LOCALLY CONFORMAL PARALLEL $G_2$ AND $\text{Spin}(7)$ MANIFOLDS

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Abstract. We characterize compact locally conformal parallel $G_2$ (respectively, $\text{Spin}(7)$) manifolds as fiber bundles over $S^1$ with compact nearly Kähler (respectively, compact nearly parallel $G_2$) fiber. A more specific characterization is provided when the local parallel structures are flat.

1. Introduction

Locally conformal parallel structures have been studied for a long time with respect to different natural geometries. The Kähler condition is the oldest example of parallel structure and locally conformal Kähler geometry is by now a quite developed subject with several interactions to other geometries [DO98, Orn04]. Other choices include locally conformal hyperkähler and quaternion Kähler metrics [OP97]. In particular, the useful technique of the quotient construction, coming from symplectic geometry, has been described both for locally conformal Kähler and locally conformal hyperkähler metrics [GOP05, GOPP05]. By looking at further groups whose holonomy has significance, the choices of $G_2$ and $\text{Spin}(7)$ appear as deserving of attention.

In fact locally conformal parallel $G_2$ and $\text{Spin}(7)$ structures are both natural classes in the frameworks of the the symmetries considered in [FG82, Fer86]. More recently, conformal parallel $G_2$ structures have been studied in details on solvmanifolds [CF04]. A locally conformal parallel structure is encoded in the Lee form $\theta$, the 1-form representing the irreducible component of the covariant derivative of the fundamental form in the standard representation. When $\theta$ can be assumed to be parallel for one metric in the conformal class the geometry of the manifold is described, in the Kähler and hyperkähler settings, by a fibration over a circle [OV03, Ver04, GOPP05]. In fact, we obtain here the corresponding characterization in the $G_2$ and $\text{Spin}(7)$ cases.

Theorem A A compact Riemannian 7-manifold $M$ admits a locally conformal parallel $G_2$ structure if and only if there exists a fibre bundle $M \to S^1$ with abstract fibre $N/\Gamma$, where $N$ is a compact simply connected nearly Kähler 6-manifold and $\Gamma$ is a finite subgroup of complex isometries of $N$ acting freely. Moreover, the cone $C(N/\Gamma)$ covers $M$ with cyclic infinite covering transformations group.

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Theorem B A compact Riemannian 8-manifold $M$ admits a locally conformal parallel Spin(7) structure if and only if there exists a fibre bundle $M \to S^1$ with abstract fibre $N/\Gamma$, where $N$ is a compact simply connected nearly parallel $G_2$ manifold and $\Gamma$ is a finite subgroup of isometries of $N$ acting freely. Moreover, the cone $C(N/\Gamma)$ covers $M$ with cyclic infinite covering transformations group.

When the local parallel structures are flat then this statement simplifies. In the $G_2$ case, one gets a trivial circle bundle whose fibre is the standard nearly Kähler 6-sphere $S^6$ (Theorem 3.13). In the Spin(7) case, the fibre is a spherical space form $S^7/\Gamma$, where $\Gamma$ is a finite subgroup of Spin(7) (Theorem 3.14). In particular, the problem of describing all locally conformal parallel flat Spin(7) structures is thus reduced to the possibility of recognizing, inside the list of finite subgroup of SO(8), the ones which are contained in Spin(7).

An independent proof of Theorem A has recently been given by M. Verbitsky in [Ver05]. Moreover, the statements given here concerning the universal coverings have been recently formulated for manifolds of any dimension in the setting of geometric structures of vectorial type by I. Agricola and T. Friedrich in [AF05].

2. Preliminaries

Let $e_1, \ldots, e_7$ be an oriented orthonormal basis of $\mathbb{R}^7$. The subgroup of GL(7) fixing the 3-form

$$\omega = e_{127} - e_{236} + e_{347} + e_{567} - e_{146} - e_{245} + e_{135}$$

is the compact Lie group $G_2 \subset SO(7)$, and $\omega$ corresponds to a real spinor so that $G_2$ can be identified with the isotropy group of a non-trivial real spinor.

