SHARP $L^2 \rightarrow L^q$ BOUNDS ON SPECTRAL PROJECTORS FOR LOW REGULARITY METRICS

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Abstract. We establish $L^2 \rightarrow L^q$ mapping bounds for unit-width spectral projectors associated to elliptic operators with $C^s$ coefficients, in the case $1 \leq s \leq 2$. Examples of Smith-Sogge [6] show that these bounds are best possible for $q$ less than the critical index. We also show that $L^\infty$ bounds hold with the same exponent as in the case of smooth coefficients.

1. Introduction

The goal of this paper is to study the $L^p$ norms of eigenfunctions, and approximate eigenfunctions, of elliptic second order differential operators with low regularity coefficients, on compact manifolds without boundary. We consider the eigenvalues $-\lambda^2$ and eigenfunctions $\phi$ for an equation

$$(1) \quad d^*(a d\phi) + \lambda^2 \rho \phi = 0.$$ 

Here we assume $\rho > 0$ is a real, positive measurable function, and $a_x : T_x^*(M) \rightarrow T_x(M)$ is the transformation associated to a real symmetric form on $T_x^*(M)$, also strictly positive and measurable in $x$. The manifold $M$ and volume form $dx$ are assumed smooth, and $d^*$ is the transpose of the differential $d$ with respect to $dx$. This setting includes the most general elliptic second order operator on $M$, assumed self-adjoint with respect to some measurable volume form $\rho \, dx$, and assumed to annihilate constants, and hence of the form $\rho^{-1}d^*ad$. For limited regularity $a$ and $\rho$ we pose the problem as above to avoid domain considerations.

If we consider the real quadratic forms

$$Q_0(f, g) = \int_M f g \rho \, dx, \quad Q_1(f, g) = Q_0(f, g) + \int_M a(df, dg) \, dx,$$

then

$$Q_0(f, f) = \|f\|_{L^2(M, \rho\, dx)}^2, \quad Q_1(f, f) \approx \|f\|_{H^1(M)}^2,$$

hence $Q_0$ is compact relative to $Q_1$ by Rellich’s lemma. By the standard argument of simultaneously diagonalizing $Q_0$ and $Q_1$, there exists a complete orthonormal basis $\phi_j$ for $L^2(M, \rho \, dx)$ consisting of eigenfunctions for (1), with $\lambda_j \rightarrow \infty$.

The object of this paper is to establish bounds on the $L^2 \rightarrow L^q$ operator norm of the unit-width spectral projectors for (1). Let $\Pi_{\lambda}$ be the projection of $L^2(M, \rho \, dx)$ onto the subspace spanned by the eigenfunctions of (1) for which $\lambda_j \in [\lambda, \lambda + 1]$. In
the case that the coefficients $\rho$ and $a$ are $C^\infty$, the following estimates hold, and are best possible in terms of the exponent of $\lambda$,

\begin{align}
\|\Pi_\lambda f\|_{L^q(M)} &\leq C \lambda^{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})} \|f\|_{L^2(M)}, \quad 2 \leq q \leq q_n, \\
\|\Pi_\lambda f\|_{L^q(M)} &\leq C \lambda^{n(\frac{1}{2} - \frac{1}{q})-\frac{1}{2}} \|f\|_{L^2(M)}, \quad q_n \leq q \leq \infty,
\end{align}

where

$$q_n = \frac{2(n+1)}{n-1}.$$

For $C^\infty$ metrics the estimates at $q = q_n$ are due to Sogge [8]. The estimate for $q = \infty$ is related to the spectral counting remainder estimates of Avakumović-Levitan-Hörmander; it can also be obtained from Sogge’s estimate by Sobolev embedding. The case $q = 2$ is of course trivial, and all other values of $q$ follow from these endpoints by interpolation.

In [5], both estimates (2) and (3) were established on the full range of $q$ for the case that both $a$ and $\rho$ are of class $C^1$. On the other hand, Smith-Sogge [6] and Smith-Tataru [7] constructed examples, for each $0 < s < 2$, of functions $a$ and $\rho$ with coefficients of class $C^s$ (Lipschitz in case $s = 1$) for which there exist eigenfunctions $\phi_\lambda$ such that for all $q \geq 2$

$$\|\phi_\lambda\|_{L^q(M)} \geq C \lambda^{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})(1+\sigma)} \|\phi_\lambda\|_{L^2(M)},$$

where $C > 0$ is independent of $\lambda$, and where

$$\sigma = \frac{2-s}{2+s}.$$

For $2 < q < \frac{2(n+2s-1)}{n-1}$, this shows that the spectral projection estimates for $C^s$ metrics with $s < 2$ can be strictly worse than in the $C^2$ case.

