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WEAK TYPE (1,1) ESTIMATES FOR A CLASS OF DISCRETE ROUGH MAXIMAL FUNCTIONS

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Abstract. We prove weak type $(1,1)$ estimate for the maximal function associated with the sequence $[m^\alpha]$, $1<\alpha<1+\frac{1}{1000}$. As a consequence, the sequence $[m^\alpha]$ is universally L^1-good.

1. Introduction and statement of the result

Let

$$\mathcal{M}^* f(x) = \sup_{M>0} \frac{1}{M} \sum_{0<m<M} f(x-[m^\alpha]),\ x\in\mathbb{Z}.$$

The aim of this note is to prove the weak type $(1,1)$ of the maximal function $\mathcal{M}^*$ for $1<\alpha<1+\frac{1}{1000}$. Thus we provide a counterexample of arithmetic set type to the conjecture of J. Rosenblatt and M. Wierdl, see [4]. We use an approach similar to those of M. Christ [5], see also [8, 11]. We reduce the problem of the weak type $(1,1)$ of $\mathcal{M}^*$ to the regularity estimates for the convolution of a certain measure μ_M supported by the sequence $[m^\alpha]$ and its reflection $\tilde{\mu}_M$. This is closely connected to the problem of representation of a given integer as a difference of two numbers of the form $[m^\alpha]$. In order to obtain necessary estimates, we use B. I. Segals approach, [7], [9], see also [6]. The ℓ^p $(1<p\le\infty)$ boundedness of the maximal function $\mathcal{M}^*$ has been established in [1, 3] and [2].

Our main result is the following

Theorem 1.1. *Let $1<\alpha<1+\frac{1}{1000}$. Then the operator $\mathcal{M}^*$ defined above is of weak type $(1,1)$.*

Recall that a sequence of integers $\{a_n\}_{n\in\mathbb{N}}$ is *universally L^1-good* if the following property holds: for any measure preserving ergodic flow $\{T^s\}_{s\in\mathbb{Z}}$ on any probability space $(\Omega,\mathcal{F},\mu)$ and $f\in L^1(\Omega,\mu)$ the averages

$$\frac{1}{N}\sum_{n\le N} f\circ T^{a_n} \to \int f d\mu,$$

μ-a.e. as $N\to\infty$.

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Corollary 1.2. *The sequence* $[m^\alpha]$, $1 < \alpha < 1 + \frac{1}{1000}$ *is universally* L^1*-good.*

Proof. By now, the classical argument can be found in [1]. □

Remark 1.3. *The range of* $1 < \alpha < 1 + \frac{1}{1000}$ *can be improved by the method used in the paper.*

2. Some lemmas

For a fixed integer $Q \geq 1$, denote $x_P = x - \frac{P}{Q}$, $P = 0, 1, \dots, Q-1$. In our application Q will be $M^{\frac{1}{1000}}$.

Lemma 2.1. *Let* $M \leq m \leq 2M$, *and* $x_P \geq M$, *and*

$$f_P(m) = j_1(x_P + m^\alpha)^{1/\alpha} + j_2 m^\alpha,$$

where $|j_1| \leq M^{\frac{1}{100}+\alpha-1}$ *and* $|j_2| \leq M^{\frac{1}{100}}$. *Then there exist* $m_0 \in (M, 2M)$ *such that for*

$$|m - m_0| \geq M^{\frac{99}{100}} \tag{2.2}$$

we have

$$c_\alpha M^{-\frac{103}{100}} \leq f_P''(m) \leq C_\alpha M^{-\frac{98}{100}}.$$

Proof. Straightforward calculation shows that

$$f_P''(m) = (\alpha - 1)m^{\alpha-2}\left(\frac{x_P j_1}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}} + j_2\alpha\right)$$

Denote $A(m) = (\frac{x_P j_1}{(x_P+m^\alpha)^{2-\frac{1}{\alpha}}} + j_2\alpha)$. Assume that for every $M \leq m \leq 2M$ we have $|A(m)| \geq \frac{1}{100}M^{-\frac{3}{100}}$. Then $|f_P''(m)| \geq c_\alpha M^{-\frac{103}{100}}$ and the lower bound follows. Assume that $|A(m_0)| \leq \frac{1}{100}M^{-\frac{3}{100}}$ and observe that for $|x_P| \geq M$ and $|m_0 - m| \geq M^{\frac{99}{100}}$ we have, using mean value theorem,

$$|A(m) - A(m_0)| = \left|\frac{x_P}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}} - \frac{x_P}{(x_P + m_0^\alpha)^{2-\frac{1}{\alpha}}}\right| \geq \frac{1}{10}M^{-\frac{3}{100}}.$$

