NAVIER-STOKES EQUATIONS IN ARBITRARY DOMAINS : 
THE FUJITA-KATO SCHEME

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Abstract. Navier-Stokes equations are investigated in a functional setting in 3D open sets \( \Omega \), bounded or not, without assuming any regularity of the boundary \( \partial \Omega \). The main idea is to find a correct definition of the Stokes operator in a suitable Hilbert space of divergence-free vectors and apply the Fujita-Kato method, a fixed point procedure, to get a local strong solution.

1. Introduction

Since the pioneering work by Leray [3] in 1934, there have been several studies on solutions of Navier-Stokes equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + \nabla \pi + (u \cdot \nabla)u &= 0 \quad \text{in} \quad [0,T] \times \Omega, \\
\text{div } u &= 0 \quad \text{in} \quad [0,T] \times \Omega, \\
u &= 0 \quad \text{on} \quad [0,T] \times \partial \Omega, \\
u(0) &= u_0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

Fujita and Kato [2] in 1964 gave a method to construct so called mild solutions in smooth domains \( \Omega \), producing local (in time) smooth solutions of (NS) in a Hilbert space setting. These solutions are global in time if the initial value \( u_0 \) is small enough in a certain sense. The case of non smooth domains has been studied by Deuring and von Wahl [1] in 1995 where they considered domains \( \Omega \subset \mathbb{R}^3 \) with Lipschitz boundary \( \partial \Omega \). They found local smooth solutions using results contained in Shen’s PhD thesis [4]. Their method does not cover the critical space case as in [2]. One of the difficulty there was to understand the Stokes operator, and in particular its domain of definition.

In Section 2, we give a “universal” definition of the Stokes operator, for any domain \( \Omega \subset \mathbb{R}^3 \) (Definition 2.4). In Section 3, we construct a mild solution of (NS) with a method similar to Fujita-Kato’s [2] (Theorem 3.5) for initial values \( u_0 \) in the critical space \( D(A^{\frac{1}{2}}) \). We show in Section 4 that this mild solution is a strong solution, i.e. (NS) is satisfied almost everywhere.

2. The Stokes operator

Let \( \Omega \) be an open set in \( \mathbb{R}^3 \). The space

\[
L^2(\Omega)^3 = \{ u = (u_1, u_2, u_3); u_i \in L^2(\Omega), \ i = 1, 2, 3 \}
\]

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endowed with the scalar product
\[ \langle u, v \rangle = \int_{\Omega} u \cdot \overline{v} = \sum_{i=1}^{3} \int_{\Omega} u_i \overline{v_i} \]
is a Hilbert space. Define
\[ G = \{ \nabla p; p \in L^2_{loc}(\Omega) \text{ with } \nabla p \in L^2(\Omega)^3 \}; \]
the set \( G \) is a closed subspace of \( L^2(\Omega)^3 \). Let
\[ H = G^\perp = \{ u \in L^2(\Omega)^3; \langle u, \nabla p \rangle = 0, \forall p \in L^2_{loc}(\Omega) \text{ with } \nabla p \in L^2(\Omega)^3 \}. \]
The space \( H \), endowed with the scalar product \( \langle \cdot, \cdot \rangle \) is a Hilbert space. We have the following Hodge decomposition
\[ L^2(\Omega)^3 = H^\perp \oplus G. \]
We denote by \( \mathbb{P} \) the projection from \( L^2(\Omega)^3 \) onto \( H : \mathbb{P} \) is the usual Helmholtz projection. We denote by \( J \) the canonical injection \( H \hookrightarrow L^2(\Omega)^3 : J' = \mathbb{P} \) (\( J' \) being the adjoint of \( J \)) and \( \mathbb{P} J \) is the identity on \( H \). Let now \( \mathcal{D}(\Omega)^3 = \mathcal{C}_c^\infty(\Omega)^3 \) and
\[ D = \{ u \in \mathcal{D}(\Omega)^3; \text{div} u = 0 \}. \]
It is clear that \( D \) is a closed subspace of \( \mathcal{D}(\Omega)^3 \). We denote by \( J_0 : D \hookrightarrow \mathcal{D}(\Omega)^3 \) the canonical injection : \( J_0 \subset J \). Let \( \mathbb{P}_1 \) be the adjoint of \( J_0 : \mathbb{P}_1 = J_0' : \mathcal{D}'(\Omega)^3 \rightarrow \mathcal{D}' \). We have \( \mathbb{P} \subset \mathbb{P}_1 \). The following theorem characterizes the elements in \( \ker \mathbb{P}_1 \).

