

LEXIFYING IDEALS

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1. Introduction

Let $B = k[x_1, \dots, x_n]$ be a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i . What are the possible Hilbert functions of graded ideals in B ? This question was answered by Macaulay [Ma], who showed that for every graded ideal there exists a lexicographic ideal with the same Hilbert function. Lexicographic ideals are highly structured: they are defined combinatorially and it is easy to derive the inequalities characterizing their possible Hilbert functions. Macaulay's Theorem also plays an important role in the study of graded B -ideals; for example,

- Hartshorne's [Ha] proof that the Hilbert scheme is connected uses lexicographic ideals in an essential way.
- The homological properties of lexicographic ideals are combinatorially tractable [EK]. This leads to results by Bigatti [Bi], Hulett [Hu], Pardue [Pa], showing that the lexicographic ideals have extremal Betti numbers.

Let M be a monomial ideal. We say that a graded ideal in B/M is *lexifiable* if there exists a lexicographic ideal in B/M with the same Hilbert function. We call M and B/M *Macaulay-Lex* if every graded ideal in B/M is lexifiable. The following results are well known: Macaulay's Theorem [Ma] says that 0 is a Macaulay-Lex ideal, Kruskal-Katona's Theorem [Ka, Kr] says that (x_1^2, \dots, x_n^2) is a Macaulay-Lex ideal, and Clements-Lindström's Theorem [CL] says that $(x_1^{e_1}, \dots, x_n^{e_n})$ is a Macaulay-Lex ideal if $e_1 \leq \dots \leq e_n \leq \infty$. These theorems are well-known and have many applications in Commutative Algebra, Combinatorics, and Algebraic Geometry.

It is easy to construct examples like Example 2.13, where a given Hilbert function is not attained by any lexicographic ideal in the degrees of the minimal generators of M . This motivated us to slightly weaken the definition: Let q be the maximal degree of a minimal monomial generator of M ; we call M and B/M *pro-lex* if every graded ideal generated in degrees $\geq q$ in B/M is lexifiable. There exist examples of non pro-lex rings; see Example 3.15. We have started working in a new direction of research along the lines of the following problem.

Problem 1.1. *Find classes of pro-lex monomial ideals.*

Theorem 5.1 shows that if M is Macaulay-Lex and N is lexicographic, then $M+N$ is Macaulay-Lex. Theorem 4.1 shows that if M is Macaulay-Lex, then it stays Macaulay-Lex after we add extra variables to the ring B . In Section 3 we prove:

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Theorem 1.2. *Let $P = (x_1^{e_1}, \dots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \dots \leq e_n \leq \infty$ (here $x_i^\infty = 0$), and M be a compressed monomial ideal in B/P generated in degrees $\leq p$. If $n = 2$, assume that M is (B/P) -lex. Set $\Upsilon = B/(M + P)$. Then Υ is pro-lex above p , that is, for every graded ideal Γ in Υ generated in degrees $\geq p$ there exists an Υ -lex ideal Θ with the same Hilbert function.*

In the case when $M = P = 0$, Theorem 1.2 is Macaulay's Theorem [Ma]; in the case when $M = 0$, Theorem 1.2 is Clements-Lindström's Theorem [CL]. Examples 3.14 and 3.15 show that there are obstructions to generalizing Theorem 1.2.

We make use of ideas of Bigatti [Bi], Clements and Lindström [CL], and Green [Gr]. Our proofs are algebraic, and we avoid computations using generic forms (used in [Gr]) and combinatorial counting (used in [CL]). In Section 2 we introduce definitions and notation used throughout the paper.

2. Lexification

The notation in this section will be used throughout the paper. We introduce several definitions.

Let k be a field and $B = k[x_1, \dots, x_n]$ be graded by $\deg(x_i) = 1$ for all i . We denote by B_d the k -vector space spanned by all monomials of degree d . Denote $\mathbf{m} = (x_1, \dots, x_n)_1$ the k -vector space spanned by the variables. We order the variables $x_1 > \dots > x_n$, and denote by \succ_{lex} the homogeneous lexicographic order on the monomials. We say that an ideal is p -generated if it has a system of generators of degree p .

A monomial $x_1^{a_1} \dots x_n^{a_n}$ has *exponent vector* $\mathbf{a} = (a_1, \dots, a_n)$ and is denoted by $\mathbf{x}^{\mathbf{a}}$. For a monomial m , we set $\max(m) = \max\{i \mid x_i \text{ divides } m\}$. An ideal is called *monomial* if it can be generated by monomials; such an ideal has a unique minimal system of monomial generators.

Notation 2.1. *Let M be a monomial ideal. Set $\Upsilon = B/M$. Vector spaces in Υ (and sometimes ideals) are denoted by greek letters. For example, we denote by C_d a subspace of B_d , and we denote by τ_d a subspace of Υ_d .*

Definition 2.2. *A monomial is a product of powers of the variables, so it can be considered as an element in either B or Υ . We say that a monomial is an Υ -monomial if it does not vanish in Υ , that is, it is not in M . We say that a monomial is an Υ_d -monomial if it is an Υ -monomial of degree d . Furthermore, we say that τ_d is an Υ_d -monomial space if it can be spanned by Υ_d -monomials. We denote by $\{\tau_d\}$ the set of Υ_d -monomials contained in τ_d . The cardinality of this set is $|\tau_d| = \dim_k \tau_d$. By $\mathbf{m}\tau_d$ we mean the k -vector subspace $\left(\mathbf{m}(\tau_d)\right)_{d+1}$ of Υ_{d+1} .*

Definition 2.3. *Let L be a monomial ideal in Υ minimally generated by Υ -monomials l_1, \dots, l_r . We say that L is Υ -lex, (Υ -lexicographic), if the following property is satisfied:*

$$\left. \begin{array}{l} m \text{ is an } \Upsilon\text{-monomial} \\ m \succ_{lex} l_i \text{ and } \deg(m) = \deg(l_i), \text{ for some } 1 \leq i \leq r \end{array} \right\} \implies m \in L.$$

The Υ_d -lex-segment $\lambda_{d,p}$ of length p in degree d is defined as the k -vector space spanned by the lexicographically first (greatest) p monomials in Υ_d . We say that λ_d

is a *lex-segment* in Υ_d if there exists a p such that $\lambda_d = \lambda_{d,p}$. For a Υ_d -monomial space τ_d , we say that $\lambda_{d,|\tau_d|}$ is its Υ_d -lexification.