Hence for $|m_0 - m| \geq M^{\frac{99}{100}}$ we have $|A(m)| \geq \frac{1}{100}M^{-\frac{3}{100}}$. The lower bound for $f_P''(m)$ follows. Direct calculation easily shows the upper bound

$$|f_P''(m)| \leq C_\alpha M^{-\frac{98}{100}}.$$

□

Corollary 2.3. *Let* $M^{\frac{1}{200}} \leq k \leq 2M^{\frac{1}{200}}$, $\phi \in C_c^\infty(-4, 4)$ *and*

$$S(j_1, j_2) = \sum_{M^{1-\frac{1}{200}}k \leq m \leq M^{1-\frac{1}{200}}(k+1)} \phi(\tfrac{m}{M})e^{2\pi i f_P(m)}.$$

Then, under the same assumptions as in Lemma 2.1, we have

$$|S(j_1, j_2)| \leq C_\alpha M^{1-\frac{1}{200}}M^{-\frac{1}{200}}.$$

Proof. We use the following

Theorem 2.4 (Van der Corput, [12]). *Let a, b, k be positive integer numbers such that $a < b$, $l \geq 2$. Let $f \in C^l([a,b])$. Denote $r = \inf_{x\in[a,b]} |f^{(l)}(x)|$ and $R = \sup_{x\in[a,b]} |f^{(l)}(x)|$. Then*

$$|\sum_{m=a}^{b} e^{f(m)}|$$
$$< 21(b-a)\left(\left(\frac{r}{R^2}\right)^{-1/(\kappa-2)} + (r(b-a)^l)^{-2/\kappa} + \left(\frac{r(b-a)}{R}\right)^{-2/\kappa}\right),$$

where $\kappa = 2^l$.

Then we take $\phi(t) = 1$ for $0 \leq t \leq 2$, fix k, j_1, j_2 and apply the above theorem to estimate two sums $S(j_1, j_2)$ taken over the intervals

$$\max\{m_0 + M^{1-\frac{1}{100}}, M^{1-\frac{1}{200}}k\} \leq m \leq M^{1-\frac{1}{200}}(k+1)$$

and

$$M^{1-\frac{1}{200}}k \leq m \leq \min\{M^{1-\frac{1}{200}}(k+1), m_0 - M^{\frac{99}{100}}\}.$$

We apply the above theorem with $l = 2$, $r \geq M^{-\frac{103}{100}}$, $R \leq M^{-\frac{98}{100}}$, $b - a = M^{1-\frac{1}{200}}$ and we see that the sum is bonded by $M^{1-\frac{1}{200}}M^{-\frac{1}{4}}$. Consequently, remember (2.2), we have $|S(j_1, j_2)| \leq C_\alpha M^{1-\frac{1}{200}}M^{-\frac{1}{200}}$. See [7].

The case of nonconstant ϕ follows in a standard way by Abel summation formula. □

Lemma 2.5. *Let, for a fixed $\varphi \in C_c^\infty(1,2)$ with $\int \varphi = 1$, μ_M be a measure on $\mathbb{Z}$ defined as follows*

$$\mu_M(x) = \frac{1}{M}\sum_{m\in\mathbb{Z}} \delta_0(x - [m^\alpha])\varphi\left(\tfrac{[m^\alpha]}{M^\alpha}\right),$$

where δ_0 stands for Dirac's delta. Then, for $\tilde{\mu}(x) = \mu(-x)$,

$$\mu_M * \tilde{\mu}_M(x) = \rho_M(x) + O(M^{-\alpha-\frac{1}{1000}}),$$

where $\rho_M(0) = M^{-1}$ and for $x \neq 0$ we have

$$0 \leq \rho_M(x) \leq CM^{-\alpha}.$$

Furthermore, for $x \geq CM$ and $x + h \geq CM$,

$$|\rho_M(x+h) - \rho_M(x)| \leq C\frac{|h|}{M^{2\alpha}}. \tag{2.6}$$

Similar statement holds if $x \leq -CM$ and $x + h \leq -CM$.

Proof. We start with the proof of (2.6). Since $\mu_m * \tilde{\mu}_m$ is symmetric, it suffices to consider the case $x \geq CM$ and $x + h \geq CM$. We fix $Q = M^{\frac{1}{1000}}$, and define

$$u_k = M^{1-\frac{1}{200}}k,$$
$$x_P = x - \tfrac{P}{Q},\ P = 0, 1, \ldots, Q-1,$$
$$A_k = \frac{1}{\alpha(x_P + u_k^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

Let F be a C^∞ function such that $0 \le F \le 1$ and

$$F(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x \ge 1. \end{cases} \tag{2.7}$$

