WEAK TYPE (1,1) ESTIMATES FOR A CLASS OF DISCRETE ROUGH MAXIMAL FUNCTIONS

ROMAN URBAN AND JACEK ZIENKIEWICZ

ABSTRACT. We prove weak type (1,1) estimate for the maximal function associated with the sequence \([m^\alpha]\), \(1 < \alpha < 1 + \frac{1}{1000}\). As a consequence, the sequence \([m^\alpha]\) is universally \(L^1\)-good.

1. Introduction and statement of the result

Let

\[ M^* f(x) = \sup_{M > 0} \frac{1}{M} \sum_{0 < m < M} f(x - [m^\alpha]), \quad x \in \mathbb{Z}. \]

The aim of this note is to prove the weak type (1,1) of the maximal function \(M^*\) for \(1 < \alpha < 1 + \frac{1}{1000}\). Thus we provide a counterexample of arithmetic set type to the conjecture of J. Rosenblatt and M. Wierdl, see [4]. We use an approach similar to those of M. Christ [5], see also [8, 11]. We reduce the problem of the weak type (1,1) of \(M^*\) to the regularity estimates for the convolution of a certain measure \(\mu_M\) supported by the sequence \([m^\alpha]\) and its reflection \(\tilde{\mu}_M\). This is closely connected to the problem of representation of a given integer as a difference of two numbers of the form \([m^\alpha]\). In order to obtain necessary estimates, we use B. I. Segals approach, [7], [9], see also [6]. The \(\ell^p\) (\(1 < p \leq \infty\)) boundedness of the maximal function \(M^*\) has been established in [1, 3] and [2].

Our main result is the following

**Theorem 1.1.** Let \(1 < \alpha < 1 + \frac{1}{1000}\). Then the operator \(M^*\) defined above is of weak type (1,1).

Recall that a sequence of integers \(\{a_n\}_{n \in \mathbb{N}}\) is universally \(L^1\)-good if the following property holds: for any measure preserving ergodic flow \(\{T^s\}_{s \in \mathbb{Z}}\) on any probability space \((\Omega, \mathcal{F}, \mu)\) and \(f \in L^1(\Omega, \mu)\) the averages

\[ \frac{1}{N} \sum_{n \leq N} f \circ T^{a_n} \rightarrow \int f d\mu, \]

\(\mu\)-a.e. as \(N \rightarrow \infty\).
Corollary 1.2. The sequence \([m^\alpha], 1 < \alpha < 1 + \frac{1}{1000}\) is universally \(L^1\)-good.

Proof. By now, the classical argument can be found in [1]. \qed

Remark 1.3. The range of \(1 < \alpha < 1 + \frac{1}{1000}\) can be improved by the method used in the paper.

2. Some lemmas

For a fixed integer \(Q \geq 1\), denote \(x_P = x - \frac{P}{Q}\), \(P = 0, 1, \ldots, Q-1\). In our application \(Q\) will be \(M^{\frac{1}{200}}\).

Lemma 2.1. Let \(M \leq m \leq 2M\), and \(x_P \geq M\), and
\[
f_P(m) = j_1(x_P + m^\alpha)^{1/\alpha} + j_2 m^\alpha,
\]
where \(|j_1| \leq M^{\frac{1}{100}} + \alpha - 1\) and \(|j_2| \leq M^{\frac{1}{100}}\). Then there exist \(m_0 \in (M, 2M)\) such that for
\[
|m - m_0| \geq M^{\frac{99}{100}}
\]
we have
\[
c_\alpha M^{-\frac{101}{100}} \leq f_P''(m) \leq C_\alpha M^{-\frac{98}{100}}.
\]

Proof. Straightforward calculation shows that
\[
f_P''(m) = \frac{(x_P + m^\alpha)^{2-\frac{1}{\alpha}} - x_P^2}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}} + j_2 \alpha \frac{x_P^2}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}}
\]
Denote \(A(m) = \frac{x_P j_1}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}} + j_2 \alpha\). Assume that for every \(M \leq m \leq 2M\) we have \(|A(m)| \geq \frac{1}{100} M^{-\frac{3}{100}}\). Then \(|f_P''(m)| \geq c_\alpha M^{-\frac{101}{100}}\) and the lower bound follows. Assume that \(|A(m_0)| \leq \frac{1}{100} M^{-\frac{3}{100}}\) and observe that for \(|x_P| \geq M\) and \(|m_0 - m| \geq M^{\frac{99}{100}}\) we have, using mean value theorem,
\[
|A(m) - A(m_0)| = \left| \frac{x_P}{(x_P + m^\alpha)^{2-\frac{1}{\alpha}}} - \frac{x_P}{(x_P + m_0^\alpha)^{2-\frac{1}{\alpha}}} \right| \geq \frac{1}{10} M^{-\frac{3}{100}}.
\]
Hence for \(|m_0 - m| \geq M^{\frac{99}{100}}\) we have \(|A(m)| \geq \frac{1}{100} M^{-\frac{3}{100}}\). The lower bound for \(f_P''(m)\) follows. Direct calculation easily shows the upper bound
\[
|f_P''(m)| \leq C_\alpha M^{-\frac{100}{100}}.
\]
\qed

Corollary 2.3. Let \(M^{\frac{1}{100}} \leq k \leq 2M^{\frac{1}{100}}\), \(\phi \in C_c^\infty(-4,4)\) and
\[
S(j_1, j_2) = \sum_{M^{1-\frac{1}{100}} k \leq m \leq M^{1-\frac{1}{100}} (k+1)} \phi\left( \frac{m}{M} \right) e^{2\pi if_P(m)}.
\]
Then, under the same assumptions as in Lemma 2.1, we have
\[
|S(j_1, j_2)| \leq C_\alpha M^{1-\frac{1}{100}} M^{-\frac{1}{100}}.
\]