A Riemannian manifold $(M^7, g)$ is a $G_2$ manifold if its structure group reduces to $G_2$, or equivalently if there exists a 3-form $\omega$ on $M$ locally given by (1). A $G_2$ manifold is said to be parallel if $\omega$ is parallel (namely, if Hol$(g) \subset G_2$) and locally conformal parallel if the 1-form $\theta$ defined by

$$\theta = -\frac{1}{3} \ast (\ast d\omega \wedge \omega)$$

satisfies

$$d\omega = \frac{3}{4} \theta \wedge \omega, \quad d\ast \omega = \theta \wedge \ast \omega.$$ 

The latter terminology is justified by the fact that if (2) holds, then $\theta$ is closed [Cab96] and the functions locally defined by $\theta = df$ gives rise to local parallel $G_2$ structures. As usual, the choice of constants is a matter of habits and convenience.

Finally, a $G_2$ structure with co-closed Lee form is called a Gauduchon $G_2$ structure.

Next, let $e_0, \ldots, e_7$ be an oriented orthonormal basis of $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. The subgroup of GL(8) fixing the 4-form

$$\phi = e_0 \wedge \omega + \ast_7 \omega$$
is the compact Lie group $\text{Spin}(7) \subset \text{SO}(8)$. Like in the $G_2$ case, $\phi$ corresponds to a real spinor and $\text{Spin}(7)$ can be identified with the isotropy group of a non-trivial real spinor.

A $\text{Spin}(7)$ structure on $(M^8, g)$ is a reduction of its structure group to $\text{Spin}(7)$, or equivalently a nowhere vanishing 4-form $\phi$ on $M$, locally written as (3). A $\text{Spin}(7)$ manifold $(M, g, \phi)$ is said to be parallel if $\text{Hol}(g) \subset \text{Spin}(7)$, or equivalently if $\phi$ is parallel. The Lee form of a $\text{Spin}(7)$ manifold is now 

$$\Theta = -\frac{1}{7} \ast (\ast d\phi \wedge \phi)$$

and if $d\phi = \Theta \wedge \phi$ then $d\Theta = 0$ [Cab95] and $\phi$ is locally conformal parallel. Again, a $\text{Spin}(7)$ structure with co-closed Lee form will be called a Gauduchon $\text{Spin}(7)$ structure.

3. Structure of compact manifolds

We will need the following fact concerning the homotheties of a cone over a compact Riemannian manifold.

**Theorem 3.1.** [GOPP05] Let $N$ be a compact Riemannian manifold, and denote by $\mathcal{C}(N)$ its Riemannian cone. Let $\Gamma$ be a discrete subgroup of homotheties of $\mathcal{C}(N)$ acting freely and properly discontinuously. Then $\Gamma$ is a finite central extension of $\mathbb{Z}$, and the finite part is a finite subgroup of $\text{Isom}(N)$.

Recall that an almost Hermitian $(N^6, J, g)$ with Kähler form $F$ is nearly Kähler if $dF$ decomposes as a $(3, 0) + (0, 3)$-form. Similarly a $G_2$ manifold $(N^7, \omega, g)$ is nearly parallel if $d\omega = \lambda \ast \omega$ with constant $\lambda > 0$. In both cases $g$ is Einstein with positive scalar curvature. It is easy to see that $N$ is nearly Kähler or nearly parallel $G_2$ if and only if its Riemannian cone $\mathcal{C}(N)$ is a parallel $G_2$ or $\text{Spin}(7)$ manifold, respectively [Bär93].

**3.1. Proof of Theorem A.** Denote by $[\omega]$ the conformal class of the fundamental 3-form, and observe that we can more appropriately define locally conformal parallel $G_2$ manifolds as pairs $(M, [\omega])$ such that any representative of the conformal class is locally conformal parallel. Of course, one has an associated conformal class $[g]$ of Riemannian metrics. Note also that any $G_2$ manifold $M$ can be viewed through its Lee form as a Weyl manifold. Thus, whenever $M$ is compact, a Gauduchon $G_2$ structure can be obtained [Gau95, FI03].

**Examples 3.2.** Consider a nearly Kähler manifold $N^6$, with fundamental form $F$. Then $N \times S^1$ admits a locally conformal parallel $G_2$ structure defined by [CS02]

$$\omega = dt \wedge F + dF.$$  

(4)

Thus the four known compact nearly Kähler 6-manifolds, namely $S^6$, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold $F = \text{U}(3)/\text{(U}(1) \times \text{U}(1) \times \text{U}(1))$ give rise to products with $S^1$ that admit a locally conformal parallel $G_2$ structure [Hit01, Sal01]. Further examples can be obtained by choosing $N$ to be the twistor orbifold of the
weighted projective planes $\mathbb{CP}^2_{p_0,p_1,p_2}$, the case $(p_0,p_1,p_2) = (1,1,1)$ corresponding to the flag manifold $F$. These orbifolds $N$ admit a nearly Kähler structure and therefore their products with $S^1$ are locally conformal parallel $G_2$ manifolds.