Define $\Psi^{\mathrm{u}}_{k,P}(x)$ as a periodic, with period 1, extension of

$$\Psi^{\mathrm{u}}_{k,P}(x) = \begin{cases} F\left(\frac{Q}{A_k}(x - \frac{10A_k}{Q})\right) & \text{if } 0 \le x \le \frac{1}{2}, \\ F\left(\frac{Q}{A_k}(-A_k(1+\frac{10}{Q}) - x)\right) & \text{if } -\frac{1}{2} \le x \le 0. \end{cases} \tag{2.8}$$

Let $G \in C_c^\infty(-1,1)$ and $\sum_{s\in\mathbb{Z}} G(x-s) \equiv 1$. Define,

$$\Psi_P(x) = \sum_{s\in\mathbb{Z}} G(Q(x_P + s)) \in C_c^\infty(\mathbb{T}).$$

It is easy to see that

$$\sum_{P=0}^{Q-1} \Psi_P(x) \equiv 1. \tag{2.9}$$

Observe that by (2.9),

$$\begin{aligned} &\mu_M * \tilde{\mu}_M(x) \le CM^{-\alpha-1} + \\ &\frac{1}{M^2} \sum_{P=0}^{Q-1} \sum_{k=M^{\frac{1}{200}}}^{2M^{\frac{1}{200}}-1} \sum_{m=M^{1-\frac{1}{200}}k}^{M^{1-\frac{1}{200}}(k+1)} 1_A(x+[m^\alpha])\Psi_P(m^\alpha)\varphi\left(\tfrac{m^\alpha}{M^\alpha}\right)\varphi\left(\tfrac{x+m^\alpha}{M^\alpha}\right) \\ &\le CM^{-\alpha-1} + I; \end{aligned}$$

error term $CM^{-\alpha-1}$ appears because of replacing $\varphi(\frac{[m^\alpha]}{M^\alpha})$ by $\varphi(\frac{m^\alpha}{M^\alpha})$ and is easily estimated by Taylor's formula. We will prove the estimate

$$\begin{aligned} &I \le \\ &\frac{1}{M^2} \sideset{}{^*}\sum_{P=0}^{Q-1} \sum_{k=M^{\frac{1}{200}}}^{2M^{\frac{1}{200}}-1} \sum_{m=M^{1-\frac{1}{200}}k}^{M^{1-\frac{1}{200}}(k+1)} \Psi^{\mathrm{u}}_{k,P}((x_P+m^\alpha)^{1/\alpha})\Psi_P(m^\alpha)\varphi\left(\tfrac{m^\alpha}{M^\alpha}\right)\varphi\left(\tfrac{x+m^\alpha}{M^\alpha}\right), \end{aligned} \tag{2.10}$$

where $A = \{[m^\alpha] : m \in \mathbb{N}\}$, and $*$ in the sign of summation above denotes that the first term with $P = 0$ is taken two times: with $x_P = x$ and $x_P = x+1$. In order to prove (2.10) we need to show that the conditions (here $\|x\| = \min_{k\in\mathbb{Z}} |x-k|$, denotes the distance of x to the nearest integer)

$$\|m^\alpha - \tfrac{P}{Q}\| \le \tfrac{1}{Q},\ x + [m^\alpha] = [y^\alpha],\ M^{1-\frac{1}{200}}k \le m \le M^{1-\frac{1}{200}}(k+1), \tag{2.11}$$
$$M \le y, m \le 2M$$

imply that

$$y - A_k(1+\tfrac{10}{Q}) \le (x_P + m^\alpha)^{1/\alpha} \le y + \tfrac{10A_k}{Q} \text{ for } P \ne 0, \tag{2.12}$$

and one of the following estimates for $P = 0$,

$$y - A_k(1+\tfrac{10}{Q}) \le (x_0 + m^\alpha)^{1/\alpha} \le y + \tfrac{10A_k}{Q} \tag{2.13}$$

or

$$y - A_k(1 + \tfrac{10}{Q}) \le (x_1 + m^\alpha)^{1/\alpha} \le y + \tfrac{10A_k}{Q}, \tag{2.14}$$

and consequently

$$1_A(x + [m^\alpha]) \le \begin{cases} \Psi^{\mathrm{u}}_{k,P}((x_P + m^\alpha)^{1/\alpha}) & \text{for } P \ne 0, \\ \Psi^{\mathrm{u}}_{k,0}((x_0 + m^\alpha)^{1/\alpha}) + \Psi^{\mathrm{u}}_{k,0}((x_0 + 1 + m^\alpha)^{1/\alpha}) & \text{for } P = 0, \end{cases}$$

which implies (2.10). In order to obtain (2.12)–(2.14) we notice that a number $y \in \mathbb{N}$ satisfies $[y^\alpha] = x + [m^\alpha] =: z$ if and only if there exists $\theta \in [0, 1)$ such that the first of the equalities below holds

$$y = (z + \theta)^{1/\alpha} = z^{1/\alpha} + \frac{\theta}{\alpha z^{1-\frac{1}{\alpha}}} + \frac{\eta}{M^{2\alpha-1}},\ |\eta| < 1.$$