**Theorem 2.1 (de Rham).** Let \( T \in \mathcal{D}'(\Omega)^3 \) such that \( \mathbb{P}_1 T = 0 \) in \( \mathcal{D}' \). Then there exists \( S \in (\mathcal{C}_c^\infty(\Omega))^3 \) such that \( T = \nabla S \). Conversely, if \( T = \nabla S \) with \( S \in (\mathcal{C}_c^\infty(\Omega))^3 \), then \( \mathbb{P}_1 T = 0 \) in \( \mathcal{D}' \).

We denote by \( H^1_0(\Omega)^3 \) the closure of \( \mathcal{D}(\Omega)^3 \) with respect to the scalar product \( \langle u, v \rangle_1 = \langle u, v \rangle + \sum_{i=1}^{3} \langle \partial_i u, \partial_i v \rangle \). By Sobolev embeddings, we have \( H^1_0(\Omega)^3 \hookrightarrow L^6(\Omega)^3 \). Define
\[ V = H \cap H^1_0(\Omega)^3. \]
The space \( V \) is a closed subspace of \( H^1_0(\Omega)^3 \); endowed with the scalar product \( \langle \cdot, \cdot \rangle_1 \), \( V \) is a Hilbert space.

**Proposition 2.2.** The space \( V \) is dense in \( H \).

**Proof.** Let \( u \in H \) be in the orthogonal of \( V \) with respect to \( H \), i.e.
\[ \langle u, v \rangle = 0 \quad \text{for all } v \in V. \]
(2.1)
Since \( D \subset V \), (2.1) implies also
\[ \langle u, v \rangle = 0 \quad \text{for all } v \in D. \]
It means that \( u \), viewed as an element of \( \mathcal{D}' \), is 0. By Theorem 2.1, there exists a distribution \( S \in \mathcal{D}(\Omega)^3 \) such that \( J u = \nabla S \). Since \( J u \in L^2(\Omega)^3 \), so is \( \nabla S \) and therefore, \( u = \mathbb{P} J u = \mathbb{P} \nabla S = 0 \). \( \square \)
The canonical injection \( J : \mathcal{V} \hookrightarrow H^1_0(\Omega)^3 \) is the restriction of \( J \) to \( \mathcal{V} \). We denote by \( \tilde{P} \) the adjoint of \( \tilde{J} \) : since \( \tilde{J} \) is the restriction of \( J \) to \( \mathcal{V} \), \( \tilde{P} \) is an extension of \( P \) to \( \mathcal{V}' \). On \( \mathcal{V} \times \mathcal{V} \) we define now the form \( a \) by \( a(u,v) = \sum_{i=1}^3 \langle \partial_i \tilde{J}u, \partial_i \tilde{J}v \rangle : a \) is a bilinear, symmetric, \( \delta + a \) is a coercive form on \( \mathcal{V} \times \mathcal{V} \) for all \( \delta > 0 \), then defines a bounded self-adjoint operator \( A_0 : \mathcal{V} \to \mathcal{V}' \) by \( (A_0u)(v) = a(u,v) \) with \( \delta + A_0 \) invertible for all \( \delta > 0 \).

**Proposition 2.3.** For all \( u \in \mathcal{V} \), \( A_0u = \tilde{P}(-\Delta^\Omega_D)\tilde{J}u \), where \( \Delta^\Omega_D \) denotes the Dirichlet-Laplacian on \( H^1_0(\Omega)^3 \).