**Proposition 3.3.** Let $(M, [\omega])$ be a compact locally conformal parallel $G_2$ manifold. If $\omega$ is a Gauduchon $G_2$ form, $\theta$ its Lee form and $g$ the metric with Levi-Civita connection $\nabla^g$, then $\nabla^g \theta = 0$. Moreover:

1. The Riemannian universal cover of $M$ is isometric, up to homotheties, to the product $N \times \mathbb{R}$, where $N$ is a compact 6-dimensional Einstein manifold with positive scalar curvature.

2. The scalar curvature of $g$ is the positive constant $s = \frac{15}{8} |\omega|^2$.

3. The Lee flow preserves the Gauduchon $G_2$ structure, that is $L_\theta \omega = 0$, where $\theta^\sharp$ denotes the Lee vector field.

4. If $\nabla$ is the unique $G_2$ connection with totally skew-symmetric torsion $T$, then $\nabla^\omega \omega = 0$.

**Proof:** The Weyl structure of $M$ defines a Weyl connection $D$ such that $Dg = \theta \otimes g$. Since $\theta$ is closed and the Ricci tensor vanishes, Theorem 3 of [Gau95] can be applied. This gives $\nabla^g \theta = 0$ as well as property i), where the compactness of $N$ is due to Myers Theorem. Statement ii) is a direct consequence of the formula (4.19) in [FI03]. To prove iii), observe that $L_\theta \omega = \nabla^g \omega + \frac{3}{4} |\omega|^2 \omega$. Using the fact that locally $\theta = -4df$ and that the Weyl connection $D$ coincides locally with the Levi-Civita connection of the parallel $G_2$ structure $(\omega' = e^{3f} \omega, g' = e^{2f} g)$ we get

\begin{align*}
D_\theta X &= \nabla^g_{\theta} X - \frac{1}{4} |\theta|^2 X, \\
D \omega &= \frac{3}{4} \theta \otimes \omega.
\end{align*}

On the other hand, using the first equation in (5) we calculate $D_\theta \omega = \nabla^g_{\theta} \omega + \frac{3}{4} |\theta|^2 \omega$. Compare with the second equation in (5) to conclude $0 = \nabla^g \omega = L_\theta \omega$. Finally iv) follows from the fact that the unique $G_2$ connection having as torsion a 3-form is given by [FI03]

\begin{align*}
g(\nabla_X Y, Z) &= g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z),
\end{align*}

where $T = \frac{1}{4} * (\theta \wedge \omega) = -\frac{1}{4} i_\theta (\phi^\omega)$. The last two equalities lead to $\nabla \phi = 0$ and hence to $\nabla T = 0$. \qed

**Corollary 3.4.** A 7-dimensional compact Riemannian manifold $M$ carries a locally conformal parallel $G_2$ structure if and only if $M$ admits a covering which is a Riemannian cone over a compact nearly Kähler 6-manifold and such that the covering transformations are homotheties preserving the corresponding parallel $G_2$ structure.

**Proof:** According to Proposition 3.3, the universal covering $\tilde{M}$ is isometric to $N \times \mathbb{R}$ with the product metric $\tilde{g} = g_N + dt^2$ and $\tilde{\theta} = dt$. The local conformal transformations making the structures parallel give rise to a global $G_2$ structure on $\tilde{M}$. Therefore

\begin{align*}
\tilde{g}' &= e^{2t} \tilde{g} = e^{2t} (g_N + dt^2), \\
\tilde{\omega}' &= e^{3t} \tilde{\omega},
\end{align*}
defines a parallel $G_2$ structure on $\tilde{M}$. Observe that $\tilde{g}'$ is the cone metric and use [Bär93] to get the conclusion.

**Remark 3.5.** The universal covering used in the proof of Corollary 3.4 can be replaced by any Riemannian covering of $M$ such that the Lee form is exact, say $\tilde{\theta} = df$. Then $f : \tilde{M} \to \mathbb{R}$ is a Riemannian submersion, because $|\tilde{\theta}|$ is constant. Moreover, $f$ is globally trivial, for the Lee flow acts by isometries and $\tilde{M}$ is complete.

The previous Remark together with Theorem 3.1 gives that among all the cones covering a compact $M$, one can always choose the one whose covering transformations group is cyclic infinite.