Thus,

$$z^{1/\alpha} \in \left(y - \frac{1}{\alpha z^{\frac{\alpha-1}{\alpha}}} - \frac{1}{M^{2\alpha-1}}, y + \frac{1}{M^{2\alpha-1}} \right).$$

Since by (2.11), $\Psi_P(m^\alpha) \ne 0$, we can write

$$x + [m^\alpha] = x - \frac{P}{Q} + m^\alpha + \frac{\eta_0}{Q} \text{ for } P \ne 0$$

for some $|\eta_0| < 1$. Then there exists η_1, $|\eta_1| \le 1$ such that

$$z^{1/\alpha} = (x_P + m^\alpha + \tfrac{\eta_0}{Q})^{1/\alpha} = (x_P + m^\alpha)^{1/\alpha} + \tfrac{\eta_1}{QM^{\alpha-1}}.$$

Similar statement holds for $P = 0$, possibly with x replaced by x_1. Hence,

$$(x_P + m^\alpha)^{1/\alpha} \in \left(y - \frac{1}{\alpha z^{\frac{\alpha-1}{\alpha}}} - \frac{1}{M^{2\alpha-1}} - \frac{1}{QM^{\alpha-1}}, y + \frac{1}{M^{2\alpha-1}} + \frac{1}{QM^{\alpha-1}} \right).$$

In particular, for $M^{1-\frac{1}{200}}k < m < M^{1-\frac{1}{200}}(k+1)$ we have,

$$(x_P + m^\alpha)^{1/\alpha} \in \left(y - \frac{1}{\alpha z^{\frac{\alpha-1}{\alpha}}} - \frac{2}{QM^{\alpha-1}}, y + \frac{2}{QM^{\alpha-1}} \right). \tag{2.15}$$

Since for some η_2 with $|\eta_2| \le 1$,

$$\frac{1}{z^{\frac{\alpha-1}{\alpha}}} = \frac{1}{(x + m^\alpha)^{\frac{\alpha-1}{\alpha}}} + \frac{\eta_2}{M^{2\alpha-1}}, \tag{2.16}$$

it follows from (2.15) and (2.16) that

$$\begin{aligned}(x_P + m^\alpha)^{1/\alpha} \in &\left(y - \frac{1}{\alpha(x + m^\alpha)^{\frac{\alpha-1}{\alpha}}} - \frac{1}{M^{2\alpha-1}} - \frac{2}{QM^{\alpha-1}}, y + \frac{2}{QM^{\alpha-1}} \right) \\ &\subset \left(y - \frac{1}{\alpha(x + u_k^\alpha)^{\frac{\alpha-1}{\alpha}}} - \frac{3}{QM^{\alpha-1}}, y + \frac{3}{QM^{\alpha-1}} \right).\end{aligned}$$

Thus we get (2.12)–(2.14).

Expanding $\Psi^{\mathrm{u}}_{k,P}$ and Ψ_P in (2.10) into the Fourier series we obtain,

$$\begin{aligned}\mu_M{*}\tilde{\mu}_M(x) \\ \leq &\frac{1}{M^2}\sum_{P=0}^{Q-1}{}^{*}\sum_{k=M^{\frac{1}{200}}}^{2M^{\frac{1}{200}}-1}\sum_{(j_1,j_2)\neq(0,0)} c^{(k,P)}_{j_1}c^{(P)}_{j_2}\sum_{m=M^{1-\frac{1}{200}}k}^{M^{1-\frac{1}{200}}(k+1)}\varphi\left(\tfrac{m^\alpha}{M^\alpha}\right)\varphi\left(\tfrac{x+m^\alpha}{M^\alpha}\right)e^{2\pi i f_P(m)} \\ &+\frac{1}{M^2}\sum_{P=0}^{Q-1}{}^{*}\sum_{k=M^{\frac{1}{200}}}^{2M^{\frac{1}{200}}-1}\sum_{m=M^{1-\frac{1}{200}}k}^{M^{1-\frac{1}{200}}(k+1)} A_k(1+\tfrac{20}{Q})\tfrac{1}{Q}\varphi(\tfrac{m^\alpha}{M^\alpha})\varphi(\tfrac{x+m^\alpha}{M^\alpha}) =: I_1+I_2,\end{aligned}$$

where $c^{(k,P)}_{j_1}$ and $c^{(P)}_{j_2}$ are Fourier coefficients of $\Psi^{\mathrm{u}}_{k,P}$ and Ψ_P, and moreover, we have used the fact that for $kM^{1-\frac{1}{200}} \leq m < (k+1)M^{1-\frac{1}{200}}$,

$$0 < c^{(k,P)}_0 < A_k(1+\tfrac{22}{Q}) \leq \frac{1}{\alpha(x+m^\alpha)^{\frac{\alpha-1}{\alpha}}} + CM^{1-\alpha-\frac{1}{1000}}$$

and $c^{(P)}_0 = Q^{-1}$.