For simplicity, we sometimes say *lex* instead of Υ -lex if it is clear over which ring we work.

Example 2.4. The ideal (a^2, ab, b^2) is *lex* in the ring $k[a, b, c, d]/(ac, ad)$, and its generators span a *lex-segment*. The k -vector space spanned by a^2, ab, b^2 is the lexification of the k -vector space spanned by b^2, c^2, cd . However, the ideal is not *lex* in $k[a, b, c, d]$.

Proposition 2.5. If τ_d is an Υ_d -lex-segment, then $\mathbf{m}\tau_d$ is an Υ_{d+1} -lex-segment.

The proofs of Propositions 2.5, 2.7, and 2.8, are easy and very similar to the proofs in the polynomial case.

Definition 2.6. A monomial m' is said to be in the *big shadow* of a monomial m if $m' = \frac{x_i m}{x_j}$ for some x_j dividing m and some $i \leq j$. A monomial ideal in Υ is Υ -Borel if it contains all Υ -monomials in the big shadows of its minimal Υ -monomial generators. Ideals that are B -Borel are usually called *strongly stable* or *0-Borel fixed*. We say that a monomial space τ_d is Υ_d -Borel if it contains all Υ_d -monomials in the big shadows of its monomial generators.

Proposition 2.7. If τ_d is Υ_d -Borel, then $\mathbf{m}\tau_d$ is Υ_{d+1} -Borel.

Proposition 2.8. If τ_d is an Υ_d -lex-segment, then it is Υ_d -Borel.

Notation 2.9. Let Γ be a graded ideal in Υ . It decomposes as a direct sum of its components $\Gamma = \bigoplus_{d \geq 0} \Gamma_d$. Its Hilbert function $\text{Hilb}_\Gamma^\Upsilon : \mathbf{N} \cup 0 \rightarrow \mathbf{N} \cup 0$ is defined by

$$\text{Hilb}_\Gamma^\Upsilon(d) = \dim_k(\Gamma_d) \quad \text{for all } d \geq 0.$$

We use the notation $|\Gamma_d|^\Upsilon = \text{Hilb}_\Gamma^\Upsilon(d)$; and for simplicity, we write $|\Gamma_d|$ if it is clear over which ring we work.

Definition 2.10. We say that a graded ideal R in Υ is *lexifiable* if there exists an Υ -lex ideal with the same Hilbert function as R . The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *Macaulay-Lex* if every graded ideal in Υ is *lexifiable*.

Given an Υ_d -monomial space τ_d , the number $|\mathbf{m}\tau_d|$ measures the dimension of $(\tau_d)_{d+1}$. In the proofs one usually works in a fixed degree, so we need the next variation of Definition 2.10.

Definition 2.11. We say that an Υ_d -monomial space τ_d is Υ_d -lexifiable if its lexification λ_d has the property that $|\mathbf{m}\lambda_d| \leq |\mathbf{m}\tau_d|$. The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *d-pro-lex*, if every Υ_d -monomial space is Υ_d -lexifiable.

Example 2.12. This example shows that the order of the variables can make a difference. The ideal (ab) is not *lexifiable* in the ring $k[a, b]/(ab^2)$ for the *lex* order with $a > b$, but it is *lexifiable* for the *lex* order with $b > a$.

Example 2.13. The ideal (ab) is not *lexifiable* in the ring $k[a, b]/(a^2b, ab^2)$ in any *lex* order.

It is easy to construct many examples like Example 2.13. This observation suggests that in order to obtain positive results we need to slightly relax Definition 2.10; the following definition seems reasonable to us:

Definition 2.14. Let q be the maximal degree of a minimal monomial generator of M . The monomial ideal M and the quotient ring $\Upsilon = B/M$ are called *pro-lex* if every graded ideal generated in degrees $\geq q$ in Υ is *lexifiable*.

In the examples we usually denote the variables by a, b, c, d for simplicity.

3. Compression

The following definition generalizes a definition introduced by Clements and Lindström [CL], who used it over a quotient of a polynomial ring modulo pure powers of the variables.

Definition 3.1. Let E be a monomial ideal in B . A $(B/E)_d$ -monomial space τ_d is called *i-compressed* (or *i-compressed in $(B/E)_d$*) if we have the disjoint union

$$\{\tau_d\} = \coprod_{0 \leq j \leq d} x_i^{d-j} \{\sigma_j\}$$

and each σ_j is a *lex-segment* in $(B/(E, x_i))_j$. We say that a k -vector space τ_d is $(B/E)_d$ -compressed (or *compressed*) if it is a $(B/E)_d$ -monomial space and is *i-compressed* for all $1 \leq i \leq n$. A monomial ideal W in B/E is called *compressed* if W_d is compressed for all $d \geq 0$. The term “compressed” is used by Clements and Lindström [CL].

Example 3.2. The ideal

$$(a^3, a^2b, a^2c, ab^2, abc, b^3, b^2c)$$

is compressed in the ring $k[a, b, c]$. The ideal

$$M = (a^3, a^2b, a^2c, a^2d, ab^2, abc, abd, b^3, b^2c)$$

is Borel, but not compressed (it is not *c-compressed* because $b^3 \in M$ and $ad^2 \notin M$).