**Proposition 3.6.** Let $(M, [\omega])$ be a compact locally conformal parallel $G_2$ manifold. Then there exists a covering $\tilde{M} \xrightarrow{Z} M$, where $\tilde{M}$ is a globally conformal parallel $G_2$ manifold.

**Proof:** As a consequence of the property $\nabla g^g \theta = 0$, we can always assume the Lee form to be of constant length and harmonic. Moreover, the Lee field preserves $\omega$, so that it is Killing and the Lee flow acts as $G_2$ isometries. Next, Corollary 3.4 gives a globally conformal parallel covering $C(N) \xrightarrow{\Gamma} M$, where $N$ is a compact nearly Kähler 6-manifold. Denote by $\rho : \Gamma \to \mathbb{R}^+$ the map given by the dilation factors of elements of $\Gamma$. The isometries in $\Gamma$ are then $\ker \rho$ and by Theorem 3.1 we obtain

$$\Gamma/\ker \rho \simeq \text{Im}(\rho) \simeq \mathbb{Z}.$$ 

Define $\tilde{M} \xrightarrow{Z} M$ as $\tilde{C}(N/\ker \rho) \xrightarrow{\Gamma/\ker \rho} M$. □

If $C(N) \xrightarrow{Z} M$ is the covering given by Proposition 3.6 and $\pi$ the projection of the cone onto its radius, it is easy to check that $\pi$ is equivariant with respect to the action of the covering maps on $C(N)$ and the action of $n \in \mathbb{Z}$ on $t \in \mathbb{R}$ given by $n + t$. From all of this, Theorem A follows.

### 3.2. Proof of Theorem B.

Again, denote by $[\phi]$ the conformal class of the fundamental 4-form, and note that locally conformal parallel Spin(7) manifolds can be viewed as pairs $(M, [\omega])$ such that any representative of the conformal class is locally conformal parallel. Of course, one has an associated conformal class $[g]$ of Riemannian metrics.

**Remark 3.7.** Like in the $G_2$ case, any compact Spin(7) manifold admits a Gauduchon Spin(7) structure [Iva04]. In fact, if $(N^7, \omega)$ is any nearly parallel $G_2$ manifold, the product $N^7 \times S^1$ carries a natural locally conformal parallel Spin(7) structure defined by

$$\phi = dt \wedge \omega + *_{N^7} \omega$$

where $*_{N^7}$ is the Hodge star operator on $N^7$. 


Examples 3.8. Through the above formula one obtains several classes of examples of locally conformal parallel Spin(7) manifolds. In fact, according to [FKMS97] nearly parallel $G_2$ manifolds split in 4 classes, in correspondence with the number of Killing spinors. These 4 classes are (I) proper nearly parallel $G_2$ manifolds, (II) Sasakian-Einstein manifolds, (III) 3-Sasakian manifolds and (IV) the class containing only $S^7$. In particular, the class (I) contains all squashed 3-Sasakian metrics, including $S^7_{sq}$ with its Einstein metric. Moreover, we have in this class the homogeneous real Stiefel manifold $SO(5)/SO(3)$ and the Allof-Wallach spaces [CMS96]. The class (II) includes $S^7$ with any of its differentiable structures and any Einstein metric in the families constructed in [BGK05]. The class (III) is known to contain examples with arbitrarily large second Betti number [BGMR98]. Note that the topological $S^7$ can be endowed with nearly parallel $G_2$ structures in any of the above mentioned 4 classes.

These 4 classes correspond to the sequence of inclusions $\text{Spin}(7) \supset \text{SU}(4) \supset \text{Sp}(2) \supset \{\text{id}\}$. From this point of view, the products $N \times S^1$ carry a proper locally conformal parallel Spin(7) structure, a locally conformal Kähler Ricci-flat metric, a locally conformal hyperkähler metric and a locally conformal flat metric according to whether $N$ belongs to the class (I), (II), (III) or (IV).

One can easily check that if $I,J,K$ denote the hypercomplex structure on $N \times S^1$ (class (III)), then the Spin(7) form $\phi$ is related to the Kähler forms $\omega_I, \omega_J, \omega_K$ by $2\phi = \omega_I^2 + \omega_J^2 - \omega_K^2$. Thus the locally conformal hyperkähler property $d\omega_I = \Theta \wedge \omega_I, \ldots$ implies $\Theta = \Theta$. This holds of course also for $N$ in class (II), where the conformality factors $f_\alpha$ on $N \times S^1$ can be choosen in the same way for both the local Kähler metrics and the local parallel Spin(7) forms.

