AN ENDPOINT \((1, \infty)\) BALIAN-LOW THEOREM

JOHN J. Benedetto, Wojciech Czaja, Alexander M. Powell, and Jacob Sterbenz

Abstract. It is shown that a \((1, \infty)\) version of the Balian-Low Theorem holds. If \(g \in L^2(\mathbb{R})\), \(\Delta_1(g) < \infty\) and \(\Delta_\infty(\widehat{g}) < \infty\), then the Gabor system \(\mathcal{G}(g, 1, 1)\) is not a Riesz basis for \(L^2(\mathbb{R})\). Here, \(\Delta_1(g) = \int |t|^2 |g(t)|^2 dt\) and \(\Delta_\infty(\widehat{g}) = \sup_{N>0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma\).

1. Introduction

Given a square integrable function \(g \in L^2(\mathbb{R})\), and constants \(a, b > 0\), the associated Gabor system, \(\mathcal{G}(g, a, b) = \{g_{m,n}\}_{m,n \in \mathbb{Z}}\), is defined by
\[
g_{m,n}(t) = e^{2\pi iam t} g(t - bn).
\]
Gabor systems provide effective signal decompositions in a variety of settings ranging from eigenvalue problems to applications in communications engineering. Background on the theory and applications of Gabor systems can be found in [16], [12], [13], [3].

We shall use the Fourier transform defined by \(\hat{g}(\gamma) = \int g(t) e^{-2\pi i \gamma t} dt\), where the integral is over \(\mathbb{R}\). Depending on the context, \(|\cdot|\) will denote either the Lebesgue measure of a set, or the modulus of a function or complex number.

The Balian-Low Theorem is a classical manifestation of the uncertainty principle for Gabor systems.

Theorem 1.1 (Balian-Low). Let \(g \in L^2(\mathbb{R})\). If
\[
\int |t|^2 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^2 |\hat{g}(\gamma)|^2 d\gamma < \infty,
\]
then \(\mathcal{G}(g, 1, 1)\) is not an orthonormal basis for \(L^2(\mathbb{R})\).

The Balian-Low Theorem has a long history and some of the original references include [1], [19], [2]. The theorem still holds if “orthonormal basis” is replaced by “Riesz basis”. For this and other generalizations of the Balian-Low Theorem, we refer the reader to the survey articles [6], [9], as well as [4], [5], [7], [8], [10], [14], [17]. The issue of sharpness in the Balian-Low Theorem was investigated in [5], where the following was shown.

Received by the editors March 3, 2005.

Key words and phrases. Gabor analysis, Balian-Low Theorem, time-frequency analysis.

2000 Mathematics Subject Classification. Primary 42C15, 42C25; Secondary 46C15.

The first author was supported in part by NSF DMS Grant 0139759, ONR Grant N000140210398.

The second author was supported by European Commision Grant MEIF-CT-2003-500685.

The third author was supported in part by NSF DMS Grant 0219233.
Theorem 1.2. If $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$, and $d > 2$, then there exists a function $g \in L^2(\mathbb{R})$ such that $G(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and

$$\int \frac{1 + |t|^p}{\log^d(2 + |t|)} |g(t)|^2 dt < \infty \quad \text{and} \quad \int \frac{1 + |\gamma|^q}{\log^d(2 + |\gamma|)} |\hat{g}(\gamma)|^2 d\gamma < \infty.$$ 

When $(p, q) = (2, 2)$, this says that the Balian-Low Theorem no longer holds if the weights $(t^2, \gamma^2)$ are weakened by appropriate logarithmic terms. In view of Theorem 1.2, it is also natural to ask if there exist versions of the Balian-Low Theorem for the general $(p, q)$ case corresponding to the weights $(t^p, \gamma^q)$. The best that is known is the following.

Theorem 1.3. Suppose $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < \infty$ and let $\epsilon > 0$. If

$$\int |t|^{(p+\epsilon)} |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^{(q+\epsilon)} |\hat{g}(\gamma)|^2 d\gamma < \infty$$

then $G(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The above theorem follows by combining Theorem 4.4 of [11] and Theorem 1 in [15]. The $\epsilon > 0$ can, of course, be removed in the case $(p, q) = (2, 2)$, by the Balian-Low Theorem.