In this paper, we consider the case of coefficients $a$ and $\rho$ of class $C^s$ for $1 \leq s < 2$ (Lipschitz in case $s = 1$.) We start by establishing the following bound, which by the examples of [6] is best possible on the indicated range of $q$.

**Theorem 1.** Assume that the coefficients $a$ and $\rho$ are either of class $C^s$ for some $1 < s < 2$, or Lipschitz class if $s = 1$. Let $\Pi_\lambda$ denote the $L^2$-projection onto the subspace spanned by eigenfunctions of (1) with $\lambda_j \in [\lambda, \lambda+1]$. Then

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})(1+\sigma)} \|f\|_{L^2(M)}, \quad 2 \leq q \leq q_n.$$

Applying Sobolev embedding to the estimate at $q = q_n$ would not yield the correct bound for $q = \infty$. However, the proof of Theorem 1 also yields no-loss estimates on small sets. Precisely, we will establish the following local estimate, with constant uniform over the balls $B$.

**Theorem 2.** Let $B_R \subset M$ be a ball of radius $R = \lambda^{-\sigma}$. Then under the same conditions as Theorem 1

$$\|\Pi_\lambda f\|_{L^q(B_R)} \leq C \lambda^{n(\frac{1}{2} - \frac{1}{q})-\frac{1}{2}} \|f\|_{L^2(M)}, \quad q_n \leq q \leq \infty.$$
Interpolating with the trivial $L^2$ estimate establishes the estimate (2) on such balls $B_R$. Since the constant $C$ in (4) is uniform for all balls $B_R$, we obtain the same global $L^2 \to L^\infty$ mapping properties in the case of Lipschitz coefficients as in the case of smooth coefficients,

$$\|\Pi_\lambda f\|_{L^\infty(M)} \leq C \lambda^{\frac{n}{q_n}} \|f\|_{L^2(M)}.$$  
A corollary of this result is the Hörmander multiplier theorem on compact manifolds for functions of elliptic operators with Lipschitz coefficients, as shown by results of Duong-Ouhabaz-Sikora [1], section 7.2. We note that, in related work, Ivrii [2] has obtained the sharp spectral counting remainder estimate for operators with coefficients of regularity slightly stronger than Lipschitz.

The proof of Theorem 2 that we will present requires that $q$ be not too large, but in all dimensions works for $q = q_n$. We therefore show here how heat kernel estimates permit us to deduce (4) for all $q \geq q_n$ from the case $q = q_n$. For this, let $H_\lambda$ denote the heat kernel at time $\lambda^{-2} \leq 1$ for the diffusion system associated to (1). By Theorem 6.3 of Saloff-Coste [4], the integral kernel $h_\lambda$ of $H_\lambda$ satisfies

$$|h_\lambda(x, y)| \leq C \lambda^n \exp(-c \lambda^2 d(x, y)^2).$$

By Young’s inequality, then for $q_n \leq q \leq \infty$

$$\|\Pi_\lambda f\|_{L^q(B_R)} \leq C \lambda^{n \left(\frac{1}{q_n} - \frac{1}{q}\right)} \|H_\lambda^{-1} \Pi_\lambda f\|_{L^q(B_{R^*})} + C_N \lambda^{-N} \|H_\lambda^{-1} \Pi_\lambda f\|_{L^2(M \setminus B^*_R)}$$

$$\leq C \lambda^{n \left(\frac{1}{q_n} - \frac{1}{q}\right) - \frac{1}{2}} \|f\|_{L^2(M)}$$

where we use (4) at $q = q_n$ with $B_R$ replaced by its double $B_{R^*}$, and the fact that $\|H_\lambda^{-1} \Pi_\lambda f\|_{L^2} \approx \|\Pi_\lambda f\|_{L^2}$ since $\exp(\lambda^2 / \lambda^2) \approx 1$ for $\lambda \in [\lambda, \lambda + 1]$.