Let

$$\rho_M(x) = \frac{1}{M^2}\sum_{m=1}^{\infty}\varphi(\tfrac{m^\alpha}{M^\alpha})\varphi(\tfrac{x+m^\alpha}{M^\alpha})\frac{1}{\alpha(x+m^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

It is easy to see that

(1) $|\rho_M(x) - I_2| \leq CM^{-\alpha}Q^{-1}$,
(2) ρ_M satisfies conditions (2.6).

Therefore, we have to show that for $1<\alpha<1+\frac{1}{1000}$ we have $|I_1| \leq M^{-\alpha-\frac{1}{1000}}$.

Notice that independently of (k,P) and (P), (see [10, chapter 1], for example)

$$\sum_{|j_1|\geq M^{\alpha-1+\frac{1}{100}}} |c^{(k,P)}_{j_1}| \leq M^{-4} \text{ and } \sum_{|j_2|\geq M^{\frac{1}{100}}} |c^{(P)}_{j_2}| \leq M^{-4}.$$

Therefore, it suffices to take j_1, j_2 satisfying assumptions of Corollary 2.3 in the sum defining I_1. Since also $\sum_{j_1,j_2}|c^{(k,P)}_{j_1}||c^{(P)}_{j_2}| \leq C\log M$ (the proof is an easy exercise, [10, chapter 1]) we have

$$\begin{aligned}|I_1| \leq &\frac{QM^{\frac{1}{200}}\log M}{M^2}\sup_{(j_1,j_2)\neq(0,0)}|S(j_1,j_2)| \\ &\leq QM^{-\frac{1}{200}}M^{-1-\frac{1}{200}}M^{\frac{1}{200}}\log M \leq M^{-\alpha}M^{-\frac{1}{1000}} \text{ for } 1<\alpha\leq 1+\frac{1}{1000},\end{aligned}$$

where the indices j_1, j_2 in the $\sup|S(j_1,j_2)|$ are as in Corollary 2.3. Hence, we obtain the upper bound in

$$-CM^{-\alpha}M^{-\frac{1}{1000}} \leq \mu_M * \tilde{\mu}_M(x) - \rho_M(x) \leq CM^{-\alpha}M^{-\frac{1}{1000}}. \tag{2.17}$$

To obtain the lower bound in (2.17) we repeat the proof with the following changes.

- We replace $\psi^{\mathrm{u}}_{k,P}$ in (2.8) by the function $\psi^{\mathrm{l}}_{k,P}$, defined below,

$$\psi^{\mathrm{l}}_{k,P}(x) = \begin{cases} F(\frac{Q}{A_{k+1}}(-A_{k+1}(1-\frac{20}{Q})-x)) & \text{for } -\frac{1}{2}\leq x\leq -\frac{20A_{k+1}}{Q}, \\ F(\frac{Q}{A_{k+1}}(x+\frac{20A_{k+1}}{Q})) & \text{for } -\frac{20A_{k+1}}{Q}\leq x\leq \frac{1}{2},\end{cases}$$

where F is defined in (2.7).

- Observe that similarly to (2.12), for $kM^{1-\frac{1}{200}} \le m \le (k+1)kM^{1-\frac{1}{200}}$, the inequalities
$$y - A_{k+1}(1-\tfrac{10}{Q}) \le (x_P + m^\alpha)^{1/\alpha} \le y - \tfrac{10A_{k+1}}{Q} \text{ for } P \ne 0$$
and $\|m^\alpha - \frac{P}{Q}\| \le \frac{1}{Q}$ imply $x + [m^\alpha] = [y^\alpha]$. Hence, we have that for $P \ne 0$,
$$\Psi^1_{k,P}((x_P + m^\alpha)^{1/\alpha})\Psi_P(m^\alpha) \le 1_A(x + [m^\alpha])\Psi_P(m^\alpha).$$
- We do not take into account $P = 0$ in the sumation in (2.10).

We leave the details for the reader.