This note shows the existence of a Balian-Low Theorem in the case $(p, q) = (1, \infty)$, and thus extends Theorems 1.1 and 1.3. To define what this means, let $g \in L^2(\mathbb{R})$ and $1 \leq p < \infty$ and set

$$\Delta_p(g) = \int |t|^p |g(t)|^2 dt \quad \text{and} \quad \Delta_\infty(g) = \sup_{N > 0} \int |t|^N |g(t)|^2 dt.$$ 

With this notation, the classical Balian-Low Theorem says that if $\Delta_2(g) < \infty$ and $\Delta_2(\hat{g}) < \infty$ then $G(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

Our main result of this note is the following theorem.

Theorem 1.4. Let $g \in L^2(\mathbb{R})$ and suppose that $G(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$. Then

$$\Delta_1(g) = \infty \quad \text{or} \quad \Delta_\infty(\hat{g}) = \infty.$$ 

This yields the following $(1, \infty)$ version of the classical Balian-Low Theorem.

Corollary 1.5. Let $g \in L^2(\mathbb{R})$ and suppose

$$\Delta_1(g) < \infty \quad \text{and} \quad \Delta_\infty(\hat{g}) < \infty.$$ 

Then $G(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

2. Background

A collection $\{e_n\}_{n \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall f \in L^2(\mathbb{R}), \quad A\|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 \leq B\|f\|_{L^2(\mathbb{R})}^2.$$ 

$A$ and $B$ are the frame constants associated to the frame. If $\{e_n\}_{n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, but is no longer a frame if any element is removed, then we say that $\{e_n\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$. Riesz bases are also known as exact frames or bounded
unconditional bases, e.g., see [3]. The Zak transform is an important tool for studying
Riesz bases given by Gabor systems.

Given \( g \in L^2(\mathbb{R}) \), the Zak transform is formally defined by
\[
\forall (t, \gamma) \in Q = \{0, 1\}^2, \quad Zg(t, \gamma) = \sum_{n \in \mathbb{Z}} g(t - n)e^{2\pi im\gamma}.
\]
This defines a unitary operator from \( L^2(\mathbb{R}) \) to \( L^2(Q) \). Further background on the Zak
transform, as well as the next theorem, can be found in [3], [16].

**Theorem 2.1.** Let \( g \in L^2(\mathbb{R}) \). \( G(g, 1, 1) \) is a Riesz basis for \( L^2(\mathbb{R}) \) with frame
constants \( 0 < A \leq B < \infty \) if and only if \( A \leq |Zg(t, \gamma)|^2 \leq B \) for a.e. \((t, \gamma) \in Q\).

A function \( g \in L^2(\mathbb{R}) \) is said to be in the homogeneous Sobolev space of order \( s > 0 \),
denoted \( \dot{H}^s(\mathbb{R}) \), if \( ||g||^2_{\dot{H}^s(\mathbb{R})} \equiv \int |\gamma|^s |\hat{g}(\gamma)|^2 d\gamma < \infty \). Since the condition \( \Delta_1(g) < \infty \)
in Theorem 1.5 is equivalent to \( \hat{g} \in \dot{H}^{1/2}(\mathbb{R}) \), we shall need some results on \( \dot{H}^{1/2}(\mathbb{R}) \).
The following alternate characterization of \( \dot{H}^{1/2}(\mathbb{R}) \) will be useful, e.g., [18].

**Theorem 2.2.** If \( f \in \dot{H}^{1/2}(\mathbb{R}) \) then
\[
||f||^2_{\dot{H}^{1/2}(\mathbb{R})} = \frac{1}{4\pi^2} \iint |f(x) - f(y)|^2 \frac{dxdy}{|x - y|^2}.
\]

We let \( 1_S(t) \) denote the characteristic function of a set \( S \subseteq \mathbb{R} \), and let \( S^c \) denote
the complement of \( S \subseteq \mathbb{R} \). Given \( f \in L^2(\mathbb{R}) \), the symmetric-decreasing rearrangement
\( f^* \) of \( f \) is defined by
\[
f^*(t) = \int_0^\infty 1_{S_x}(t) dx,
\]
where \( S_x = (-s_x/2, s_x/2) \) and \( s_x = |\{t : |f(t)| > x\}|. An important property of a
symmetric-decreasing rearrangement is that it decreases the \( \dot{H}^{1/2}(\mathbb{R}) \) norm of functions, [18].

**Theorem 2.3.** If \( f \in \dot{H}^{1/2}(\mathbb{R}) \) then
\[
||f||_{\dot{H}^{1/2}(\mathbb{R})} \geq ||f^*||_{\dot{H}^{1/2}(\mathbb{R})}.
\]

This has the following useful corollary, [18].