If we interpolate the estimate of Theorem 1 at $q = q_n$ with the estimate (5), then we obtain the following.

**Corollary 3.** Under the same conditions as Theorem 1

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{n \left(\frac{1}{q_n} - \frac{1}{q}\right) - \frac{1}{2} + \frac{s}{q}} \|f\|_{L^2(M)}, \quad q_n \leq q \leq \infty.$$  

For $q_n < q < \infty$, however, the exponent is strictly larger than that predicted by the examples of [6]. It is not currently known what the sharp exponent is for this range.

The key idea in our proof is that a $C^s$ function is well approximated on sets of diameter $R = \lambda^{-1}$ by a $C^2$ function, up to an error which is suitably bounded when dealing with eigenfunctions localized to frequency $\lambda$. In effect, rescaling by $R$ reduces matters to a $C^2$ situation, where no-loss estimates hold. The loss of $\lambda^{-\frac{s}{q}}$ comes from adding up the bounds over $\approx R^{-1}$ disjoint sets.

This scaling parameter $R$ occurs in the examples of Smith-Sogge [6] and Smith-Tataru [7]. The idea of scaling by $R$ to prove $L^p$ estimates was first used by Tataru in [9], to establish Strichartz-type estimates for time-dependent wave equations with $C^s$ coefficients, yielding improved existence theorems for a class of quasilinear hyperbolic equations.

**Notation.** By a $C^s$ function on $\mathbb{R}^n$, for $1 < s \leq 2$ we understand a continuously differentiable function $f$ such that

$$\|f\|_{C^s} = \|f\|_{L^\infty(\mathbb{R}^n)} + \|df\|_{L^\infty(\mathbb{R}^n)} + \sup_{h \in \mathbb{R}^n} |h|^{1-s} \|df(\cdot + h) - df(\cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$
Thus, $C^s$ coincides with $C^{1,s-1}$ for $s \in (1,2]$. For $s = 1$, we use $C^1$ to mean Lipschitz. For $0 < s < 1$ we take $C^s$ to be the standard H"older class.

We use $d$ to denote the differential taking functions to covector fields, and $d^*$ its adjoint with respect to $dx$. When working on $\mathbb{R}^n$, $d = (\partial_1, \ldots, \partial_n)$, and $d^*$ is the standard divergence operator.

The notation $A \lesssim B$ means $A \leq C B$, where $C$ is a constant that depends only on the $C^s$ norm of $a$ and $\rho$, as well as on universally fixed quantities, such as the manifold $M$ and the non-degeneracy of $a$ and $\rho$. In particular, $C$ can be taken to depend continuously on $a$ and $\rho$ in the $C^s$ norm, so our estimates are uniform under small $C^s$ perturbations of $a$ and $\rho$.

2. Scaling Arguments

Our starting point is the following square-function estimate for solutions to the Cauchy problem. For $C^\infty$ coefficients this was established by Mockenhaupt-Seeger-Sogge [3]. The version we need for $C^{1,1}$ metrics is Theorem 1.3 of [5]. That theorem was stated under the condition $F = 0$ and for coefficients which are constant for large $x$, but these conditions are easily dropped by the Duhamel principle and a partition of unity argument.

**Theorem 4.** Suppose that $a$ and $\rho$ are defined globally on $\mathbb{R}^n$, and that
\[
\|a^{ij} - \delta^{ij}\|_{C^{1,1}(\mathbb{R}^n)} + \|\rho - 1\|_{C^{1,1}(\mathbb{R}^n)} \leq c_0,
\]
where $c_0$ is a small constant depending only on $n$. Let $u$ solve the Cauchy problem
\[
\rho(x) \partial_t^2 u(t,x) - d^*(a(x) du(t,x)) = F(t,x), \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x).
\]
Then
\[
\|u\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{H^{\frac{n}{4}+1}} + \|u_1\|_{H^{\frac{n}{4}-1}} + \|F\|_{L^1_t L^1_x}^{\frac{1}{2}}.
\]

We first deduce the following corollary which is more useful for our purposes.