Remark 3.9. The existence of a Killing spinor on a nearly parallel $G_2$ manifold turns out to be equivalent to the existence of a parallel spinor with respect to the unique $G_2$ connection with totally skew-symmetric torsion [FI02]. The different classes of structures correspond to the fact that the holonomy group of the torsion connection is contained in $\text{Sp}(1)$, $\text{SU}(3)$ and $G_2$, respectively.

An easy consequence of the considerations in [Iva04] is that the unique Spin(7) connection with totally skew-symmetric torsion on a locally conformal parallel Spin(7) manifold is determined by the formula

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z)$$

where

$$T = -\frac{1}{6} *(\Theta \wedge \phi) = -\frac{1}{6} i_{\Theta}(\ast \phi).$$

Now, in the exactly same way as in the $G_2$ case, we have the following.

Proposition 3.10. Let $(M^8, [\phi])$ be a compact locally conformal parallel Spin(7) manifold, where $\phi$ is a Gauduchon Spin(7) structure. Let $\Theta$ be its Lee form, $g$ be the metric and $\nabla^g$ its Levi-Civita connection. Then $\nabla^g \Theta = 0$. Moreover:
(1) The Riemannian universal cover is isometric, up to homothety, to the product $N^7 \times \mathbb{R}$, where $N$ is a compact 7-dimensional Einstein manifold with positive scalar curvature.

(2) The scalar curvature of $g$ is the positive constant $s = \frac{21}{36} |\Theta|^2$.

(3) The Lee flow preserves the Gauduchon $\text{Spin}(7)$ structure, that is $\mathcal{L}_\Theta \phi = 0$, where $\Theta$ denotes the Lee vector field.

(4) If $\nabla$ denotes the unique $\text{Spin}(7)$ connection with totally skew-symmetric torsion $T$, then $\nabla T = 0$.

Moreover, similarly to the $G_2$ case, we get:

**Corollary 3.11.** A 8-dimensional compact Riemannian manifold $M$ carries a locally conformal parallel $\text{Spin}(7)$ structure if and only if $M$ admits a covering which is a Riemannian cone over a compact nearly parallel $G_2$ manifold and such that the covering transformations are homotheties preserving the corresponding parallel $\text{Spin}(7)$ structure.

**Proposition 3.12.** Let $(M, [\phi])$ be a compact locally conformal parallel $\text{Spin}(7)$ manifold. Then there exists a covering $\tilde{M} \to M$, where $\tilde{M}$ is a globally conformal parallel $\text{Spin}(7)$ manifold.

Then Theorem B follows.

### 3.3. Locally conformal flat structures and further remarks

The following statements specialize Theorems A and B whenever the local metrics are flat.

**Theorem 3.13.** Let $(M, [\omega])$ be a compact locally conformal parallel flat $G_2$ manifold. Then $(M, [\omega]) = (S^6 \times S^1, [\omega_0 = \eta \wedge F_0 + dF_0])$, where $F_0$ is the standard nearly Kähler structure on $S^6$.

**Proof:** The flatness of the local parallel $G_2$ structures is equivalent to the flatness of the Weyl connection of $M$, and from Corollary at page 11 in [Gau95] the associated compact simply connected nearly Kähler manifold is isometric to the round sphere $S^6$. Thus Theorem A implies that $M$ is a fibre bundle over $S^1$ with fibre $S^6/\Gamma$, where $\Gamma$ is a finite subgroup of $\text{SO}(7)$. Since $M$ is orientable, $\Gamma$ must be trivial, and the connectedness of $\text{SO}(7)$ implies that $M$ is isometric to the trivial bundle $S^6 \times S^1$. The $G_2$ structure is given by $[\omega] = [\eta \wedge F + dF]$, where $F$ is a nearly Kähler structure on the round sphere $S^6$. But the standard nearly Kähler structure on $S^6$ inherited from the imaginary octonions is the only nearly Kähler structure on $S^6$ compatible with the round metric [Fri05], and this ends the proof. \(\square\)

**Theorem 3.14.** Let $(M, [\phi])$ be a compact locally conformal parallel flat $\text{Spin}(7)$ manifold. Then $(M, [\phi]) = (N \times S^1, [\phi_0 = \eta \wedge \omega_0 + *N \omega_0])$, where $N = S^7/\Gamma$ is a spherical space form and $\omega_0$ is the standard nearly parallel $G_2$ structure on $S^7$. 
Then the 4-form on $\mathbb{R}^8$ is given, in terms of octonion multiplication, by (see [HL82])

$$\phi(x, y, z, v) = \langle x \cdot (\bar{y} \cdot z) - z \cdot (\bar{y} \cdot x), v \rangle.$$ 