In order to prove the estimate for $\rho_M(x)$ for $0 < |x| \le CM$ we argue as follows. Assume that the following equality holds
$$x + [m^\alpha] = [(m+s)^\alpha] \text{ where } s > 0,\ x > 0.$$
Then we have
$$x - 1 \le (m+s)^\alpha - m^\alpha \le x + 1 \tag{2.18}$$
and hence
$$cM^{\alpha-1} \le csM^{\alpha-1} \le x \le CsM^{\alpha-1}. \tag{2.19}$$
Observe that for the increasing function $g(m) = (m+s)^\alpha - m^\alpha$ we have
$$g(m+1) - g(m) \approx sM^{\alpha-2} \lesssim 1, \text{ for } |x| \le CM,$$
and thus for a fixed s there are at most $1 + s^{-1}M^{2-\alpha} \approx Mx^{-1}$ different consecutive values of m for wchich (2.18) can hold. Since moreover, by (2.19), s has to satisfy $s \approx xM^{1-\alpha}$, the total number of solutions of (2.18) is bounded from above by
$$CMx^{-1}(xM^{1-\alpha}) = CM^{2-\alpha}.$$
Hence we easily obtain $\rho_M(x) \le CM^{-\alpha}$ for $0 < |x| < CM$ and the lemma follows. □

3. Proof of Theorem 1.1

Let
$$M^*f(x) = \sup_{n>0} |\mu_{2^n} * f(x)|.$$
With no loss of generality it suffices to show the weak type $(1,1)$ of M^*. We will use the argument of [5] and [11] adapted to our setting. Let $\lambda > 0$ and $f \in \ell^1(\mathbb{Z})$. Let $N = 2^n$, $n \in \mathbb{N}$. We consider the Calderón-Zygmund decomposition
$$f = g + \sum f_{s,j} = \sum b_s + g,$$
where $|g| < \lambda$ and b_s contains terms $f_{s,j}$ supported by $Q_{s,j}$ with $|Q_{s,j}| \simeq 2^{\alpha s}$, $\sum_{s,j} |Q_{s,j}| \le \lambda^{-1}\|f\|_{\ell^1}$ and $\|f_{s,j}\|_{\ell^1} \simeq \lambda |Q_{s,j}|$. We do not assume $\int f_{s,j} = 0$, instead we decompose further each b_s writing $b_s(x) = b_s^{(N)}(x) + B_s^{(N)}(x) + g_s^{(N)}(x)$, where $b_s^{(N)}(x) = \chi_{\{|b_s|>\lambda N\}}(x) b_s(x)$, and for $h_s^{(N)}(x) = b_s(x) - b_s^{(N)}(x)$ we have $B_s^{(N)}(x) = h_s^{(N)}(x) - g_s^{(N)}(x)$ and $g_s^{(N)}(x) = \sum_j \frac{\chi_{Q_{s,j}}(x)}{2^{\alpha s}} \int_{Q_{s,j}} h_s^{(N)}$. Consequently,
$$f = g + \sum_s g_s^{(N)} + \sum_s B_s^{(N)} + \sum_s b_s^{(N)}. \tag{3.1}$$

Observe that $\frac{1}{2^{\alpha s}}\int_{Q_{s,j}}|h_s^{(N)}|\leq C\lambda$ and, since $Q_{s,j}$ are mutually disjoint, we get

$$|g(x)|+\sum_s|g_s^{(N)}(x)|\leq C\lambda. \tag{3.2}$$

We have

$$\{x:\mu_N*|b_s^{(N)}|(x)>0\}=\bigcup_{m\approx N}([m^\alpha]+\{x:|b_s^{(N)}|(x)>0\}).$$

Thus,

$$\begin{aligned}|\{x:\mu_N*|b_s^{(N)}|(x)>0\}|=&\sum_{m\approx N}|\{x:|b_s^{(N)}|(x)>0\}|\\ =&N|\{x:|b_s^{(N)}|(x)>\lambda N\}|.\end{aligned}$$

Consequently (remember that $N=2^n$),

$$\begin{aligned}\sum_s\sum_{N\text{ - dyadic}}|\{x:\mu_N*|b_s^{(N)}|(x)>0\}|\leq&\sum_s\sum_{N\text{ - dyadic}}N|\{x:|b_s|(x)>\lambda N\}|\\ \leq&\sum_s\frac{1}{\lambda}\|b_s\|_{\ell^1}\leq\frac{1}{\lambda}\|f\|_{\ell^1}.\end{aligned}\tag{3.3}$$

Moreover, since for the fixed dyadic N the supports of $B_s^{(N)}(x)$ are mutually disjoint, it is easy to see, for a fixed $x\in Q_{s_0,j_0}$,

$$\begin{aligned}\sum_{N\text{ dyadic}}\sum_s N^{-1}B_{n-s}^{(N)}(x)^2\leq&\Bigg(\sum_{\{N\text{ - dyadic}:N\lambda\geq|b_{s_0}(x)|\}}N^{-1}|b_{s_0}(x)|^2\\ &+\lambda^2\sum_{N\text{ - dyadic}}N^{-1}\chi_{\{\operatorname{supp}b_{s_0}\}}(x)\Bigg)\\ \leq&C\lambda|b_{s_0}(x)|+\lambda^2\chi_{\{\operatorname{supp}b_{s_0}\}}(x)\\ \leq&C\lambda\sum_s(|b_s(x)|+C\lambda\chi_{\{\operatorname{supp}b_s\}}(x).\end{aligned}\tag{3.4}$$