**Corollary 2.4.** If \( S \subset \mathbb{R} \) is a measurable set of positive and finite measure then
\( ||1_S||_{\dot{H}^{1/2}(\mathbb{R})} = \infty \).

3. Proof of the \((1, \infty)\) Balian-Low Theorem

The proof of Theorem 1.4 requires the following preliminary technical theorem.

**Theorem 3.1.** Let \( f \) be a non-negative measurable function supported in the interval
\([-1, 1]\) and suppose that there exist constants \( 0 < A \leq B < \infty \) such that
\[
(3.1) \quad A \leq |f(x) \pm f(x - 1)| \leq B, \quad \text{a.e. } x \in [-1, 1].
\]

Then \( ||f||_{\dot{H}^{1/2}(\mathbb{R})} = \infty \).
**Proof.** We begin by defining the measurable sets

\[ S = \{ x \in [0, 1] : f(x - 1) \leq f(x) \}, \]

\[ T = S^c \cap [0, 1] = \{ y \in [0, 1] : f(y) < f(y - 1) \}, \]

and note that (3.1) implies

\[ A \leq f(x) - f(x - 1), \quad \text{a.e. } x \in S, \]  

\[ A \leq f(y) - f(y - 1), \quad \text{a.e. } y \in T. \]  

We break up the proof into two cases depending on whether or not \( S \) is a proper non-trivial subset of \([0, 1]\).

**Case I.** We shall first consider the case where

\[ 0 < |S| < 1, \]

and hence that \( 0 < |T| < 1. \)

Define the following capacity type integral over the product set \( S \times T. \)

\[ I = \int_S \int_T \frac{1}{|x - y|^2} dy dx. \]

Conditions (3.2) and (3.3) allow one to bound \( I \) in terms of the \( \dot{H}^{1/2}(\mathbb{R}) \) norm of \( f \) as follows.

\[ I \leq \frac{1}{4A^2} \int_S \int_T \frac{|f(x) - f(x - 1) + f(y - 1) - f(y)|^2}{|x - y|^2} dy dx \]
\[ \leq \frac{1}{2A^2} \left( \int_S \int_T \frac{|f(x) - f(y)|^2}{|x - y|^2} dy dx + \int_S \int_T \frac{|f(y - 1) - f(x - 1)|^2}{|x - y|^2} dy dx \right) \]
\[ \leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x - y|^2} dy dx \]
\[ = \frac{4\pi^2}{A^2} \|f\|_{\dot{H}^{1/2}}^2. \]

It therefore suffices to show that \( I = \infty. \)

Since by the Lebesgue differentiation theorem almost every point of \( T \) is a point of density, it follows from (3.4) that we may chose \( a \in (0, 1) \) such that \( a \) is point of density of \( T \) which satisfies either

\[ 0 < |S \cap [0, a]| < a \]  

or

\[ 0 < |S \cap [a, 1]| < 1 - a. \]

Without loss of generality, we assume (3.6). If (3.7) holds then our arguments proceed analogously; for example in the first subcase below we would symmetrize about \( x = 1 \) instead of \( x = 0. \)

To estimate \( I \), we shall proceed separately depending on whether \( \int_0^a \frac{1_{S(x)}}{|x-a|} dx \) is finite or infinite.
Subcase i. Suppose \( \int_0^a \frac{1_{S(x)}}{|x-a|} \, dx < \infty \). It will be convenient to work with the following set

\[ \tilde{S} = (S \cup (-S)) \cap [-a, a]. \]

By (3.6) we have \(|\tilde{S}| = 2|S \cap [0, a]| \neq 0\). It follows from Corollary 2.4 and the definition of \( \tilde{S} \) that

\[
\tilde{I} \equiv \int_{-a}^{a} \int_{-\infty}^{\infty} \frac{1_{\tilde{S}(x)}1_{(\tilde{S})^{c}(y)}}{|x-y|^2} \, dy \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|1_{\tilde{S}}(x) - 1_{\tilde{S}}(y)|^2}{|x-y|^2} \, dx \, dy
\]

\[
= 4\pi^2 ||1_{\tilde{S}}||^2_{H^s(\mathbb{R})} = \infty.
\]