Then the 4-form on $\mathbb{R}^8 - \{0\}$

$$\tilde{\phi}(x, y, z, v) = \frac{\phi(x, y, z, v)}{(|x|^2 + |y|^2 + |z|^2 + |v|^2)^2}$$

descends to a locally conformal parallel flat Spin(7) structure on $S^7 \times S^1$. For any finite $\Gamma \subset \text{Spin}(7)$ that acts freely on $S^7$ we get an example of locally conformal parallel Spin(7) structure on $(S^7/\Gamma) \times S^1$ as well as a new nearly parallel $G_2$ structure (respectively Sasaki-Einstein) on the factor $S^7/\Gamma$.

A class of finite subgroups of Spin(7) can be described by recalling that Spin(7) is generated by the right multiplication on the octonions $\mathbb{O}$ by elements of $S^6 \subset \text{Im}(\mathbb{O})$. Let $1 \leq m \leq 7$ and let $V_m(\mathbb{R}^7)$ be the Stiefel manifold of orthonormal $m$-frames in $\mathbb{R}^7$. Any $\sigma \in V_m(\mathbb{R}^7)$ gives rise to mutually orthogonal $\sigma_1, \ldots, \sigma_m \in S^6$ generating a finite subgroup $G_{\sigma(m)} \subset \text{Spin}(7)$ of order $2^{m+1}$. This is due to the identity $(o\bar{y})x = -(o\bar{x})y$ that holds for any $o \in \mathbb{O}$ and any orthogonal octonions $x, y$. For example, the choice of $\sigma(4) = \{i, j, k, l\}$ gives rise to the group $G_{\sigma(4)}$:

$$
\begin{align*}
o &\rightarrow \pm o, \quad o \rightarrow \pm oi, \quad o \rightarrow \pm oj, \quad o \rightarrow \pm ok, \\
o &\rightarrow \pm ol, \quad o \rightarrow \pm (oi)j, \quad o \rightarrow \pm (oi)k, \quad o \rightarrow \pm (oi)l, \\
o &\rightarrow \pm (aj)k, \quad o \rightarrow \pm (aj)l, \quad o \rightarrow \pm (ok)l, o \rightarrow \pm ((oi)j)k, \\
o &\rightarrow \pm ((oi)k)l, \quad o \rightarrow \pm ((oi)j)l, \quad o \rightarrow \pm ((aj)k)l, \quad o \rightarrow \pm ((oi)j)k)l.
\end{align*}
$$
Other examples include the finite subgroup of $\text{Sp}(2) \subset \text{Spin}(7)$ obtained by looking at $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$. A simple example is given by the group:

$$\sigma = (q, q') \mapsto (\pm(q, q'), \pm(qi, q'i), \pm(qj, q'j), \pm(qk, q'k)).$$

**Remark 3.16.** Note that most of these finite groups do not act freely on $S^7$ and it would be interesting to recognize which of the quotients are actually smooth and can appear as spherical space forms, thus fitting in J. Wolf classification [Wol84, Wol01]. For any such space form $S^7/\Gamma$ the product manifold $(S^7/\Gamma) \times S^1$ will carry a locally conformal parallel flat $\text{Spin}(7)$ structure. Moreover, all compact locally conformal parallel flat $\text{Spin}(7)$ manifold could be obtained in this way selecting from the J. Wolf classification [Wol84, Wol01] of finite subgroups of $SO(8)$ those which are finite subgroups of $\text{Spin}(7)$.

Another way of producing examples is to look at the locally conformal parallel $\text{Spin}(7)$ and non locally conformal Kähler Ricci flat $S^7_{\text{sq}} \times S^1$. Since the isometry group of the squashed sphere $S^7_{\text{sq}}$ is $\text{Sp}(2) \cdot \text{Sp}(1)$, any finite $\Gamma \subset \text{Sp}(2) \cdot \text{Sp}(1)$ acting freely gives rise to such a structure on $(S^7_{\text{sq}}/\Gamma) \times S^1$. These are pure examples in the sense that they are not locally conformal Kähler Ricci flat. More generally, any 3-Sasakian structure in dimension 7 gives rise to a ‘squashed’ nearly parallel $G_2$ structure [FKMS97], that admits exactly one Killing spinor and similar arguments apply.

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