Hence by (3.4), we have

$$\lambda^{-2}\sum_s\sum_{N\text{ - dyadic}}N^{-1}\|B_{n-s}^{(N)}\|_{\ell^2}^2\leq\frac{C}{\lambda}\|f\|_{\ell^1}. \tag{3.5}$$

We will use the following lemma,

Lemma 3.6. *Let $N=2^n$, $n\in\mathbb{N}$. For sufficiently small $\delta>0$ we have the following estimates, see* [5]*,*

$$\|\mu_N*B_{n-s}^{(N)}\|_{\ell^2}^2\leq C\lambda\|B_{n-s}^{(N)}\|_{\ell^1}2^{-\delta s}+2^{-n}\|B_{n-s}^{(N)}\|_{\ell^2}^2 \tag{3.7}$$

and for $s_1>s_2$,

$$|\langle\mu_N*B_{n-s_1}^{(N)},\mu_N*B_{n-s_2}^{(N)}\rangle|\leq C\lambda\|B_{n-s_2}^{(N)}\|_{\ell^1}2^{-\delta s_1}. \tag{3.8}$$

Proof. By Lemma 2.5,

$$\mu_N * \tilde{\mu}_N(x) = C\rho_N(x) + 2^{-n}\delta_0(x) + O(2^{-n(\alpha+\frac{1}{1000})}),$$

where ρ_N satisfies

$$|\rho_N(x)| \le \frac{C}{N^\alpha} = \frac{C}{2^{n\alpha}} \text{ and } \rho_N(0) = 0.$$

Moreover, for $|x| > CM$ and $|x+h| > CM$,

$$|\rho_N(x+h) - \rho_N(x)| \le \frac{C}{2^{n\alpha}}\frac{|h|}{2^{n\alpha}}. \tag{3.9}$$

Let for $s \le n$, $\operatorname{supp} B_s^{(N)} \subset [0, 4\cdot 2^{n\alpha}]$ and $\operatorname{supp}\rho_n \subset [0, 4\cdot 2^{n\alpha}]$. Denote by $F_{s,j}$ the restriction of $B_s^{(N)}$ to $Q_{s,j}$ By the estimate $\|B_s^{(N)}\|_{\ell^1} \le \lambda 2^{s\alpha}$ we have,

$$\begin{aligned} A :=&|\langle \mu_N * B^{(N)}_{n-s_1}, \mu_N * B^{(N)}_{n-s_2}\rangle| = |\langle \mu_N * \tilde{\mu}_N * B^{(N)}_{n-s_1}, B^{(N)}_{n-s_2}\rangle| \\ \le& 2^{-n\alpha}2^{-\frac{n}{1000}}\|B^{(N)}_{n-s_1}\|_{\ell^1}\|B^{(N)}_{n-s_2}\|_{\ell^1} + \left\langle \sum_{j_1} |\rho_n * F_{n-s_1,j_1}|, |B^{(N)}_{n-s_2}| \right\rangle \\ &+ N^{-1}|\langle B^{(N)}_{n-s_1}, B^{(N)}_{n-s_2}\rangle|. \end{aligned} \tag{3.10}$$

Observe that for $s_1 \ne s_2$ the supports of $B^{(N)}_{n-s_1}$, $B^{(N)}_{n-s_2}$ are disjoint and consequently the third summand is equal to zero. Consider the second summand in (3.10). By the regularity estimate (3.9) and the fact that $\int F_{n-s,j} = 0$ we get in a standard way

$$|\rho_N * F_{n-s_1,j}(x)| \le \frac{2^{(n-s_1)\alpha}}{2^{n\alpha}}\frac{\|F_{n-s_1,j}\|_{\ell^1}}{2^{n\alpha}}$$

for $|x - x_{s_1,j}| > CN + C2^{(n-s_1)\alpha}$, where $x_{s_1,j}$ denotes the center of $Q_{s_1,j}$. Moreover, for any x, $|\rho_N * F_{n-s_1,j}(x)| \le N^{-\alpha}\|F_{n-s_1,j}\|_{\ell^1} \le \lambda N^{-\alpha}2^{(n-s_1)\alpha}$. Consequently, we have

$$\begin{aligned} \sum_j |\rho_N * F_{n-s_1,j}(x)| \le& \sum_{\{j:|x-x_{s_1,j}|\le CN+C2^{(n-s_1)\alpha}\}} 2^{-n\alpha}\|F_{n-s_1,j}\|_{\ell^1} \\ &+ 2^{-s_1\alpha}\sum_{\{j:|x-x_{s_1,j}|> CN+C2^{(n-s_1)\alpha}\}} 2^{-n\alpha}\|F_{n-s_1,j}\|_{\ell^1} \\ \le& C\lambda\max\{N^{1-\alpha}, 2^{-s_1\alpha}\} + C\lambda 2^{-s_1\alpha} \le C\lambda 2^{-s_1\delta}. \end{aligned}$$