The symmetric definition of \( \tilde{S} \) implies that

\[
\tilde{I} = 2 \int_{0}^{a} \int_{-\infty}^{\infty} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx = 2(I_1 + I_2 + I_3),
\]

where

\[
I_1 \equiv \int_{0}^{a} \int_{-\infty}^{0} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx \leq \int_{0}^{a} \int_{-\infty}^{0} \frac{1}{|y|^2} \, dy \, dx < \infty,
\]

\[
I_2 \equiv \int_{0}^{a} \int_{a}^{\infty} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx \leq \int_{0}^{a} \frac{1_{\tilde{S}}(x)}{|x-a|} \, dx < \infty,
\]

\[
I_3 \equiv \int_{0}^{a} \int_{-a}^{0} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx.
\]

A simple calculation for \( I_3 \) shows that

\[
I_3 = \int_{0}^{a} \int_{-a}^{0} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x-y|^2} \, dy \, dx + \int_{0}^{a} \int_{0}^{a} \frac{1_{\tilde{S}}(x)1_{(\tilde{S})^{c}}(y)}{|x+y|^2} \, dy \, dx \leq 2I,
\]

where the inequality for the second term in the middle of (3.9) follows from the fact that \(|x-y| \leq |x+y|\) in the square \([0, a] \times [0, a]\).

It follows from (3.8) and (3.9) that

\[ \infty = \tilde{I} \leq 2I_1 + 2I_2 + 4I. \]

Since \( I_1 \) and \( I_2 \) are finite, we have \( I = \infty \), as desired.

Subcase ii. Suppose \( \int_{0}^{a} \frac{1_{S(x)}}{|x-a|} \, dx = \infty \). Define

\[
I_D = \int_{0}^{a} \int_{0}^{a} \frac{1_{S}(x)1_{T}(y)}{|x-y|^2} \, dy \, dx \leq I,
\]

where \( D = \{(x, y) \in \mathbb{R}^2 : x < y\} \). To compute a lower bound for \( I_D \) first note that since \( a \) is a point of density of \( T \), there exists a sufficiently large constant \( 0 < C < \infty \) such that

\[ |a-x| \leq C \, |T \cap [x, a]|, \quad a.e. \, x \in [0, 1]. \]
Therefore for $a.e. \ x \in [0, a)$
\[
\frac{1}{|a-x|} \leq \frac{C|T \cap [x,a]|}{|a-x|^2} = C \frac{T \cap [x,a]}{|a-x|^2} \cdot \min_{y \in [x,a]} \left\{ \frac{1}{|x-y|^2} \right\} \\
\leq C \int_x^a \frac{1}{|x-y|^2} dy.
\]
This implies that
\[
\infty = \int_0^a \frac{1}{|a-x|} dx \leq C \int_0^a \int_x^a \frac{1}{|x-y|^2} dy dx = C I_D,
\]
and it follows that $I_D = \infty$, and hence $I = \infty$, as desired.

Case II. We conclude by addressing the cases where $|S| = 0$ or $|S| = 1$. Without loss of generality we only consider $|S| = 1$, and hence assume that $S = [0,1]$ up to a set of measure zero. It follows from (3.2) and the positivity of $f$ that
\[
A \leq f(x), \ a.e. \ x \in [0,1].
\]
This, together with the fact that $f$ is supported in $[-1,1]$, implies that
\[
\infty = \int_1^{\infty} \int_0^1 \frac{1}{|x-y|^2} dxdy \\
\leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \\
= \frac{4\pi^2}{A^2} \|f\|_{H^{1/2}(\mathbb{R})}^2,
\]
as desired. This completes the proof.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We proceed by contradiction. Assume that $g \in L^2(\mathbb{R})$, that $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ with frame constants $0 < A \leq B < \infty$, and that $\Delta_1(g) < \infty$ and $\Delta_\infty(\mathcal{G}) < C < \infty$, for some constant $C$.