**Proof.** For all \( u, v \in \mathcal{V} \), we have

\[
(A_0u)(v) = \sum_{i=1}^3 \langle \partial_i \tilde{J}u, \partial_i \tilde{J}v \rangle = \langle (-\Delta^\Omega_D)\tilde{J}u, \tilde{J}v \rangle_{H^{-1},H^1_0}
\]

The first two equalities come from the definition of \( A_0 \) and \( a \). The third equality comes from the definition of the Dirichlet-Laplacian on \( H^1_0(\Omega)^3 \) and the fact that for \( v \in \mathcal{V} \), \( \tilde{J}v = v \). The last equality is due to \( \tilde{J}'\varphi = \tilde{P}\varphi \) in \( \mathcal{V}' \) for all \( \varphi \in H^{-1}(\Omega)^3 \). This shows that \( A_0u \) and \( \tilde{P}(-\Delta^\Omega_D)\tilde{J}u \) are two continuous linear forms on \( \mathcal{V} \) which coincide on \( \mathcal{V} \), they are then equal. \( \square \)

**Definition 2.4.** The operator \( A \) defined on its domain \( D(A) = \{ u \in \mathcal{V} ; A_0u \in \mathcal{H} \} \) by \( Au = A_0u \) is called the Stokes operator.

**Theorem 2.5.** The Stokes operator is self-adjoint in \( \mathcal{H} \), generates an analytic semigroup \( (e^{-tA})_{t \geq 0} \), \( D(A^{\frac{1}{2}}) = \mathcal{V} \) and satisfies

\[
D(A) = \{ u \in \mathcal{V} ; \exists \pi \in (C_c^\infty(\Omega))' : \nabla \pi \in H^{-1}(\Omega) \text{ and } -\Delta u + \nabla \pi \in \mathcal{H} \}
\]

\[
Au = -\Delta u + \nabla \pi.
\]

**Remark 2.6.** Since \( H^1_0(\Omega)^3 \hookrightarrow L^3(\Omega)^3 \), it is clear by interpolation and dualization that \( \tilde{P} \) maps \( L^p(\Omega)^3 \) to \( D(A')' \) for \( \frac{3}{2} \leq p \leq 2 \), \( 0 \leq s \leq \frac{1}{2} \) and \( s = -\frac{3}{4} + \frac{3}{2p} \). Since \( A \) is self-adjoint, one has \((\delta + A_0)^{-s}D(A')' = \{(\delta + A_0)^{-s}u ; u \in D(A')' = \mathcal{H} \). In particular, \((\delta + A_0)^{-\frac{1}{4}}\mathcal{P}_1 \) maps \( L^3(\Omega)^3 \) into \( \mathcal{H} \).

### 3. Mild solution to the Navier-Stokes system

Let \( T > 0 \).

Define the following Banach space

\[
\mathcal{E}_T = \left\{ u \in C([0,T]; D(A^{\frac{1}{2}})) \cap C^1([0,T]; D(A^{\frac{1}{2}})) \right\}
\]

such that \( \sup_{0 < s < T} \| s^{\frac{1}{2}} A^{\frac{1}{2}} u(s) \|_{\mathcal{H}} + \sup_{0 < s < T} \| s A^{\frac{1}{2}} u'(s) \|_{\mathcal{H}} < \infty \)
endowed with the norm
\[ \|u\|_{\mathcal{E}_T} = \sup_{0 < s < T} \|A^{\frac{1}{2}}u(s)\|_{\mathcal{H}} + \sup_{0 < s < T} \|s^{\frac{1}{2}}A^{\frac{3}{2}}u(s)\|_{\mathcal{H}} + \sup_{0 < s < T} \|sA^{\frac{1}{2}}u(s)\|_{\mathcal{H}}. \]

Let \( \alpha \) be defined by \( \alpha(t) = e^{-tA}u_0 \) where \( u_0 \in D(A^{\frac{1}{2}}) \). Then \( \alpha \in \mathcal{E}_T \). Indeed, it is clear that \( \alpha \in \mathcal{C}([0, T]; D(A^{\frac{1}{2}})) \). We also have that \( t^\frac{1}{2}A^{\frac{3}{2}}\alpha(t) = t^\frac{1}{2}A^{\frac{3}{2}}e^{-tA}A^{\frac{1}{2}}u_0 \) is bounded on \((0, T)\) since \((e^{-tA})_{t \geq 0}\) is an analytic semigroup. Moreover, one has \( \alpha'(t) = -Ae^{-tA}u_0 \) which yields \( tA^{\frac{1}{2}}\alpha'(t) = -tAe^{-tA}A^{\frac{1}{2}}u_0 \) continuous on \([0, T]\), bounded in \( \mathcal{H} \). For \( u, v \in \mathcal{E}_T \), we define now
\[ \Phi(u, v)(t) = \int_0^t e^{-(t-s)A}(-\frac{1}{2}\overline{P})(u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s))ds, \quad 0 < t < T. \]