Lemma 3.3. If τ_d is *i-compressed* in $(B/E)_d$ then $\mathbf{m}\tau_d$ is *i-compressed* in $(B/E)_{d+1}$. If τ_d is $(B/E)_d$ -lex, then it is $(B/E)_d$ -compressed.

Definition 3.4. A B -monomial ideal K is called *compressed-plus-powers* if $K = M + P$, where $P = (x_1^{e_1}, \dots, x_n^{e_n})$ with $2 \leq e_1 \leq e_2 \leq \dots \leq e_n \leq \infty$ and the monomial ideal M is compressed in B/P . Sometimes, when we need to be more precise, we say that K is *compressed-plus- P* . Furthermore, we say that K is *lex-plus- P* if M is lex in B/P .

Notation 3.5. Throughout this section we use the following notation and assumptions:

- $P = (x_1^{e_1}, \dots, x_n^{e_n})$ with $2 \leq e_1 \leq e_2 \leq \dots \leq e_n \leq \infty$.
- The ideal $K = M + P$ is a *compressed-plus- P* monomial ideal in B ; here M is compressed in B/P .
- If $n = 2$ we assume in addition that K is *lex-plus- P* .

- Except in Theorem 3.12, we assume that M is generated by monomials in degree p .
- Set $\Upsilon = B/K$.

For a $(B/P)_d$ -monomial space A_d set

$$\begin{aligned} t_i(A_d) &= \left| \{ m \in \{A_d\} \mid \max(m) \leq i \} \right| \\ s_i(A_d) &= \left| \{ m \in \{A_d\} \mid \max(m) = i \text{ and } x_i^{e_i-1} \text{ divides } m \} \right| \\ r_{i,j}(A_d) &= \left| \{ m \in \{A_d\} \mid \max(m) \leq i \text{ and } x_i^j \text{ does not divide } m \} \right|. \end{aligned}$$

The formula in the following lemma is a generalization of a formula introduced by Bigatti [Bi], who used it for B -Borel ideals.

Lemma 3.6. *Let A_d be a $(B/P)_d$ -monomial space.*

1. *If A_d is compressed and $n \geq 3$, then A_d is $(B/P)_d$ -Borel.*
2. *If A_d is $(B/P)_d$ -Borel, then*

$$\left| \{\mathbf{m}A_d\} \right| = \sum_{i=1}^n t_i(A_d) - s_i(A_d) = \sum_{i=1}^n r_{i,e_i-1}(A_d).$$

Proof. First, we prove (1). Let $m \in \{A_d\}$ and m' be a $(B/P)_d$ -monomial in its big shadow. Hence $m' = \frac{x_i m}{x_j}$ for some x_j dividing m and some $i \leq j$. There exists an index $1 \leq q \leq n$ such that $q \neq i, j$. Note that that m and m' have the same q -exponents. Since A_d is q -compressed and $m' \succ_{lex} m$, it follows that $m' \in \{A_d\}$. Therefore, A_d is $(B/P)_d$ -Borel.

Now, we prove (2). We will show that $\{\mathbf{m}A_d\}$ is equal to the set

$$\begin{aligned} & \coprod_{i=1}^n x_i \{ m \in \{A_d\} \mid \max(m) \leq i \} \\ & \setminus \coprod_{i=1}^n x_i \{ m \in \{A_d\} \mid \max(m) = i \text{ and } x_i^{e_i-1} \text{ divides } m \}. \end{aligned}$$

Denote by \mathcal{P} the set above. Let $w \in A_d$. For $j \geq \max(w)$ we have that $x_j w \in \mathcal{P}$. Let $j < \max(w)$. Then $v = x_j \frac{w}{x_{\max(w)}} \in A_d$. So, $x_j w = x_{\max(w)} v \in \mathcal{P}$. \square

Lemma 3.7 is a generalization of a result by M. Green [Gr], who proved a particular case of it over a polynomial ring (in the case $M = 0$). Green's proof is entirely different than ours; he makes a computation with generic linear forms. It is not clear how to apply his computation to the case $M \neq 0$.

Lemma 3.7. *Let τ_d be an n -compressed Borel Υ_d -monomial space, and let λ_d be a lex-segment in Υ_d with $|\{\lambda_d\}| \leq |\{\tau_d\}|$. Let L_d and T_d be the $(B/P)_d$ -monomial spaces such that $\{L_d\} = \{\lambda_d\} \coprod \{M_d\}$ and $\{T_d\} = \{\tau_d\} \coprod \{M_d\}$. For each $1 \leq i \leq n$ and each $1 \leq j \leq e_i$ we have*

$$r_{i,j}(L_d) \leq r_{i,j}(T_d).$$

Proof. Set $R = B/P$. By Lemma 3.6, M_d is R_d -Borel. Therefore, both L_d and T_d are R_d -Borel and n -compressed.

First, we consider the case $i = n$. Clearly, $r_{n,e_n}(L_d) = |L_d| = |T_d| = r_{n,e_n}(T_d)$ (if $e_n = \infty$, then we consider $r_{n,d+1}$ here). We induct on j decreasingly. Suppose that $r_{i,j+1}(L_d) \leq r_{i,j+1}(T_d)$ holds by induction.

If $\{T_d\}$ contains no monomial divisible by x_n^j then

$$r_{i,j}(L_d) \leq r_{i,j+1}(L_d) \leq r_{i,j+1}(T_d) = r_{i,j}(T_d).$$

Suppose that $\{T_d\}$ contains a monomial divisible by x_n^j . Denote by $e = x_1^{b_1} \dots x_n^{b_n}$, with $b_n \geq j$, the lex-smallest monomial in T_d that is divisible by x_n^j . Let $0 \leq q \leq j-1$. Since T_d is R_d -Borel, it follows that $c_q = x_{n-1}^{b_n-q} \frac{e}{x_n^{b_n-q}} \in T_d$. This is the lex-smallest monomial that is lex-greater than e and x_n divides it at power q . Let the monomial $a = x_1^{a_1} \dots x_{n-1}^{a_{n-1}} x_n^q \in R_d$ be lex-greater than e . Since T_d is n -compressed and a is lex-greater (or equal) than c_q , it follows that $a \in T_d$.