Thus we estimate the second summand in (3.10) by

$$\|B^{(N)}_{n-s_2}\|_{\ell^1}\left\|\sum_j |\rho_N * F_{n-s_1,j}|\right\|_{\ell^\infty} \le C\lambda 2^{-s_1\delta}\|B_{n-s_2}\|_{\ell^1}.$$

Finally, we get

$$A \le C\lambda 2^{-s_1\delta}\|B_{n-s_2}\|_{\ell^1} + C\lambda\|B_{n-s_2}\|_{\ell^1}2^{-\frac{n}{1000}} \le C\lambda\|B_{n-s_2}\|_{\ell^1}2^{-\delta s_1}.$$

The assumption $\operatorname{supp} B^{(N)}_{n-s} \subset [0, 4\cdot 2^{n\alpha}]$ can be removed in a standard way. The estimates (3.7), (3.8) follow. □

Now we are ready to give

Proof of Theorem 1.1. Using Lemma 3.6 we have,

$$
\begin{aligned}
&\lambda^2|\{x : \max_N |\sum_{s\geq 0} B^{(N)}_{n-s} * \mu_N(x)| \geq c\lambda\}| \leq \sum_x \max_N |\sum_{s\geq 0} B^{(N)}_{n-s} * \mu_N(x)|^2 \\
&\leq \sum_x \sum_N |\sum_{s\geq 0} B^{(N)}_{n-s} * \mu_N(x)|^2 \\
&\leq \sum_{N,s\geq 0} \|\mu_N * B^{(N)}_{n-s}\|^2_{\ell^2} + 2 \sum_{N,s_1>s_2} |\langle \mu_N * B^{(N)}_{n-s_1}, \mu_N * B^{(N)}_{n-s_2}\rangle| \qquad (3.11)\\
&\leq \sum_{N,s\geq 0} C\lambda \|B^{(N)}_{n-s}\|_{\ell^1} 2^{-\delta s} + 2^{-n}\|B^{(N)}_{n-s}\|^2_{\ell^2} + 2 \sum_{N,s_1>s_2} C\lambda \|B^{(N)}_{n-s_2}\|_{\ell^1} 2^{-\delta s_1} \\
&\leq C\lambda \|f\|_{\ell^1},
\end{aligned}
$$

where the second summand in the last inequality is estimated by (3.5).

By (3.1) we have,

$$
\begin{aligned}
\{\sup_N |\mu_N * f(x)| > 4C\lambda\} &\subset \{\sup_N |\mu_N * (|g| + \sum_s |g^{(N)}_s|)(x)| > C\lambda\} \\
&\cup \{\sup_N |\mu_N * (\sum_{s>0} |B^{(N)}_{n-s}|)(x)| > C\lambda\} \cup \{\sup_N |\mu_N * (\sum_{s>0} |B^{(N)}_{n+s}|)(x)| > C\lambda\} \\
&\cup \{\sup_N \sum_s \mu_N * |b^{(N)}_s| > C\lambda\} =: S_1 \cup S_2 \cup S_3 \cup S_4.
\end{aligned}
$$

In the above sum, by (3.2), the first set is empty if the constant C is sufficiently large. Since $\operatorname{supp} \mu_N * B^{(N)}_s \subset \cup_{s,j} Q^{***}_{s,j}$ for $s \geq n$, the set S_3 is a subset of $\cup_{s,j} Q^{***}_{s,j}$ and consequently $|S_3| \leq |\cup_{s,j} Q^{***}_{s,j}| \leq C \sum_{s,j} |Q_{s,j}| \leq \frac{C}{\lambda}\|f\|_{\ell^1}$. Moreover by (3.11) we have $|S_2| \leq \frac{C}{\lambda}\|f\|_{\ell^1}$. The set $S_4 \subset \bigcup_{N,s}\{\mu_N * |b^{(N)}_s|(x) > 0\}$. Hence by (3.3) $|S_4| \leq \sum_{N,s} |\{\mu_N * |b^{(N)}_s|(x) > 0\}| \leq \frac{C}{\lambda}\|f\|_{\ell^1}$. The theorem follows. □

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