**Notation 3.1.** Let \( X, Y \) be Banach spaces. For a bounded linear operator \( S : X \to Y \), we denote by \( \|S\|_{\mathcal{L}(X; Y)} \) the norm of \( S \), i.e.
\[ \|S\|_{\mathcal{L}(X; Y)} = \sup\{\|Sx\|_Y : \forall x \in X \text{ with } \|x\|_X \leq 1\}. \]

If \( X = Y \), we adopt the notation \( \|S\|_{\mathcal{L}(X)} \) instead of \( \|S\|_{\mathcal{L}(X; Y)} \). For a bilinear operator \( B : X \times X \to Y \), we denote by \( \|B\|_{\mathcal{L}(X \times X; Y)} \) the norm of \( B \), i.e.
\[ \|B\|_{\mathcal{L}(X \times X; Y)} = \sup\{\|B(x, x')\|_Y : \forall x, x' \in X \text{ with } \|x\|_X \leq 1 \text{ and } \|x'\|_X \leq 1\}. \]

**Notation 3.2.** For \( u, v \in L^2(\Omega)^3 \), we denote by \( u \otimes v \) the matrix defined by
\[ (u \otimes v)_{i,j} = u_i v_j, \quad 1 \leq i, j \leq 3. \]

**Remark 3.3.** If \( u, v \) are sufficiently smooth vector fields such that \( \text{div}u = 0 \), then
\[ \text{div}(u \otimes v) := \sum_{i=1}^3 \partial_i (u, v) = \sum_{i=1}^3 u_i \partial_i v = (u \cdot \nabla)v. \]

**Proposition 3.4.** The transform \( \Phi \) is bilinear, symmetric, continuous from \( \mathcal{E}_T \times \mathcal{E}_T \) to \( \mathcal{E}_T \) and the norm of \( \Phi \) is independent of \( T \).

**Proof.** The fact that \( \Phi \) is bilinear and symmetric is clear. Moreover, \( \Phi(u, v) = e^{-A} * f \), where \( f \) is defined by
\[ f(s) = (-\frac{1}{2}\overline{P})(u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s)), \quad s \in [0, T]. \]

For \( u, v \in \mathcal{E}_T \), it is clear that \( (u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s) \in L^2(\Omega)^3 \) and therefore \( (\delta + A_0)^{-\frac{3}{2}} f(s) \in \mathcal{H} \) with \( \sup_{0 < s < T} s^{\frac{1}{2}} \|(\delta + A_0)^{-\frac{3}{2}} f(s)\|_{\mathcal{H}} \leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \). We have then
\[ \Phi(u, v) = e^{-A} * f = (\delta + A)^{\frac{3}{2}} e^{-A} * ((\delta + A_0)^{-\frac{3}{2}} f) \]
and therefore
\[ \|A^{\frac{1}{2}}\Phi(u, v)(t)\|_{\mathcal{H}} \leq \int_0^t \|A^{\frac{1}{2}}(\delta + A)^{\frac{3}{2}} e^{-(t-s)A} \|_{\mathcal{L}(\mathcal{H})} \|((\delta + A_0)^{-\frac{3}{2}} f)\|_{\mathcal{H}} ds \]
\[ \leq c \left( \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \]
\[ \leq c \left( \int_0^1 \frac{1}{\sqrt{1-s}} \frac{1}{\sqrt{s}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \]
\[ \leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}. \]
Continuity with respect to \( t \in [0, T] \) of \( t \mapsto A^{\frac{1}{2}} \Phi(u, v)(t) \) is clear once we have proved the boundedness. We also have

\[
\|A^{\frac{1}{2}} \Phi(u, v)(t)\|_{\mathcal{H}} \leq \int_0^t \|A^{\frac{1}{2}}(\delta + A)^{\frac{1}{2}} e^{-((t-s)A)}\|_{L^2(\mathcal{H})} \|((\delta + A_0)^{-\frac{1}{2}} f(s))\|_{\mathcal{H}} ds \\
\leq c \left( \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{\sqrt{\sigma}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq ct^{-\frac{1}{4}} \left( \int_0^1 \frac{1}{(1-\sigma)^{\frac{1}{2}}} \frac{1}{\sqrt{\sigma}} d\sigma \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq ct^{-\frac{1}{4}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]