**Corollary 5.** Suppose that $f$ satisfies an equation on $\mathbb{R}^n$ of the form
\[
d^*(a df) + \mu^2 \rho f = d^* g_1 + g_2.
\]
If $a$ and $\rho$ satisfy the condition of Theorem 4, then
\[
\|f\|_{L^\infty_n} \lesssim \mu^{\frac{n}{4}} \left( \|f\|_{L^2} + \mu^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1} \|g_2\|_{L^2} \right).
\]

**Proof.** Let $S_\xi = S_\xi(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \leq r$. Let $a_\mu = S_{c\mu} a$, for $c$ to be chosen suitably small. Then
\[
\|(a - a_\mu) df\|_{L^2} \lesssim c^{-2} \mu^{-1} \|df\|_{L^2}, \quad \mu^2 \|f - \rho f\|_{L^2} \lesssim c^{-2} \mu \|f\|_{L^2},
\]
and thus we may replace $a$ and $\rho$ by $a_\mu$ and $\rho_\mu$ at the expense of absorbing the above two terms into $g_1$ and $g_2$, which does not change the size of the right hand side of (7).

Next, let $f_{<\mu} = S_{c\mu} f$. Since
\[
\|S_{c\mu} a_\mu\|_{L^2} \lesssim (c\mu)^{-1},
\]
and similarly for $[S_{c\mu}, \rho_\mu]$, we can absorb the commutator terms into $g_1$ and $g_2$, and since all terms are localized to frequencies less than $\mu$ we can write
\[
d^*(a_\mu df_{<\mu}) + \mu^2 \rho_\mu f_{<\mu} = g_{<\mu},
\]
where 
\[ \|g_{<\mu}\|_{L^2} \lesssim \mu \|f\|_{L^2} + \|df\|_{L^2} + \mu \|g_1\|_{L^2} + \|g_2\|_{L^2} \]
Since \( \|d^*(a_{\mu} df_{<\mu})\|_{L^2} \lesssim (c\mu)^2 \|f_{<\mu}\|_{L^2} \), for \( c \) suitably small the \( L^2 \) norm of the left hand side of (8) is comparable to \( \mu^2 \|f_{<\mu}\|_{L^2} \), hence we have
\[ \|f_{<\mu}\|_{L^2} \lesssim \mu^{-1} (\|f\|_{L^2} + \mu^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1} \|g_2\|_{L^2}) \]
Sobolev embedding now implies (7) if \( f \) is replaced on the left hand side by \( f_{<\mu} \). In fact there is a gain of \( \mu^{-\frac{1}{2}} \), since \( \frac{1}{q_n} = n(\frac{1}{2} - \frac{1}{q_n}) - \frac{1}{2} \).

If we let \( f_{>\mu} = f - S_{c-1} f \), then similar arguments let us write
\[ (9) \]
\[ d^*(a_{\mu} df_{>\mu}) + \mu^2 \rho_{\mu} f_{>\mu} = d^* g_{>\mu} \]
where now \( g_{>\mu} \), like \( f_{>\mu} \), is frequency localized to frequencies larger than \( c^{-1} \), and
\[ \|g_{>\mu}\|_{L^2} \lesssim \|f\|_{L^2} + \mu^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1} \|g_2\|_{L^2} \]
Taking the inner product of both sides of (9) against \( f_{>\mu} \) yields
\[ \|df_{>\mu}\|_{L^2}^2 - 4\mu^2 \|f_{>\mu}\|_{L^2}^2 \lesssim \|g_{>\mu}\|_{L^2} \|df_{>\mu}\|_{L^2} \]
and by the frequency localization of \( f_{>\mu} \) we obtain
\[ \|f_{>\mu}\|_{L^2} \lesssim \|f\|_{L^2} + \mu^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1} \|g_2\|_{L^2} \]
Since \( n(\frac{1}{2} - \frac{1}{q_n}) = \frac{1}{q_n} + \frac{1}{2} \leq 1 \), Sobolev embedding yields (7) if \( f \) is replaced on the left hand side by \( f_{>\mu} \). As above, there is in fact a gain of \( \mu^{-\frac{1}{2}} \) for this term.