For a monomial u , we denote by $x_n \notin u$ the property that x_n^j does not divide u . By what we proved above, it follows that

$$(3.8) \quad \left| \{u \in \{T_d\} \mid x_n \notin u, u \succ_{lex} e\} \right| = \left| \{u \in \{R_d\} \mid x_n \notin u, u \succ_{lex} e\} \right|.$$

Therefore,

$$\begin{aligned} r_{i,j}(L_d) &= |\{u \in \{L_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{L_d\} \mid x_n \notin u, u \prec_{lex} e\}| \\ &\leq |\{u \in \{R_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{L_d\} \mid x_n \notin u, u \prec_{lex} e\}| \\ &\leq |\{u \in \{R_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{L_d\} \mid u \prec_{lex} e\}| \\ &\leq |\{u \in \{R_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{T_d\} \mid u \prec_{lex} e\}| \\ &= |\{u \in \{R_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{T_d\} \mid x_n \notin u, u \prec_{lex} e\}| \\ &= |\{u \in \{T_d\} \mid x_n \notin u, u \succ_{lex} e\}| + |\{u \in \{T_d\} \mid x_n \notin u, u \prec_{lex} e\}| \\ &= r_{i,j}(T_d); \end{aligned}$$

for the third inequality we used the fact that λ_d is a lex-segment in Υ_d with $|\{\lambda_d\}| \leq |\{\tau_d\}|$; for the equality after that we used the definition of e ; for the next equality we used (3.8). Thus, we have the desired inequality in the case $i = n$.

In particular, we have proved that

$$(3.9) \quad t_{n-1}(L_d) = r_{n,1}(L_d) \leq r_{n,1}(T_d) = t_{n-1}(T_d).$$

Finally, we prove the lemma for all $i < n$. Both $\{\tau_d/x_n\}$ and $\{\lambda_d/x_n\}$ are lex-segments in Υ_d/x_n since τ_d is n -compressed. By (3.9) the inequality $t_{n-1}(L_d) \leq t_{n-1}(T_d)$ holds, and it implies the inclusion $\{\tau_d/x_n\} \supseteq \{\lambda_d/x_n\}$. The desired inequalities follow since

$$\begin{aligned} r_{i,j}(T_d) &= r_{i,j}(T_d/(x_{i+1}, \dots, x_n)) \\ &= r_{i,j}(\{\tau_d/(x_{i+1}, \dots, x_n)\} \coprod \{M_d/(x_{i+1}, \dots, x_n)\}) \\ r_{i,j}(L_d) &= r_{i,j}(L_d/(x_{i+1}, \dots, x_n)) \\ &= r_{i,j}(\{\lambda_d/(x_{i+1}, \dots, x_n)\} \coprod \{M_d/(x_{i+1}, \dots, x_n)\}). \end{aligned}$$

□

Lemma 3.10. *Let v_d be a Υ_d -monomial space. There exists a compressed monomial space τ_d in Υ_d such that $|\tau_d| = |v_d|$ and $|\mathbf{m}\tau_d| \leq |\mathbf{m}v_d|$.*

Proof. Suppose that v_d is not i -compressed. Set $z = x_i$. Since M is z -compressed in B/P , we have the disjoint union

$$\{M_d\} = \coprod_{0 \leq j \leq d} z^{d-j} \{N_j\},$$

where each N_j is a $(B/(z, P))_j$ -lex-segment.

We also have the disjoint union

$$\{v_d\} = \coprod_{0 \leq j \leq d} z^{d-j} \{\nu_j\}$$

where each ν_j is a monomial space in $B/(z, P, N_j)$. Let γ_j be the lexification of the space ν_j in $B/(z, P, N_j)$. Consider the Υ_d -monomial space τ_d defined by

$$\{\tau_d\} = \coprod_{0 \leq j \leq d} z^{d-j} \{\gamma_j\}.$$

Clearly, $|\tau_d| = |v_d|$.

Consider the $(B/P)_d$ -monomial spaces V_d and T_d such that

$$\{V_d\} = \{v_d\} \coprod \{M_d\} \quad \text{and} \quad \{T_d\} = \{\tau_d\} \coprod \{M_d\}.$$

Set $R = B/P$. The short exact sequence of k -vector subspaces of $(B/P)_{d+1}$

$$0 \rightarrow \mathbf{m}M_d \rightarrow \mathbf{m}T_d \longrightarrow \mathbf{m}T_d/\mathbf{m}M_d = \mathbf{m}\tau_d/(\mathbf{m}\tau_d \cap \mathbf{m}M_d) \rightarrow 0$$

shows that $|\mathbf{m}\tau_d| = |\mathbf{m}T_d| - |\mathbf{m}M_d|$ (here we mean $|\mathbf{m}\tau_d|^\Upsilon = |\mathbf{m}T_d|^{B/P} - |\mathbf{m}M_d|^{B/P}$). Similarly, the short exact sequence of k -vector subspaces of $(B/P)_{d+1}$

$$0 \rightarrow \mathbf{m}M_d \rightarrow \mathbf{m}V_d \longrightarrow \mathbf{m}V_d/\mathbf{m}M_d = \mathbf{m}v_d/(\mathbf{m}v_d \cap \mathbf{m}M_d) \rightarrow 0$$

shows that $|\mathbf{m}v_d| = |\mathbf{m}V_d| - |\mathbf{m}M_d|$. Therefore, the desired inequality $|\mathbf{m}\tau_d| \leq |\mathbf{m}v_d|$ is equivalent to the inequality

$$|\mathbf{m}T_d| \leq |\mathbf{m}V_d|.$$

We will prove the latter inequality.

