AN INVERSE SCATTERING PROBLEM FOR SHORT-RANGE SYSTEMS IN A TIME-PERIODIC ELECTRIC FIELD.

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Abstract. We consider a time-dependent Hamiltonian \( H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x) \) on \( L^2(\mathbb{R}^n) \), where the external electric field \( E(t) \) and the short-range electric potential \( V(t, x) \) are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value \( E_0 \) of the external electric field. When \( E_0 = 0 \), we show that the high energy limit of the scattering operators determines uniquely \( V(t, x) \). When \( E_0 \neq 0 \), the same result holds in dimension \( n \geq 3 \) for generic short-range potentials. In dimension \( n = 2 \), one has to assume a stronger decay on the electric potential.

1. Introduction

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field \( E(t) \) and a time-periodic short-range potential \( V(t, x) \) (with the same period \( T \)). For the sake of simplicity, we assume that the period \( T = 1 \).

The corresponding Hamiltonian is given on \( L^2(\mathbb{R}^n) \) by:

\[
(1.1) \quad H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x),
\]

where \( p = -i\partial_x \). When \( E(t) = 0 \), the Hamiltonian \( H(t) \) describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When \( E(t) = \cos(2\pi t) E \) with \( E \in \mathbb{R}^n \), the Hamiltonian describes the well-known AC-Stark effect in the \( E \)-direction [7].

In this paper, we assume that the external electric field \( E(t) \) satisfies:

\[
(A_1) \quad t \to E(t) \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n) , \quad E(t + 1) = E(t) \ a.e .
\]

Moreover, we assume that the potential \( V \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \), is time-periodic with period 1, and satisfies the following estimations:

\[
(A_2) \quad \forall \alpha \in \mathbb{N}^n, \forall k \in \mathbb{N}, \quad |\partial_t^k \partial_x^\alpha V(t, x)| \leq C_{k,\alpha} < x >^{-\delta - |\alpha|}, \quad \text{with } \delta > 0,
\]

where \( < x > = (1 + x^2)^{\frac{1}{2}} \). Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need \( (A_2) \) for only \( k, \alpha \) with finite order. It is well-known that under assumptions \((A_1) - (A_2)\), \( H(t) \) is essentially

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self-adjoint on $S(\mathbb{R}^n)$ the Schwartz space, [16]. We denote $H(t)$ the self-adjoint realization with domain $D(H(t))$.

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote $H_0(t)$ the free Hamiltonian:

$$(1.2) \quad H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x,$$

and let $U_0(t,s)$, (resp. $U(t,s)$) be the unitary propagator associated with $H_0(t)$, (resp. $H(t)$) (see section 2 for details).

For short-range potentials, the wave operators are defined for $s \in \mathbb{R}$ and $\Phi \in L^2(\mathbb{R}^n)$ by:

$$(1.3) \quad W^\pm(s) \Phi = \lim_{t \to \pm \infty} U(s,t) U_0(t,s) \Phi.$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field:

$$(1.4) \quad E_0 = \int_0^1 E(t) \, dt.$$

- **The case** $E_0 = 0$

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator $U$, $\mathcal{H}_c(U)$, $\mathcal{H}_{ac}(U)$, $\mathcal{H}_{sc}(U)$ and $\mathcal{H}_p(U)$ are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of $U$.

We have the following result ([10], [11], [21]):

**Theorem 1.**
Assume that hypotheses $$(A_1), (A_2)$$ are satisfied with $\delta > 1$ and with $E_0 = 0$.
Then: (i) the wave operators $W^\pm(s)$ exist for all $s \in \mathbb{R}$.

(ii) $W^\pm(s + 1) = W^\pm(s)$ and $U(s + 1, s) W^\pm(s) = W^\pm(s) U_0(s + 1, s)$.

(iii) $\text{Ran} \,(W^\pm(s)) = \mathcal{H}_{ac}(U(s + 1, s))$ and $\mathcal{H}_{sc}(U(s + 1, s)) = \emptyset$.