We now let \( f_\mu = S_{c-1} f - S_{q_n} f \), and as above write
\[ d^*(a_{\mu} df_{\mu}) + \mu^2 \rho_{\mu} f_{\mu} = g_{\mu} \]
where now \( f_{\mu} \) and \( g_{\mu} \) are localized to frequencies comparable to \( \mu \), and
\[ \|g_{\mu}\|_{L^2} \lesssim \mu \|f\|_{L^2} + \mu \|df\|_{L^2} + \mu \|g_1\|_{L^2} + \|g_2\|_{L^2} \]
Setting \( u(t,x) = \cos(\mu t) f_{\mu}(x) \), we apply (6) to deduce
\[ \|f_{\mu}\|_{L^2} \lesssim \mu^{\frac{1}{q_n}} \left( \|f_{\mu}\|_{L^2} + \mu^{-1} \|g_{\mu}\|_{L^2} \right) \]
which yields (7) for this term. \( \square \)

**Remark.** For future use, we note that in the proof of Corollary 5 the assumption that \( a \in C^{1,1} \) was used only at the last step, in order to deduce that (6) holds. The commutator and approximation bounds require only that \( a \) and \( \rho \) be Lipschitz. In particular, the bounds on \( f_{<\mu} \) and \( f_{>\mu} \) hold for Lipschitz \( a \) and \( \rho \).

**Corollary 6.** Let \( Q \) be a unit cube and \( Q^* \) its double. Suppose that \( a \) and \( \rho \) are bounded and measurable, and that there exist \( C^{1,1} \) functions \( \tilde{a} \) and \( \tilde{\rho} \) satisfying the conditions of Theorem 4 such that
\[ \|a - \tilde{a}\|_{L^\infty(Q^*)} + \|\rho - \tilde{\rho}\|_{L^\infty(Q^*)} \leq \mu^{-1} \]
Suppose that on \( Q^* \) we have
\[ d^*(a df) + \mu^2 \rho f = d^*_1 g_1 + g_2 \]
Then
\[ \|f\|_{L^\infty(Q)} \lesssim \mu^{\frac{1}{q_n}} \left( \|f\|_{L^2(Q^*)} + \mu^{-1} \|df\|_{L^2(Q^*)} + \|g_1\|_{L^2(Q^*)} + \mu^{-1} \|g_2\|_{L^2(Q^*)} \right) \]
The constant in the inequality is uniform for \( \mu \geq 1 \).

**Proof.** Let \( \phi \) be a smooth function, equal to 1 on \( Q \) and supported in \( Q^* \). Then
\[
d^\ast(a \, d(\phi f)) + \mu^2 \rho \, (\phi f) = d^\ast \left[ (a \, d\phi) f + \phi g_1 \right] + \left[ (a \, d\phi) \cdot df - (d\phi) \cdot g_1 + \phi g_2 \right]
\]
where for \( \mu \geq 1 \)
\[
\|\tilde{g}_1\|_{L^2} + \mu^{-1}\|\tilde{g}_2\|_{L^2} \lesssim \|f\|_{L^2(Q^*)} + \mu^{-1}\|df\|_{L^2(Q^*)} + \|g_1\|_{L^2(Q^*)} + \mu^{-1}\|g_2\|_{L^2(Q^*)}
\]
One may similarly absorb \((a - \tilde{a})d(\phi f)\) into \(\tilde{g}_1\), and \(\mu^2(\rho - \tilde{\rho})(\phi f)\) into \(\tilde{g}_2\). The result now follows from (7).

**Corollary 7.** Suppose that \( a \) and \( \rho \) are of class \( C^s \), with \( 0 \leq s \leq 2 \), and that
\[
\|a^{ij} - \delta^{ij}\|_{C^s(\mathbb{R}^n)} + \|\rho - 1\|_{C^s(\mathbb{R}^n)} \leq c_0,
\]
where \( c_0 \) is a small constant depending only on \( n \).

Suppose that \( R = \lambda^{-s} \), where \( \sigma = \frac{2-n}{2+s} \) and \( \lambda \geq 1 \). Assume \( Q_R \) is a cube of sidelength \( R \), \( Q_R^* \) is its double, and on \( Q_R^* \) the following equation holds
\[
d^\ast(a \, df) + \lambda^2 \rho \, f = d^\ast g_1 + g_2
\]
Then
\[
\|f\|_{L^\infty(Q_R)} \lesssim R^{-\frac{1}{2}} \lambda^{-\frac{1}{4n}} \left( \|f\|_{L^2(Q_R^*)} + \lambda^{-1}\|df\|_{L^2(Q_R^*)} + R\|g_1\|_{L^2(Q_R^*)} + R\lambda^{-1}\|g_2\|_{L^2(Q_R^*)} \right).
\]