We have the disjoint unions

$$\begin{aligned} \{V_d\} &= \coprod_{0 \leq j \leq d} z^{d-j} \{U_j\} \quad \text{and} \quad \{T_d\} = \coprod_{0 \leq j \leq d} z^{d-j} \{F_j\}, \quad \text{where} \\ \{U_j\} &= \{\nu_j\} \coprod \{N_j\} \quad \text{and} \quad \{F_j\} = \{\gamma_j\} \coprod \{N_j\} \quad \text{in the ring } B/(z, P). \end{aligned}$$

Note that each F_j is a $(B/(z, P))_j$ -lex-segment. Furthermore, we have the disjoint unions

$$\begin{aligned} \{\mathbf{m}V_d\} &= \coprod_{0 \leq j \leq d} z^{d-j+1} \{U_j + \mathbf{n}U_{j-1}\} \\ \{\mathbf{m}T_d\} &= \coprod_{0 \leq j \leq d} z^{d-j+1} \{F_j + \mathbf{n}F_{j-1}\}, \end{aligned}$$

where $\mathbf{n} = \mathbf{m}/z$. We will show that

$$|F_j + \mathbf{n}F_{j-1}| = \max\left\{|F_j|, |\mathbf{n}F_{j-1}|\right\} \leq \max\left\{|U_j|, |\mathbf{n}U_{j-1}|\right\} \leq |U_j + \mathbf{n}U_{j-1}|.$$

The first equality above holds because both F_j and $\mathbf{n}F_{j-1}$ are $(B/(z, P))_j$ -lex-segments, so $F_j + \mathbf{n}F_{j-1}$ is the longer of these two lex-segments. The last inequality is obvious. It remains to prove the middle inequality. Using the short exact sequences of k -vector subspaces of $(B/P)_j$

$$\begin{aligned} 0 \rightarrow \mathbf{n}N_{j-1} \rightarrow \mathbf{n}F_{j-1} &\longrightarrow \mathbf{n}F_{j-1}/\mathbf{n}N_{j-1} = \mathbf{n}\gamma_{j-1}/(\mathbf{n}\gamma_{j-1} \cap \mathbf{n}N_{j-1}) \rightarrow 0 \\ 0 \rightarrow \mathbf{n}N_{j-1} \rightarrow \mathbf{n}U_{j-1} &\longrightarrow \mathbf{n}U_{j-1}/\mathbf{n}N_{j-1} = \mathbf{n}\nu_{j-1}/(\mathbf{n}\nu_{j-1} \cap \mathbf{n}N_{j-1}) \rightarrow 0 \end{aligned}$$

we get $|\mathbf{n}\gamma_{j-1}| = |\mathbf{n}F_{j-1}| - |\mathbf{n}N_{j-1}|$ and $|\mathbf{n}\nu_{j-1}| = |\mathbf{n}U_{j-1}| - |\mathbf{n}N_{j-1}|$. Therefore, the desired inequality $|\mathbf{n}F_{j-1}| \leq |\mathbf{n}U_{j-1}|$ is equivalent to the inequality $|\mathbf{n}\gamma_{j-1}| \leq |\mathbf{n}\nu_{j-1}|$. The latter inequality holds since by construction γ_{j-1} is the lexification of ν_{j-1} , so $|\gamma_{j-1}| = |\nu_{j-1}|$ and by induction on the number of variables we can apply Theorem 3.11 to the ring $B/(z, P, N_j)$.

Thus, $|F_j + \mathbf{n}F_{j-1}| \leq |U_j + \mathbf{n}U_{j-1}|$. Multiplication by z^{d-j+1} is injective if $d-j+1 \leq e_i - 1$ and is zero otherwise, therefore we conclude that

$$\left| z^{d-j+1}(F_j + \mathbf{n}F_{j-1}) \right| \leq \left| z^{d-j+1}(U_j + \mathbf{n}U_{j-1}) \right|.$$

This implies the desired inequality $|\mathbf{m}T_d| \leq |\mathbf{m}V_d|$.

Note that $\{\tau_d\}$ is greater lexicographically than $\{v_d\}$. If τ_d is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lex-greater monomial space. Thus, after finitely many steps, we reach a compressed monomial space. \square

Theorem 3.11. *Let v_d be a Υ_d -monomial space and λ_d be its lexification in Υ_d . Then $|\mathbf{m}\lambda_d| \leq |\mathbf{m}v_d|$.*

Proof. The theorem clearly holds if $n = 1$. Suppose that $n = 2$. An easy calculation shows that the theorem holds, provided we do not have $e_2 \leq d + 1 < e_1$. By the assumption on the ordering of the exponents, this does not hold and we are fine.

Suppose that $n \geq 3$. First, we apply Lemma 3.10 to reduce to the compressed case. We obtain a compressed Υ_d -monomial space τ_d such that $|\tau_d| = |v_d|$ and $|\mathbf{m}\tau_d| \leq |\mathbf{m}v_d|$. Let L_d and T_d be the $(B/P)_d$ -monomial spaces such that $\{L_d\} = \{\lambda_d\} \cup \{M_d\}$ and $\{T_d\} = \{\tau_d\} \cup \{M_d\}$, where the disjoint unions take place in B/P . Both L_d and T_d are $(B/P)_d$ -compressed. We apply Lemma 3.6 to conclude that

$$\begin{aligned} |\{\mathbf{m}T_d\}| &= \sum_{i=1}^n t_i(T_d) - \sum_{i=1}^n s_i(T_d) \\ |\{\mathbf{m}L_d\}| &= \sum_{i=1}^n t_i(L_d) - \sum_{i=1}^n s_i(L_d). \end{aligned}$$