(iv) the purely point spectrum $\sigma_p(U(s + 1, s))$ is discrete outside $\{1\}$.

**Comments**
1 - The unitary operators $U(s + 1, s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems.

The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv): for $n = 3$ and $\delta > 2$, $\mathcal{H}_p(U(s + 1, s))$ is finite dimensional.
For general $\delta > 0$, $W^\pm(s)$ do not exist and we have to define other wave operators $W^\pm_{HJ}$ by solving an Hamilton-Jacobi equation.

- **The case $E_0 \neq 0$**

  This case was studied by Moller [12]: using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians); the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential $V$ satisfying $(A_2)$ has no eigenvalues [2]. The following theorem, due to Moller [12], is a time-periodic version of these results.

**Theorem 2.**

Assume that hypotheses $(A_1)$, $(A_2)$ are satisfied with $\delta > \frac{1}{2}$ and with $E_0 \neq 0$.

Then: (i) the Floquet operators have purely absolutely continuous spectrum.

(ii) the wave operators $W^\pm(s)$ exist for all $s \in \mathbb{R}$ and are unitary.

(iii) $W^\pm(s+1) = W^\pm(s)$ and $U(s+1,s) W^\pm(s) = W^\pm(s) U_0(s+1,s)$.

**The inverse scattering problem**

For $s \in \mathbb{R}$, we define the scattering operators $S(s) = W^{+\ast}(s) W^-(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s,x)$ from the scattering operators $S(s)$, $s \in [0,1]$.

In this paper, we prove the following result:

**Theorem 3.**

Assume that $E(t)$ satisfies $(A_1)$ and let $V_j$, $j = 1,2$ be potentials satisfying $(A_2)$. We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{1}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$).

Let $S_j(s)$ be the corresponding scattering operators.

Then: $\forall s \in [0,1], S_1(s) = S_2(s) \implies V_1 = V_2$.

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder’s approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to $E_0$. See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t,s)$, $s,t \in \mathbb{R}$, then by virtue of the following relation:

\[(1.5) \quad S(t) = U_0(t,s) S(s) U_0(s,t),\]
the potential $V(t, x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2} + 1$ and with $E(t) = 0$ by studying the scattering matrices in a high energy regime.

In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2. Proof of Theorem 3

2.1. The Avron-Herbst formula. First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t \in \mathbb{R}}$ be a family of selfadjoint operators on $L^2(\mathbb{R}^n)$ such that $S(\mathbb{R}^n) \subset D(H(t))$ for all $t \in \mathbb{R}$.

**Definition.**

We call propagator a family of unitary operators on $L^2(\mathbb{R}^n)$, $U(t, s)$, $t, s \in \mathbb{R}$ such that:

1. $U(t, s)$ is a strongly continuous function of $(t, s) \in \mathbb{R}^2$.
2. $U(t, s) U(s, r) = U(t, r)$ for all $t, s, r \in \mathbb{R}$.
3. $U(t, s) (S(\mathbb{R}^n)) \subset S(\mathbb{R}^n)$ for all $t, s \in \mathbb{R}$.
4. If $\Phi \in S(\mathbb{R}^n)$, $U(t, s) \Phi$ is continuously differentiable in $t, s$ and satisfies:

$$i \frac{\partial}{\partial t} U(t, s) \Phi = H(t) U(t, s) \Phi \, , \, i \frac{\partial}{\partial s} U(t, s) \Phi = -U(t, s) H(s) \Phi \, .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3].

In [12], the author gives, for $E_0 \neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all $E_0$. We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t) = O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field $E_0$, [2]; if we consider the Hamiltonian $B_0$ on $L^2(\mathbb{R}^n)$,

$$B_0 = \frac{1}{2} p^2 - E_0 \cdot x \, ,$$

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we have the following formula:

\[
e^{-itB_0} = e^{-\frac{E_0^2}{2} t^3} e^{itE_0 x} e^{-it^2 E_0 p} e^{-it^2 p}.
\]

In the next definition, we give a similar formula for time-dependent electric fields.