**Proof.** We use the notation \( f_R(x) = f(Rx) \). Then, for \( \mu = R\lambda = \lambda^{1-s} \),
\[
d^\ast(a_R \, df_R) + \mu^2 \rho_R \, f_R = R \, d^\ast g_{1,R} + R^2 g_{2,R}
\]
holds on \( Q^* \), with \( Q \) a unit cube. If \( \tilde{a} = S_{\mu^{1/2}} a_R \), then
\[
\|\tilde{a} - a_R\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}} \lambda R \|a - I\|_{C^s} = c_0 \mu^{-1}
\]
By the frequency localization, \( \tilde{a} \) satisfies the conditions of Theorem 4. We may thus apply Corollary 6 to yield
\[
\|f_R\|_{L^\infty(Q)} \lesssim (R\lambda)^\frac{1}{4n} \left( \|f_R\|_{L^2(Q^*)} + \lambda^{-1}\|df\|_{L^2(Q^*)} + R\|g_{1,R}\|_{L^2(Q^*)} + R\lambda^{-1}\|g_{2,R}\|_{L^2(Q^*)} \right)
\]
Recalling that \( \frac{1}{4n} = n(\frac{1}{2} - \frac{1}{4n}) - \frac{1}{2} \), this yields the corollary after rescaling.

**3. Proof of Theorem 1**

The proof of Corollary 7 works for all \( s \in [0,2] \), but the energy estimates of this section require that \( a \) and \( \rho \) be Lipschitz, hence we assume \( s \geq 1 \) for the remainder.

The projection \( \Pi_M f \) satisfies
\[
\|d^\ast(a \, d(\Pi_M f)) + \lambda^2 \rho \Pi_M f\|_{L^2(M, \rho \, dx)} \leq (2\lambda + 1) \|\Pi_M f\|_{L^2(M, \rho \, dx)}
\]
\[
\|d\Pi_M f\|_{L^2(M, \rho \, dx)} \lesssim (\lambda + 1) \|\Pi_M f\|_{L^2(M, \rho \, dx)}
\]
hence Theorem 1 follows from showing that, if the following holds on \( M \)
\[
d^\ast(a \, df) + \lambda^2 \rho \, f = g \tag{10}
\]
then uniformly for $\lambda \geq 1$

\[ (11) \quad \| f \|_{L^q(M)} \lesssim \lambda^{1 \over q} \left( \| f \|_{L^2(M)} + \lambda^{-1} \| df \|_{L^2(M)} + \lambda^{-1} \| g \|_{L^2(M)} \right) \]

Assume that (10) holds, and let $\phi$ be a $C^2$ bump function on $M$. Then

\[ d'(a \, d(\phi f)) + \lambda^2 \rho \, \phi f = f \, d'(a \, d\phi) + \langle a \, d\phi, df \rangle + \phi g \]

Absorbing the terms on the right into $g$ leaves the right hand side of (11) unchanged, hence by a partition of unity argument we may assume that $f$ is supported in a suitably small coordinate neighborhood on $M$.

We choose coordinate patches so that, in local coordinates, the conditions of Corollary 7 are satisfied after extending $a$ and $\rho$ to all of $\mathbb{R}^n$. Thus, we have an equation of the form (10) on $\mathbb{R}^n$, with $f$ and $g$ supported in a unit cube.

We next decompose $f = f_{< \lambda} + f_{> \lambda} + f_\delta$ as in the proof of Corollary 5. As remarked following that proof, the bounds on $f_{< \lambda}$ and $f_{> \lambda}$ hold for $a$ and $\rho$ Lipschitz, hence we are reduced to considering $f_\delta$, for which we have an equation

\[ d'(a \lambda \, d f_\delta) + \lambda^2 \rho_\delta \, f_\delta = g_\delta \]

where $a_\lambda$ and $\rho_\lambda$ are localized to frequencies smaller than $c^2 \lambda$, and both $f_\lambda$ and $g_\lambda$ are localized to frequencies of size comparable to $\lambda$.

We then decompose $f_\delta = \sum_{j=1}^N \Gamma_j f_\lambda$, where each $\Gamma_j = \Gamma_j(D)$ is an order 0 multiplier, with symbol $\Gamma_j(\xi)$ supported where $|\xi| \approx \lambda$ and in a cone of suitably small angle. It then suffices to bound each $\| \Gamma_j f_\delta \|_{L^q(Q)}$ by the right hand side of (11). Without loss of generality we consider a term with $\Gamma(\xi)$ localized to a small cone about the $\xi_1$ axis.