Finally, we apply Lemma 3.7 and conclude that $|\{\mathbf{m}L_d\}| \leq |\{\mathbf{m}T_d\}|$. This inequality and short exact sequences, as in the proof of Lemma 3.10, imply the desired $|\mathbf{m}\lambda_d| \leq |\mathbf{m}v_d|$. \square

Equivalently, we obtain the following theorem, stated in the introduction:

Theorem 3.12. *Let $P = (x_1^{e_1}, \dots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \dots \leq e_n \leq \infty$ (here $x_i^\infty = 0$), and M be a compressed monomial ideal in B/P generated in degrees $\leq p$. If $n = 2$, assume that M is (B/P) -lex. Set $\Upsilon = B/(M + P)$. Then Υ is pro-lex above*

p , that is, for every graded ideal Γ in Υ generated in degrees $\geq p$ there exists an Υ -lex ideal Θ with the same Hilbert function.

Proof. We can assume that Γ is a monomial ideal by Gröbner basis theory. For each $d \geq p$, let λ_d be the lexification of Γ_d . By Theorem 3.11, it follows that $\Theta = \bigoplus_{d \geq p} \lambda_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as Γ in all degrees greater than or equal to p . \square

Remark 3.13. In the case when $M = P = 0$, Theorem 3.12 is the well-known Macaulay's Theorem [Ma]. In the case $M = 0$, Theorem 3.12 is the Clements-Lindström's Theorem [CL].

Example 3.14. It is natural to ask if a compressed ideal is Macaulay-Lex. This example shows that the answer is negative. Take $P = 0$. The ideal

$$M = (a^3, a^2b, a^2c, ab^2, abc, b^3, b^2c)$$

is compressed (and Borel) in the ring $k[a, b, c]$. The ideal (a^2, ab, b^2) in $k[a, b, c]/M$ is not lexifiable.

Example 3.15. It is natural to ask if Theorem 3.12 holds in the case when M is a B -Borel ideal. It does not. Take $P = 0$. The ideal

$$M = (a^3, a^2b, a^2c, a^2d, ab^2, abc, abd, b^3, b^2c)$$

is Borel in the ring $k[a, b, c, d]$. However it is not pro-lex because the ideal (b^2d) is not lexifiable in $k[a, b, c]/M$.

4. Adding new variables

Theorem 4.1. If B/M is Macaulay-Lex then $B[y]/M$ is Macaulay-Lex.

In this section, $W = B[y]/M$, \mathbf{m} is the k -vector space spanned by the variables in B (as in Section 2), and \mathbf{q} is the k -vector space spanned by \mathbf{m} and y .

Let V_d be a W_d -monomial space; we have the disjoint union $\{V_d\} = \coprod_{0 \leq j \leq d} y^{d-j} \{U_j\}$ where each U_j is a monomial space in W/y . Let F_j be the lexification of the space U_j in W/y . Consider the W_d -monomial space T_d defined by $\{T_d\} = \coprod_{0 \leq j \leq d} z^{d-j} \{F_j\}$. Clearly, $|T_d| = |V_d|$. We call T_d the y -compression of V_d .

Lemma 4.2. Let V_d be a W_d -monomial space, and let T_d be its y -compression. Then $|T_d| = |V_d|$ and $|\mathbf{q}T_d| \leq |\mathbf{q}V_d|$.

Proof. The proof is based on the same idea as the proof of Lemma 3.10. We write $\{V_d\} = \coprod_{0 \leq j \leq d} y^{d-j} \{U_j\}$ and $T_d = \coprod_{0 \leq j \leq d} y^{d-j} \{F_j\}$, where the F_j are B/M -lex satisfying $|F_j| = |U_j|$. Then, as in the proof of Lemma 3.10, we have the disjoint unions

$$\begin{aligned} \{\mathbf{q}V_d\} &= \coprod_{0 \leq j \leq d} y^{d-i+1} \{U_j + \mathbf{m}U_{j-1}\} \\ \{\mathbf{q}T_d\} &= \coprod_{0 \leq j \leq d} y^{d-i+1} \{F_i + \mathbf{m}F_{j-1}\}, \end{aligned}$$

and we have the inequalities

$$|F_i + \mathbf{m}F_{j-1}| = \max\{|F_j|, |\mathbf{m}F_{j-1}|\} \leq \max\{|U_j|, |\mathbf{m}U_{j-1}|\} \leq |U_j + \mathbf{m}U_{j-1}|,$$

where the middle inequality holds because B/M is Macaulay-Lex. Since multiplication by y is injective, we get

$$|y^{d-i+1}(F_i + \mathbf{m}F_{j-1})| \leq |y^{d-i+1}(U_j + \mathbf{m}U_{j-1})|.$$

□

Lemma 4.3. *Let T_d be a y -compressed W_d -monomial space. Then either T_d is W_d -lex, or there exists a W_d -monomial space F_d , such that F_d is strictly lexicographically greater than T_d , $|F_d| = |T_d|$, and $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$; (here “lexicographically greater” means that we order the monomials in $\{F_d\}$ and $\{T_d\}$ lexicographically, and then compare the two ordered sets lexicographically).*

Proof. Let r be as large as possible among the numbers such that we can write

$$T_d = y^{d-r}P \oplus \left(\bigoplus_{i>r} y^{d-i}L_i \right)$$

with P a lex segment of W_d . Such an r always exists, as we can if necessary take $r = 0$.

If $r = d$, then T_d is W_d -lex and we are done. If not, then $yP + L_{r+1}$ is not lex in W . Let m be the lex-greatest monomial of W_{r+1} such that $m \notin yP + L_{r+1}$. We consider two cases depending on whether y divides m or not.