**Definition**

We consider the family of unitary operators \( T(t) \), for \( t \in \mathbb{R} \):

\[
T(t) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p},
\]

where:

\[
\begin{align*}
 b(t) &= -\int_0^t (E(s) - E_0) \, ds - \int_0^1 \int_0^t (E(s) - E_0) \, ds \, dt, \\
 c(t) &= -\int_0^t b(s) \, ds, \\
 a(t) &= \int_0^t \left( \frac{1}{2} b^2(s) - E_0 \cdot c(s) \right) \, ds.
\end{align*}
\]

**Lemma 4.**

The family \( \{H_0(t)\}_{t \in \mathbb{R}} \) has an unique propagator \( U_0(t, s) \) defined by:

\[
U_0(t, s) = T(t) e^{-i(t-s)B_0} T^*(s).
\]

**Proof.**

We can always assume \( s = 0 \) and we make the following ansatz:

\[
U_0(t, 0) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} e^{-itB_0}.
\]

Since on the Schwartz space, \( U_0(t, 0) \) must satisfy:

\[
i \frac{\partial}{\partial t} U_0(t, 0) = H_0(t) U_0(t, 0),
\]

the functions \( a(t), b(t), c(t) \) solve:

\[
\dot{b}(t) = -E(t) + E_0, \quad \dot{c}(t) = -b(t), \quad \dot{a}(t) = \frac{1}{2} b^2(t) - E_0 \cdot c(t).
\]

We refer to [3] for details and [12] for a different formula.

In the same way, in order to define the propagator corresponding to the family \( \{H(t)\} \), we consider a Stark Hamiltonian with a time-periodic potential: \( B(t) = B_0 + V_1(t, x) \) where

\[
V_1(t, x) = e^{i c(t) \cdot p} V(t, x) e^{-i c(t) \cdot p} = V(t, x + c(t)),
\]
(we recall that $c(t)$ a is $C^1$-periodic function). Then, $B(t)$ has an unique propagator $R(t, s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t, s)$ for the family $\{H(t)\}$ is defined by:

$$U(t, s) = T(t) R(t, s) T^*(s).$$

**Comments**

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in \mathbb{R}$:

$$U_0(t + 1, s + 1) = U_0(t, s), \quad U(t + 1, s + 1) = U(t, s).$$

Thus, the wave operators satisfy $W^\pm (s + 1) = W^\pm (s)$.

### 2.2. The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder’s time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(\mathbb{R}^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.

Before giving the main result of this section, we need some notation.

- $\Phi, \Psi$ are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where $\Pi_{E_0}$ is the orthogonal hyperplane to $E_0$.
- $\Phi_{\lambda, \omega} = e^{i\sqrt{\lambda} x \cdot \omega} \Phi$, $\Psi_{\lambda, \omega} = e^{i\sqrt{\lambda} x \cdot \omega} \Psi$.

We have the following high energy asymptotics where $<,>$ is the usual scalar product in $L^2(\mathbb{R}^n)$:

**Proposition 5.**

Under the assumptions of Theorem 3, we have for all $s \in [0, 1]$,

$$< [S(s), p] \Phi_{\lambda, \omega}, \Psi_{\lambda, \omega} > = \lambda^{-\frac{1}{2}} < \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + t\omega) \, dt \right) \Phi, \Psi > + o (\lambda^{-\frac{1}{2}}).$$

**Comments**

Actually, for the case $n = 2$, $E_0 \neq 0$ and $\delta > \frac{\pi}{4}$, Proposition 5 is also valid for $\omega \in S^{n-1}$ satisfying $|\omega \cdot E_0| < |E_0|$, (see ([18], [15])).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).
Proof of Proposition 5

For example, let us prove Proposition 5 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar. More precisely, see [18] for the case $E_0 = 0$, and for the case $n = 2$, $E_0 \neq 0$, see ([17], Theorem 2.4) and ([15], Theorem 4).