By Theorem 2.1,
\[
\sqrt{A} \leq |Z g(x,w)| \leq \sqrt{B} \ a.e. \text{ on } [0,1)^2.
\]
Since $Z \mathcal{G}(x,w) = e^{2\pi i x w} Z g(-w,x)$ we have
\[
\sqrt{A} \leq |Z \mathcal{G}(x,w)| \leq \sqrt{B} \ a.e. \text{ on } [0,1)^2.
\]
Next, the assumption $\int |\gamma|^N |\mathcal{G}(\gamma)|^2 d\gamma < C$ for all $N > 0$ implies that
\[
\text{supp} \ \hat{\mathcal{G}} \subseteq [-1,1].
\]
Thus, for $(x,w) \in [0,1)^2$, we have
\[
Z \mathcal{G}(x,w) = \sum_{n \in \mathbb{Z}} \hat{\mathcal{G}}(x-n)e^{2\pi i n w} = \hat{\mathcal{G}}(x) + \hat{\mathcal{G}}(x-1)e^{2\pi i w},
\]
so that we have
\[
(3.10) \quad \sqrt{A} \leq |\hat{\mathcal{G}}(x) + \hat{\mathcal{G}}(x-1)e^{2\pi i w}| \leq \sqrt{B} \ a.e. \ (x,w) \in [0,1)^2.
\]
In particular, it follows that
\[ \sqrt{A} \leq ||\hat{g}(x)|| + ||\hat{g}(x-1)|| \leq \sqrt{B}, \quad \text{for a.e. } x \in [0,1]. \]
It now follows from Theorem 3.1 that \( |\hat{g}| \notin \dot{H}^{1/2}(\mathbb{R}) \), which implies that \( \hat{g} \notin \dot{H}^{1/2}(\mathbb{R}) \).
In other words, \( \Delta_1(g) = ||\hat{g}||^2_{\dot{H}^{1/2}(\mathbb{R})} = \infty \). This contradiction completes the proof.

Since orthonormal bases are Riesz bases with frame constants \( A = B = 1 \), Corollary 1.5 follows from Theorem 1.4.

4. Further Comments

1. Theorem 1.5 is sharp in the sense investigated in Theorem 1.2, see [5]. In fact, Theorem 1.5 no longer holds if one weakens the \( \Delta_1 \) decay hypotheses by a certain logarithmic amount. For example, if \( d>1 \) and \( \hat{g}(\gamma) = 1 \) then \( \mathcal{G}(g,1,1) \) is an orthonormal basis for \( L^2(\mathbb{R}) \), and
\[
\int \frac{|t|}{\log^2(|t|+2)}|g(t)|^2dt < \infty \quad \text{and} \quad \sup_{N>0} \int |\gamma|^N|\hat{g}(\gamma)|^2d\gamma < \infty.
\]

2. There are two noteworthy cases in which the proof of Theorem 1.4 can be significantly simplified. If one assumes that \( \mathcal{G}(g,1,1) \) is an orthonormal basis for \( L^2(\mathbb{R}) \) then the frame constants satisfy \( A = B = 1 \) and it follows from (3.10) that \( |\hat{g}(x)| = 1_R(x) \) for some set \( R \subset \mathbb{R} \) of positive and finite measure. Corollary 2.4 completes the proof in this case. Likewise, if \( \mathcal{G}(g,1,1) \) is a Riesz basis for \( L^2(\mathbb{R}) \) whose frame bounds \( A \) and \( B \) are sufficiently close to one another, e.g., \( \sqrt{B} < 3\sqrt{A} \), then a direct argument involving Theorem 2.2 and Theorem 2.3 completes the proof. The main difficulty in Theorem 1.4 and Theorem 3.1 arises when the frame constants \( A \) and \( B \) are far apart.

3. We conclude by noting that if one strengthens the hypotheses in Theorem 1.4 to \( \Delta_\infty(\hat{g}) < \infty \) and \( \Delta_{1+\epsilon}(g) < \infty \), for some \( \epsilon > 0 \), then the result is a simple consequence of the Amalgam Balian-Low Theorem. The Amalgam Balian-Low Theorem, e.g., [6], states that if \( \mathcal{G}(g,1,1) \) is a Riesz basis for \( L^2(\mathbb{R}) \) then
\[ g \notin W(C_0, l^1) \quad \text{and} \quad \hat{g} \notin W(C_0, l^1), \]
where
\[ W(C_0, l^1) = \{ f : f \text{ is continuous and } \sum_{k \in \mathbb{Z}} ||f1_{[k,k+1]}||_{L^\infty(\mathbb{R})} < \infty \}. \]
The assumptions \( \Delta_{1+\epsilon}(g) < \infty \) and \( \Delta_\infty(\hat{g}) < \infty \) imply that \( \hat{g} \) is continuous and supported in \([-1,1]\), which, in turn, implies that \( \hat{g} \in W(C_0, l^1) \).

References


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742

E-mail address: jjb@math.umd.edu

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND

Current address: Department of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, Austria

E-mail address: czaja@math.uni.wroc.pl

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240

E-mail address: apowell@math.vanderbilt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, 9500 GILMAN DRIVE, LA JOLLA, CA 92039

E-mail address: jsterbenz@math.ucsd.edu