We write

\[ d'(a \lambda \, d f_\delta) + \lambda^2 \rho_\delta \, f_\delta = \Gamma g_\delta + d'[a_\lambda, \Gamma] \, df_\delta + \lambda^2[\rho_\delta, \Gamma] \, f_\delta \]

Simple commutator estimates show that the right hand side has $L^2$ norm bounded by $\lambda \| f \|_{L^2} + \| g \|_{L^2}$, hence we are reduced to establishing

\[ (12) \quad \| f \|_{L^q(Q)} \lesssim \lambda^{1 \over q} \left( \| f \|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \| df \|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \| g \|_{L^2(\mathbb{R}^n)} \right) \]

for $f$ satisfying the equation

\[ d'(a_\lambda \, df) + \lambda^2 \rho_\lambda \, f = g \]

where $\tilde{f}(\xi)$ and $\tilde{g}(\xi)$ are localized to $|\xi| \approx \lambda$ and $\xi$ in a small cone about the $\xi_1$ axis.

By Corollary 7, for any cube $Q_R$ of sidelength $R = \lambda^{-\sigma}$, we have

\[ (13) \quad \| f \|_{L^q(Q_R)} \lesssim \lambda^{1 \over q} \left( R^{-1 \over 2} \| f \|_{L^2(Q_R)} + R^{-1 \over 2} \lambda^{-1} \| df \|_{L^2(Q_R)} + R^{1 \over 2} \lambda^{-1} \| g \|_{L^2(Q_R)} \right) \]

Let $S_R$ denote a slab of the form $\{ x \in \mathbb{R}^n : |x| - c \leq R \}$. By summing over cubes $Q_R$ contained in $S_R$, and noting $R \leq 1$, we obtain

\[ (14) \quad \| f \|_{L^q(S_R)} \lesssim \lambda^{1 \over q} \left( R^{-1 \over 2} \| f \|_{L^2(S_R)} + R^{-1 \over 2} \lambda^{-1} \| df \|_{L^2(S_R)} + \lambda^{-1} \| g \|_{L^2(S_R)} \right) \]

We will show that

\[ (15) \quad R^{-1 \over 2} \left( \| f \|_{L^2(S_R)} + \lambda^{-1} \| df \|_{L^2(S_R)} \right) \lesssim \| f \|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \| df \|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \| g \|_{L^2(\mathbb{R}^n)} \]

Given this, inequality (12) follows from (14) by adding over the $R^{-1} = \lambda^\sigma$ disjoint slabs that intersect $Q$. Also, the bound (13) implies the conclusion of Theorem 2 for $q = q_n$ (hence for all $q$ by the heat kernel arguments following that theorem.)
We establish (15) by energy inequality arguments. Let $V$ denote the vector field
\[ V = 2(\partial_1 f) a_\lambda df + (\lambda^2 \rho_\lambda f^2 - \langle a_\lambda df, df \rangle) e_1 \]
Then
\[ d^*V = 2(\partial_1 f) g + \lambda^2 (\partial_1 \rho_\lambda) f^2 - \langle (\partial_1 a_\lambda) df, df \rangle \]
Applying the divergence theorem on the set $x_1 \leq r$ yields
\[ \int_{x_1=r} V_1 dx' \lesssim \lambda^2 \| f \|_{L^2(\mathbb{R}^n)}^2 + \| df \|_{L^2(\mathbb{R}^n)}^2 + \| g \|_{L^2(\mathbb{R}^n)}^2 \]
Since $a_\lambda$ and $\rho$ are pointwise close to the flat metric, we have pointwise that
\[ V_1 \geq \frac{3}{4} |\partial_1 f|^2 + \frac{3}{4} \lambda^2 |f|^2 - |\partial_\xi f|^2 \]
The frequency localization of $\hat{f}$ to $|\xi| \leq c\lambda$ yields
\[ \int_{x_1=r} V_1 dx' \geq \frac{1}{2} \int_{x_1=r} |df|^2 + \lambda^2 |f|^2 dx' \]
Integrating this over $r$ in an interval of size $R$ yields (15).

References