• Step 1

Since $c(t)$ is periodic, $c(t) = O(1)$. Then, $V_1(t, x)$ is a short-range perturbation of $B_0$, and we can define the usual wave operators for the pair of Hamiltonians $(B(t), B_0)$:

\[
\Omega_{\pm}(s) = s - \lim_{t \to \pm \infty} R(s, t) e^{-i(t-s)B_0}.
\]

Consider also the scattering operators $S_1(s) = \Omega^+ s(s) \Omega^- s$. By virtue of (2.6) and (2.11), it is clear that:

\[
S(s) = T(s) S_1(s) T^*(s).
\]

Using the fact that $e^{-ib(s) \cdot x} p e^{ib(s) \cdot x} = p + b(s)$, we have:

\[
[S(s), p] = [S(s), p + b(s)] = T(s) [S_1(s), p] T^*(s).
\]

Thus,

\[
<S(s), p> \Phi_{\lambda, \omega}, \Psi_{\lambda, \omega} = <S_1(s), p> T^*(s) \Phi_{\lambda, \omega}, T^*(s) \Psi_{\lambda, \omega}.
\]

On the other hand,

\[
T^*(s) \Phi_{\lambda, \omega} = e^{i\sqrt{x} x \omega} e^{ic(s) \cdot (p + \sqrt{x} \omega)} e^{ib(s) \cdot x} e^{ia(s)} \Phi.
\]

So, we obtain:

\[
<S(s), p> \Phi_{\lambda, \omega}, \Psi_{\lambda, \omega} = <S_1(s), p> f_{\lambda, \omega}, g_{\lambda, \omega},
\]

where

\[
f = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Phi \quad \text{and} \quad g = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Psi.
\]

Clearly, $f$, $g$ are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.

• Step 2 : Modified wave operators

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic $U_D(t, s)$ by:

\[
U_D(t, s) = e^{-i(t-s)B_0} e^{-i \int_0^{t-s} V_1(u+s, up' + \frac{1}{2} a^2 E_0) \, du},
\]

where $p'$ is the projection of $p$ on the orthogonal hyperplane to $E_0$.

We define the modified wave operators:

\[
\Omega_{D}^\pm(s) = s - \lim_{t \to \pm \infty} R(s, t) U_D(t, s).
\]

It is clear that:

\[
\Omega_{D}^\pm(s) = \Omega^\pm(s) e^{-ig^\pm(s, p')}.
\]
where
\[(2.23) \quad g^\pm(s,p') = \int_0^{\pm\infty} V_1(u + s, u p' + \frac{1}{2} u^2 E_0) \, du .\]

Thus, if we set \(S_D(s) = \Omega_D^+(s)\Omega_D^-(s)\), one has:
\[(2.24) \quad S_1(s) = e^{-ig^+(s,p')} S_D(s) e^{ig^-(s,p')}\]

**Step 3 : High energy estimates**

Denote \(\rho = \min(1, \delta)\). We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

**Lemma 6.**

For \(\lambda \gg 1\), we have:

\[(i) \quad || \left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t,s) e^{ig^+(s,p')} f_{\lambda,\omega} || \leq C \left(1 + |(t-s)|^{\frac{1}{2} - \rho}\right).\]

\[(ii) \quad || \left( R(t,s)\Omega_D^+(s) - U_D(t,s) \right) e^{ig^+(s,p')} f_{\lambda,\omega} || = O\left(\lambda^{-\frac{1}{2}}\right), \quad \text{uniformly for } t, \ s \in \mathbb{R}.\]

**Step 4**

We denote \(F(s,\lambda,\omega) = \langle [S_1(s),p] f_{\lambda,\omega}, g_{\lambda,\omega} \rangle\). Using (2.24), we have:
\[
F(s,\lambda,\omega) = \langle e^{-ig^+(s,p')} S_D(s) e^{ig^-(s,p')} p f_{\lambda,\omega}, g_{\lambda,\omega} \rangle
= \langle [S_D(s),p] e^{ig^-(s,p')} f_{\lambda,\omega}, e^{ig^+(s,p')} g_{\lambda,\omega} \rangle
= \langle [S_D(s)-1, p - \sqrt{\lambda} \omega] e^{ig^-(s,p')} f_{\lambda,\omega}, e^{ig^+(s,p')} g_{\lambda,\omega} \rangle
= \langle (S_D(s)-1) e^{ig^-(s,p')} (pf)_{\lambda,\omega}, e^{ig^+(s,p')} g_{\lambda,\omega} \rangle
= \langle (S_D(s)-1) e^{ig^-(s,p')} (pg)_{\lambda,\omega} \rangle
:= F_1(s,\lambda,\omega) - F_2(s,\lambda,\omega).\]