Suppose that y divides m . Let u be the lex-least monomial of $yP + L_{r+1}$. Since P is lex and y does not divide m , it follows that y does not divide u . Let Q be the k -vector space spanned by $\{Q\}$, defined by

$$\{Q\} = \left(\{yP\} \cup \{L_{r+1}\} \cup \{m\} \right) \setminus \{u\}.$$

Set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i}L_i \right).$$

Now, $\{F_d\} \setminus y^{d-r-1}m = \{T_d\} \setminus y^{d-r-1}u$. Hence, $\{F_d\}$ is strictly lexicographically greater than T_d . We will compare $\{\mathbf{q}F_d\}$ and $\{\mathbf{q}T_d\}$. The set $\{\mathbf{m}y^{d-r-1}m\}$ is contained in $\{\mathbf{q}T_d\}$, so we have $\mathbf{q}F_d \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \subseteq \{y^{d-r}m\}$. Furthermore, we will show that $y^{d-r}u \notin \{\mathbf{q}F_d\}$. Suppose the opposite. Hence, there exists a q such that $y^{d-r}u = x_q \left(y^{d-r} \frac{u}{x_q} \right)$, where $\frac{u}{x_q} \in P$. But $y \frac{u}{x_q} \in yP$ is lex-smaller than u ; this contradicts the choice of u . Hence $\{\mathbf{q}T_d\} \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \supseteq \{y^{d-r}u\}$. Therefore, we have the desired inequality $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$. The lemma is proved in this case.

It remains to consider the case when m is not divisible by y . In this case, m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. Set $z = x_{\max(m)}$. In our construction we will use the set

$$N = \left\{ u \in yP \mid u \prec_{lex} m \text{ and } \left(\frac{z}{y} \right)^{e_u} u \neq 0 \text{ in } B/M \right\},$$

where e_u is the largest power of y dividing u . We will show that $N \neq \emptyset$ because $\frac{z}{y}m \in N$. Since m is the lex-greatest monomial missing in $m \notin yP + L_{r+1}$, it follows that there exists a monomial $ym' \in yP$ that is lex-smaller than m . Therefore, m' is

(non-strictly) lex-smaller than $\frac{m}{z}$. As $m' \in P$ and P is lex, it follows that $\frac{m}{z} \in P$. Thus, $\frac{y}{z}m \in N$ as desired.

We will need three of the properties of N :

Claim.

1. m is (non-strictly) lex-greater than all the monomials in $\frac{z}{y}N$.
2. $\frac{z}{y}N \cap \{L_{r+1}\} = \emptyset$.
3. $\frac{z}{y}N \cap \{yP\} \subseteq N$.

We will prove the claim. (3) is clear. (2) follows from (1) and the fact that in the considered case m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. We will prove (1). Write

$$m = x_1^{a_1} x_2^{a_2} \dots z^{a_z} \quad \text{and} \quad u = x_1^{b_1} x_2^{b_2} \dots z^{b_z} w y^{b_y},$$

where w is not divisible by x_1, \dots, z or by y . Suppose that the monomial $\frac{z}{y}u = x_1^{b_1} x_2^{b_2} \dots z^{b_z+1} w y^{b_y-1}$ is lex-greater than m . On the other hand, m is lex-greater than u . It follows that $a_j = b_j$ for $j < \max(m)$ and $b_z < a_z \leq b_z + 1$. Since the monomials have the same degree, it follows that $a_z = b_z + 1$, $w = 1$, and $b_y = 1$. Hence $m = \frac{z}{y}u$. The claim is proved.

Let Q be the k -vector space such that

$$\{Q\} = (\{yP + L_{r+1}\} \setminus N) \cup \frac{z}{y}N.$$

By the claim above, it follows that we have the disjoint union $\{Q\} = \{L_{r+1}\} \coprod yP \setminus N \coprod \frac{z}{y}N$. Clearly, $|Q| = |L_{r+1} \oplus yP|$.

We consider the set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i}L_i \right).$$

It is clear that $|F_d| = |T_d|$. Since $y^{d-r-1}m \in F_d$, we see that F_d is strictly lexicographically greater than T_d . We will show that the inequality $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$ holds. Set $U = L_{r+1} \oplus yP$ and $V = \bigoplus_{i>r+1} y^{d-i}L_i$.

Since

$$|\mathbf{q}Q| - |\mathbf{q}U| = - \left| \left\{ t \in \mathbf{q}N \setminus (\mathbf{q}N \cap \mathbf{q}(U \setminus N)) \mid \frac{z}{y}t = 0 \right\} \right| \leq 0$$

it follows that $|\mathbf{q}Q| \leq |\mathbf{q}U|$. Furthermore, we have

$$\begin{aligned} |\mathbf{q}F_d| &= |\mathbf{q}y^{d-r-1}Q| + |\mathbf{q}V| - |\mathbf{q}V \cap \mathbf{q}y^{d-r-1}Q| \\ &= |\mathbf{q}y^{d-r-1}Q| + |\mathbf{q}V| - |y^{d-r-1}(L_{r+2} \cap \mathbf{m}\{v \in Q \mid y \text{ does not divide } v\})| \\ &\leq |\mathbf{q}y^{d-r-1}U| + |\mathbf{q}V| - |y^{d-r-1}(L_{r+2} \cap \mathbf{m}\{v \in Q \mid y \text{ does not divide } v\})| \\ &\leq |\mathbf{q}y^{d-r-1}U| + |\mathbf{q}V| - |y^{d-r-1}(L_{r+2} \cap \mathbf{m}\{v \in U \mid y \text{ does not divide } v\})| \\ &= |\mathbf{q}T_d|; \end{aligned}$$

the first inequality holds because multiplication by y is injective, the second holds by set containment. \square

We are ready for the proof of Theorem 4.1:

