solvable Lie group.

Let \( P \) be the connected component of the identity in \( H^0 R \). Since \( H^0 \) is solvable, by a theorem of L. Auslander (see [7], Theorem 8.2.4) \( P \) is solvable. As the subgroup \( P \) is normalised by \( UH \) and as the latter is dense in \( M \), it follows that \( P \) is normal in \( M \). As \( M/R \) is a semisimple Lie group and \( P/R \) is a connected solvable normal subgroup, it follows that \( P = R \). This implies that \( HR \) is closed and \( R \) is open in \( HR \). Also, as \( R \) contains \( U \), \( HR \) has the form \( UC' \) for some compact subgroup \( C' \) of \( C \). Since \( \mu \) is infinitely divisible on \( BH \subset BHHR \), the minimality condition on \( M \) now implies that \( M = HR \). Also, since \( M \) is connected and \( R \) is open in \( HR \) we further get that \( M = R \). Thus \( M \) is solvable, and hence the compact connected subgroup \( C \) is abelian. Since \( H \cap U \) is trivial this further implies that \( H \) is abelian.
Now let $p : M \to C$ be the canonical projection homomorphism, and $H' = p(H)$. Then $H'$ is a dense subgroup of $C$. Since $C$ is an abelian group and its action on $U$ has no nonzero fixed point, it follows that the set of elements of $C$ whose action on $U$ admits a nonzero fixed point is a proper closed subset of $C$. Therefore there exists $h' \in H'$ whose action on $U$ has no nonzero fixed point. Let $h \in H$ be such that $p(h) = h'$. Then there exists a $u \in U$ such that $uhu^{-1} = h'$. The centraliser of $h'$ in $M$ is compact and hence the preceding conclusion implies that the centraliser of $h$ in $M$ is compact. As $H$ is abelian this shows that $H$ is compact. As $\mu$ is infinitely divisible on $BH$ the minimality condition on $M$ now implies that $H = M = C$.

Since $C$ is compact there exists a vector subgroup $V$ of $B$ such that $V$ is invariant under the action of $C$ and $B = SV$, a direct product. Hence $BC$ is a direct product of $S$ and $VC$, which shows in particular that it is a linear Lie group, namely a Lie group with a faithful finite-dimensional representation. Therefore by the general embedding theorem in [2] we get that $\mu$, which is infinitely divisible on $BC$, is embeddable on $BC$; the group involved here being a direct product of a group of rigid motions and a compact abelian group, embeddability in this case can also be obtained along the lines of the (simpler) proof in [1] for measures on the group of affine automorphisms of $\mathbb{R}^n$, $n \geq 1$.

As in the argument in [5] for the vector group case we now deduce, from the embeddability of $\mu$ on $BC$, that there exists a periodic one-parameter subgroup $\phi$ of $C$ such that $\mu$ is infinitely divisible (in fact embeddable) on $B\phi$. Then by Corollary 4.2 of [5] $\mu$ is infinitely divisible on $B$; this proves the theorem.

References