First, let us study \(F_1(s,\lambda,\omega)\). Writing \(S_D(s)-1 = (\Omega_D^+(s) - \Omega_D^-)(s)^* \Omega_D^-(s)\) and using

\[(2.25) \quad \Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s,t) \left[ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right] U_D(t,s) \, dt , \]
we obtain:

\[ S_D(s) - 1 = -i \int_{-\infty}^{+\infty} U_D(t, s)^* \left[ V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2E_0) \right] \]

\[ R(t, s) \Omega_D(s) \, dt. \]

Thus,

\[ F_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} <R(t, s) \Omega_D(s) e^{ig^-(s,p')(pf)_{\lambda,\omega}}, [V_1(t, x)- \]

\[ V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2E_0)] U_D(t, s) e^{ig^+(s,p')_{\lambda,\omega}} > dt \]

\[ = -i \int_{-\infty}^{+\infty} <U_D(t, s) e^{ig^-(s,p')(pf)_{\lambda,\omega}}, [V_1(t, x)- \]

\[ V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2E_0)] U_D(t, s) e^{ig^+(s,p')_{\lambda,\omega}} > dt \]

\[ + R_1(s, \lambda, \omega), \]

where:

\[ R_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} < [R(t, s) \Omega_D(s) - U_D(t, s)] e^{ig^-(s,p')(pf)_{\lambda,\omega}}, \]

\[ [V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2E_0)] U_D(t, s) e^{ig^+(s,p')_{\lambda,\omega}} > dt. \]

By Lemma 6, it is clear that \( R_1(s, \lambda, \omega) = O(\lambda^{-1}). \) Thus, writing \( t = \frac{\tau}{\sqrt{\lambda}} + s, \)

we obtain:

\[ F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} <U_D\left(\frac{\tau}{\sqrt{\lambda}} + s, s\right) e^{ig^-(s,p')_{\lambda,\omega}}, \]

\[ \left(V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda}E_0\right)\right) \]

\[ U_D\left(\frac{\tau}{\sqrt{\lambda}} + s, s\right) e^{ig^+(s,p')_{\lambda,\omega}} > d\tau + O(\lambda^{-1}). \]

Denote by \( f_1(\tau, s, \lambda, \omega) \) the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

\[ |f_1(\tau, s, \lambda, \omega)| \leq C (1+|\tau|)^{-\frac{1}{2}-\rho}. \]

So, by Lebesgue’s theorem, to obtain the asymptotics of \( F_1(s, \lambda, \omega), \) it suffices to determine

\[ \lim_{\lambda \to +\infty} f_1(\tau, s, \lambda, \omega). \]

Let us denote:

\[ U^\pm(t, s, p') = e^{i \int_{t}^{\pm\infty} V_1(u+s, ap' + \frac{1}{2}u^2E_0) \, du}. \]
We have :
\[
(2.31) \quad f_1(\tau, s, \lambda, \omega) = e^{-i \frac{\tau}{\sqrt{\lambda}}} B_0 \ U_- \left( \frac{\tau}{\sqrt{\lambda}}, s, p' \right) (pf)_{\lambda, \omega},
\]
\[
\left( V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, x \right) - V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0 \right) \right)
\]
\[
e^{-i \frac{\tau}{\sqrt{\lambda}}} B_0 \ U_- \left( \frac{\tau}{\sqrt{\lambda}}, s, p' \right) g_{\lambda, \omega}.
\]