Proof. Let V_d be a W_d -monomial space. If V_d is not W -lex, apply Lemmas 4.2 and 4.3 to obtain a y -compressed W_d -monomial space F_d which is strictly greater lexicographically than V_d and satisfies $|F_d| = |V_d|$ and $|\mathbf{q}F_d| \leq |\mathbf{q}V_d|$. If F_d is not W -lex, we can apply the lemmas again. After finitely many steps, the process must terminate in a lexicographic monomial space. Hence W is d -pro-lex for all degrees $d \geq 0$, and so is Macaulay-Lex. \square

5. Lexicographic quotients

Theorem 5.1. *If M is Macaulay-Lex and N is a B/M -lex ideal, then $M + N$ is Macaulay-Lex.*

The theorem follows immediately from the following result:

Proposition 5.2. *Fix a degree $d \geq 1$. If M is $(d-1)$ -pro-lex and N is a B/M -lex ideal, then $M + N$ is $(d-1)$ -pro-lex.*

Proof. Throughout this proof, for a monomial space \bar{V} in $B/(M+N)$, we denote by V the k -vector space spanned by $\{\bar{V}\}$ in B/M .

Let \bar{S}_{d-1} be a monomial space in $(B/(M+N))_{d-1}$. Let \bar{L}_{d-1} be the $B/(M+N)$ -lexification of \bar{S}_{d-1} . Set \bar{L}_d to be the k -vector space spanned by $\mathbf{m}\{\bar{L}_{d-1}\}$ and \bar{S}_d be the k -vector space spanned by $\mathbf{m}\{\bar{S}_{d-1}\}$. We will prove that

$$|\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)}.$$

First, we assume that the ideal N has no minimal generators in degree d .

Note that $N_{d-1} + L_{d-1}$ is a B/M -lex-segment. Therefore, $N_{d-1} + L_{d-1}$ is the B/M -lexification of $N_{d-1} + S_{d-1}$ in the ring B/M . Since M is $(d-1)$ -pro-lex, the following inequality holds:

$$|N_d + L_d|^{B/M} \leq |N_d + S_d|^{B/M}.$$

On the other hand,

$$\begin{aligned} |N_d + L_d|^{B/M} &= |N_d|^{B/M} + |L_d|^{B/M} - |N_d \cap L_d|^{B/M} \\ |N_d + S_d|^{B/M} &= |N_d|^{B/M} + |S_d|^{B/M} - |N_d \cap S_d|^{B/M} \end{aligned}$$

Therefore, we obtain the inequality

$$|L_d|^{B/M} - |N_d \cap L_d|^{B/M} \leq |S_d|^{B/M} - |N_d \cap S_d|^{B/M}.$$

Note that the left hand-side is equal to $|\bar{L}_d|^{B/(M+N)}$ whereas the right-hand side is equal to $|\bar{S}_d|^{B/(M+N)}$. Thus, we get the desired inequality

$$|\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)}.$$

Now, suppose that N has minimal monomial generators in degree d .

If $L_d \subseteq N_d$, then

$$0 = |\bar{L}_d|^{B/(M+N)} \leq |\bar{S}_d|^{B/(M+N)}.$$

Suppose that $L_d \not\subseteq N_d$. Set $Q = \{N_d\} \setminus \{\mathbf{m}N_{d-1}\}$. Since both $\mathbf{m}N_{d-1} + L_d$ and N_d are B/M -lex-segments, it follows that one of them contains the other. Hence $\{L_d\} \supseteq Q$, so

$$|\bar{L}_d|^{B/(M+N)} = |L_d|^{B/(M+(N_{d-1}))} - |Q|.$$

The argument above (for the case when the ideal is $(d-1)$ -generated) can be applied to N_{d-1} , and it yields

$$|L_d|^{B/(M+(N_{d-1}))} \leq |S_d|^{B/(M+(N_{d-1}))}.$$

Therefore we have

$$\begin{aligned} |\bar{L}_d|^{B/(M+N)} &= |L_d|^{B/(M+(N_{d-1}))} - |Q| \\ &\leq |S_d|^{B/(M+(N_{d-1}))} - |Q| \leq |S_d|^{B/(M+(N_{d-1}))} - |Q \cap \{S_d\}| \\ &= |\bar{S}_d|^{B/(M+N)}. \end{aligned}$$

□

Macaulay's Theorem [Ma] says that 0 is pro-lex. Hence, Theorem 5.1 applied to $M = 0$ yields the following:

Corollary 5.3. *If U is a B -lex ideal then it is Macaulay-Lex.*

Remark 5.4. *Following [Sh], we say that a monomial ideal M in B is piecewise lex if, whenever $\mathbf{x}^{\mathbf{a}} \in M$, $\mathbf{x}^{\mathbf{b}} \succ_{lex} \mathbf{x}^{\mathbf{a}}$, and $\max(\mathbf{x}^{\mathbf{b}}) \leq \max(\mathbf{x}^{\mathbf{a}})$, we have $\mathbf{x}^{\mathbf{b}} \in M$. Shakin [Sh] proved that if M is a piecewise lex ideal in B , then it is Macaulay-Lex. This result can be proved differently using our technique as follows:*

Proof. We induct on n . Let $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}$ be the minimal monomial generators of M divisible by x_n . So the lex segment L_j ending in $\mathbf{x}^{\mathbf{a}_j}$ must be contained in M . Set $N = M \cap k[x_1, \dots, x_{n-1}]$. Then N is piecewise lex and so by induction is Macaulay-Lex in $k[x_1, \dots, x_{n-1}]$. By Theorem 4.1, N is Macaulay-Lex in B . By induction on j , we conclude that $(N + L_1 + \dots + L_{j-1}) + L_j$ is Macaulay-Lex by Theorem 5.1. Hence, $M = N + L_1 + \dots + L_r$ is Macaulay-Lex as well. □

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