Continuity with respect to \( t \in [0, T] \) is clear once we have proved the boundedness. To prove the last part of the norm of \( \Phi(u, v) \) in \( \mathcal{E}_T \), we first write \( f \), using Notation 3.2 and Remark 3.3, in the following form

\[
f(s) = (-\frac{1}{2} \mathbb{P}) \text{ div } (u(s) \otimes v(s) + v(s) \otimes u(s)), \quad s \in [0, T].
\]

We have then for \( s \in [0, T] \)

\[
f'(s) = (-\frac{1}{2} \mathbb{P}) \text{ div } (u'(s) \otimes v(s) + u(s) \otimes v'(s) + v'(s) \otimes u(s) + v(s) \otimes u'(s)).
\]

For all \( s \in [0, T] \) we have

\[
s^{\frac{1}{2}} \|u'(s) \otimes v(s)\|_2 \leq (1) \|su'(s)\|_3 \|s^{\frac{1}{2}}v(s)\|_6 \\
\leq (2) \|sA^{\frac{1}{2}}u'(s)\|_{\mathcal{H}} \|s^{\frac{1}{2}}A^{\frac{1}{2}}v(s)\|_{\mathcal{H}} \\
\leq (3) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T},
\]

where the first inequality comes from the fact that \( L^3 \cdot L^6 \hookrightarrow L^2 \), the second comes from the inclusions \( D(A^{\frac{1}{2}}) \hookrightarrow L^3(\Omega)^3 \) and \( D(A^{\frac{1}{2}}) \hookrightarrow L^6(\Omega)^3 \) and the third inequality follows directly from the definition of the space \( \mathcal{E}_T \). Of course the same occurs for the other three terms \( u(s) \otimes v'(s) \), \( v'(s) \otimes u(s) \) and \( v(s) \otimes u'(s) \). Therefore, since \( A_0^{-\frac{1}{2}} \) maps \( \mathcal{V} \) to \( \mathcal{H} \), we obtain

\[
\sup_{0 < s < T} \|s^{\frac{1}{2}}(\delta + A_0)^{-\frac{1}{2}} f'(s)\|_{\mathcal{H}} \leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]

We have

\[
\Phi(u, v)(t) = \int_0^t e^{-sA} f(t-s) ds + \int_0^t e^{-(t-s)A} f(s) ds \quad t \in [0, T],
\]

and therefore

\[
\Phi(u, v)'(t) = e^{-\frac{1}{2}A} f(\frac{1}{2}) + \int_0^\frac{1}{2} (\delta + A)^{\frac{1}{2}} e^{-sA} (\delta + A_0)^{-\frac{1}{2}} f'(t-s) ds \\
+ \int_0^\frac{1}{2} -A(\delta + A)^{\frac{1}{2}} e^{-(t-s)A} (\delta + A_0)^{-\frac{1}{2}} f(s) ds,
\]
which yields
\[
\|A^{\frac{1}{4}}\Phi(u, v)'(t)\|_{\mathcal{H}} \leq \frac{c}{\sqrt{t}} \left\| (\delta + A_0)^{-\frac{1}{2}} f\left(\frac{t}{2}\right) \right\|_{\mathcal{H}} + c \left( \int_0^{\frac{t}{2}} \frac{1}{s^\frac{3}{2}} \left( \frac{1}{s^\frac{1}{2}} \|u\|_{\mathcal{E}_T} \right) ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
+ c \left( \int_0^{\frac{t}{2}} \frac{1}{(t-s)^\frac{1}{2}} \frac{1}{s^\frac{3}{2}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq \frac{c}{t} \left( \int_0^{\frac{t}{2}} \frac{d\sigma}{\left(1-\sigma\right)^{\frac{3}{2}}\sigma^\frac{1}{2}} \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]

This last inequality ensures that \(\Phi(u, v)\) is the mild solution to the Navier-Stokes system.