Using the Avron-Herbst formula (2.2), we deduce that :
\[
(2.32) \quad f_1(\tau, s, \lambda, \omega) = e^{-i \frac{\tau}{\sqrt{\lambda}}} U_- \left( \frac{\tau}{\sqrt{\lambda}}, s, p' \right) (pf)_{\lambda, \omega},
\]
\[
\left( V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, x + \frac{\tau^2}{2\lambda} E_0 \right) - V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0 \right) \right)
\]
\[
e^{-i \frac{\tau}{\sqrt{\lambda}}} g_{\lambda, \omega}.
\]

Then, we obtain :
\[
(2.33) \quad f_1(\tau, s, \lambda, \omega) = e^{-i \frac{\tau}{\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)} U_- \left( \frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega \right) (pf),
\]
\[
\left( V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, x + \frac{\tau^2}{2\lambda} E_0 \right) - V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0 \right) \right)
\]
\[
e^{-i \frac{\tau}{\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)} U_+ \left( \frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega \right) g.
\]

Since
\[
(2.34) \quad e^{-i \frac{\tau}{\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)^2} = e^{-i \frac{\tau}{\sqrt{\lambda}} \omega} e^{-i \frac{\tau}{\sqrt{\lambda}} p} e^{-i \frac{\tau}{\sqrt{\lambda}} p^2},
\]
we have
\[
(2.35) \quad f_1(\tau, s, \lambda, \omega) = e^{-i \frac{\tau}{\sqrt{\lambda}} p^2} U_- \left( \frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega \right) (pf),
\]
\[
\left( V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, x + \tau \omega + \frac{\tau^2}{2\lambda} E_0 \right) - V_1 \left( \frac{\tau}{\sqrt{\lambda}}, s, \frac{\tau}{\sqrt{\lambda}} p' + \sqrt{\lambda} \omega + \frac{\tau^2}{2\lambda} E_0 \right) \right)
\]
\[
e^{-i \frac{\tau}{\sqrt{\lambda}} p^2} U_+ \left( \frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega \right) g.
\]

Since \( | V_1(u + s, u(p' + \sqrt{\lambda} \omega) + \frac{1}{2} u^2 E_0) | \leq C (u^2 + 1)^{-\delta} \in L^1(\mathbb{R}^+, du), \) it is easy to show (using Lebesgue’s theorem again) that :
\[
(2.38) \quad \lim_{\lambda \to +\infty} s - \frac{U_+}{U_-} \left( \frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega \right) = 1.
\]

Then,
\[
(2.39) \quad \lim_{\lambda \to +\infty} f_1(\tau, s, \lambda, \omega) = \langle pf, (V_1(s, x + \tau \omega) - V_1(s, \tau \omega)) g \rangle.
\]
So, we have obtained:

\[(2.40)\quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} < pf, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau \omega) - V_1(s, \tau \omega)) \, d\tau \right) g > + o \left( \frac{1}{\sqrt{\lambda}} \right).\]

In the same way, we obtain

\[(2.41)\quad F_2(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} < f, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau \omega) - V_1(s, \tau \omega)) \, d\tau \right) pg > + o \left( \frac{1}{\sqrt{\lambda}} \right),\]

so

\[(2.42)\quad F(s, \lambda, \omega) = F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega)\]

\[(2.43)\quad = \frac{1}{\sqrt{\lambda}} < f, \left( \int_{-\infty}^{+\infty} \partial_x V_1(s, x + \tau \omega) \, d\tau \right) g > + o \left( \frac{1}{\sqrt{\lambda}} \right).\]

Using (2.19) and \(\partial_x V(s, x + \tau \omega) = e^{-i c(s) \cdot p} \partial_x V_1(s, x + \tau \omega) e^{i c(s) \cdot p}\), we obtain:

\[(2.44)\quad F(s, \lambda, \omega) = \frac{1}{\sqrt{\lambda}} < \Phi, \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + \tau \omega) \, d\tau \right) \Psi > + o \left( \frac{1}{\sqrt{\lambda}} \right).\]

References


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