**Theorem 3.5.** For all \(u_0 \in D(A^{\frac{1}{4}})\), there exists \(T > 0\) such that there exists a unique \(u \in \mathcal{E}_T\) solution of \(u = \alpha + \Phi(u, u)\) on \([0, T]\). This function \(u\) is called the mild solution to the Navier-Stokes system.

**Proof.** Let \(T > 0\). Since \(\Phi : \mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T\) is bilinear continuous, it suffices to apply Picard fixed point theorem, as in [2]. The sequence in \(\mathcal{E}_T\) \((v_n)_{n \in \mathbb{N}}\) defined by \(v_0 = \alpha\) as first term and

\[v_{n+1} = \alpha + \Phi(v_n, v_n), \quad n \in \mathbb{N}\]

converges to the unique solution \(u \in \mathcal{E}_T\) of \(u = \alpha + \Phi(u, u)\) provided \(\|A^{\frac{1}{4}}u_0\|_{\mathcal{H}}\) is small enough \((\|\alpha\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{\mathcal{E}_T \times \mathcal{E}_T}^\frac{1}{2}})\). In the case where \(\|A^{\frac{1}{4}}u_0\|_{\mathcal{H}}\) is not small (that is, if \(\|\alpha\|_{\mathcal{E}_T} \geq \frac{1}{4\|\Phi\|_{\mathcal{E}_T \times \mathcal{E}_T}^\frac{1}{2}}\)) then for \(\varepsilon > 0\), there exists \(u_{0, \varepsilon} \in D(A)\) such that \(\|A^{\frac{1}{4}}(u_0 - u_{0, \varepsilon})\|_{\mathcal{H}} \leq \varepsilon\). If we take as initial value \(u_{0, \varepsilon} \in D(A)\), we have

\[\|\alpha\|_{\mathcal{E}_T} \leq cT^\frac{1}{2} \|Au_{0, \varepsilon}\|_{\mathcal{H}} \xrightarrow{T \to 0} 0.\]

Therefore, we can find \(T > 0\) such that \(\|\alpha\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{\mathcal{E}_T \times \mathcal{E}_T}^\frac{1}{2}}\).

**4. Strong solutions**

Let \(u\) be the mild solution to the Navier-Stokes system. We show in this section that \(u\) in fact satisfies the equations of the Navier-Stokes system in an \(L^p\)-sense (for a suitable \(p\)). To begin with, we know that \(u \in \mathcal{E}_T\) and satisfies

\[u = \alpha + \Phi(u, u) = \alpha + e^{-A} \varphi(u),\]

where \(\varphi(u) = -\tilde{P}((u \cdot \nabla)u)\) and we have \(\|t^\frac{1}{2} (u(t) \cdot \nabla)u(t)\|_{\frac{3}{2}} \leq c\|u\|_{\mathcal{E}_T}^2\). Therefore, we get

\[(4.1) \quad u(0) = \alpha(0) = u_0,\]

\[(4.2) \quad \text{div}u(t) = 0 \ \text{in} \ \mathcal{L}^2, \ \text{for} \ t \in [0, T],\]

and

\[u' + Au = f \ \text{in} \ \mathcal{C}([0, T]; \mathcal{V}'),\]

which means that for all \(t \in [0, T]\),

\[\tilde{F}(u'(t) - \Delta_D^\Omega u(t) + (u(t) \cdot \nabla)u(t)) = 0.\]
Then, by Theorem 2.1, there exists \((-\pi)(t) \in (\mathcal{C}^\infty_c(\Omega))'\) such that \(\nabla \pi(t) \in H^{-1}(\Omega)^3\) and
\[
\nabla(-\pi)(t) = u'(t) - \Delta_D u(t) + (u(t) \cdot \nabla)u(t)
\]
and we have for \(0 < t < T\)
\[
-\Delta_D^2 u(t) + \nabla \pi(t) = -u'(t) - (u(t) \cdot \nabla)u(t) \in L^3(\Omega)^3 + L^2(\Omega)^3.
\]

The equation (4.3), together with (4.1) and (4.2), give the usual Navier-Stokes equations which are fulfilled in a strong sense \((a.e.)\) where we consider the expression \(-\Delta u + \nabla \pi\) undecoupled.

References


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