Proof. We use the following
**Theorem 2.4** (Van der Corput, [12]). Let \(a, b, k\) be positive integer numbers such that \(a < b\), \(l \geq 2\). Let \(f \in C^l([a, b])\). Denote \(r = \inf_{x \in [a, b]} |f^{(l)}(x)|\) and \(R = \sup_{x \in [a, b]} |f^{(l)}(x)|\). Then

\[
|\sum_{m=a}^{b} e^{f(m)}| < 21(b - a) \left( \frac{r}{R^2} \right)^{1/(\kappa - 2)} + \left( r(b - a)^l \right)^{-2/\kappa} + \left( \frac{r(b - a)}{R} \right)^{-2/\kappa},
\]

where \(\kappa = 2^l\).

Then we take \(\phi(t) = 1\) for \(0 \leq t \leq 2\), fix \(k, j_1, j_2\) and apply the above theorem to estimate two sums \(S(j_1, j_2)\) taken over the intervals

\[
\max\{m_0 + M^{1/\alpha}, M^{1/\alpha}k\} \leq m \leq M^{1/\alpha}(k + 1)
\]

and

\[
M^{1/\alpha}k \leq m \leq \min\{M^{1/\alpha}(k + 1), m_0 - M^{1/\alpha}\}.
\]

We apply the above theorem with \(l = 2\), \(r \geq M^{1/\alpha}, R \leq M^{1/\alpha}, b - a = M^{1/\alpha}\) and we see that the sum is bounded by \(M^{1/\alpha}M^{-1/4}\). Consequently, remember (2.2), we have \(|S(j_1, j_2)| \leq C_M M^{1 - \frac{1}{2\pi}} M^{-\frac{1}{2}}\). See [7].

The case of nonconstant \(\phi\) follows in a standard way by Abel summation formula. \(\square\)

**Lemma 2.5.** Let, for a fixed \(\varphi \in C^\infty_c(1, 2)\) with \(\int \varphi = 1\), \(\mu_M\) be a measure on \(\mathbb{Z}\) defined as follows

\[
\mu_M(x) = \frac{1}{M} \sum_{m \in \mathbb{Z}} \delta_0(x - [m^\alpha])\varphi\left(\frac{[m^\alpha]}{M^\alpha}\right),
\]

where \(\delta_0\) stands for Dirac’s delta. Then, for \(\hat{\mu}(x) = \mu(-x)\),

\[
\mu_M * \hat{\mu}_M(x) = \rho_M(x) + O(M^{-\frac{1}{1+\alpha}}),
\]

where \(\rho_M(0) = M^{-1}\) and for \(x \neq 0\) we have

\[
0 \leq \rho_M(x) \leq CM^{-\alpha}.
\]

Furthermore, for \(x \geq CM\) and \(x + h \geq CM\),

\[
|\rho_M(x + h) - \rho_M(x)| \leq C\frac{|h|}{M^{2\alpha}}.
\]

Similar statement holds if \(x \leq -CM\) and \(x + h \leq -CM\).

**Proof.** We start with the proof of (2.6). Since \(\mu_m * \hat{\mu}_m\) is symmetric, it suffices to consider the case \(x \geq CM\) and \(x + h \geq CM\). We fix \(Q = M^{1/\alpha}\), and define

\[
u_k = M^{1/\pi}k,
\]

\[
\nu_P = x - \frac{P}{Q}, P = 0, 1, \ldots, Q - 1,
\]

\[
A_k = \frac{1}{\alpha(\nu_P + \nu_k^{\frac{\alpha}{2-\alpha}})}.
\]
Let $F$ be a $C^\infty$ function such that $0 \leq F \leq 1$ and
\begin{equation}
F(x) = \begin{cases} 
1 & \text{if } x \leq 0, \\
0 & \text{if } x \geq 1.
\end{cases}
\end{equation}
Define $\Psi_{k,P}(x)$ as a periodic, with period 1, extension of
\begin{equation}
\Psi_{k,P}(x) = \begin{cases} 
F \left( \frac{Q}{A_k} (x - \frac{10A_k}{Q}) \right) & \text{if } 0 \leq x \leq \frac{1}{2}, \\
F \left( \frac{Q}{A_k} (-A_k(1 + \frac{10}{Q}) - x) \right) & \text{if } -\frac{1}{2} \leq x \leq 0.
\end{cases}
\end{equation}
Let $G \in C^\infty_c(-1,1)$ and $\sum_{s \in \mathbb{Z}} G(x-s) \equiv 1$. Define,
\begin{equation}
\Psi_P(x) = \sum_{s \in \mathbb{Z}} G(Q(x_P+s)) \in C^\infty_c(\mathbb{T}).
\end{equation}
It is easy to see that
\begin{equation}
\sum_{P=0}^{Q-1} \Psi_P(x) \equiv 1.
\end{equation}
Observe that by (2.9),
\begin{align*}
\mu_M \ast \tilde{\mu}_M(x) & \leq CM^{-\alpha-1} + \\
& \frac{1}{M^2} \sum_{P=0}^{Q-1} \sum_{k=M \frac{m}{T^\alpha}}^{2M \frac{m}{T^\alpha} - 1} \sum_{m=M^1 - \frac{1}{T^\alpha} k}^{M^1 - \frac{1}{T^\alpha} (k+1)} 1_A(x + [m^\alpha]) \Psi_P(m^\alpha) \varphi \left( \frac{m^\alpha}{M^\alpha} \right) \varphi \left( \frac{x+m^\alpha}{M^\alpha} \right) \\
& \leq CM^{-\alpha-1} + I;
\end{align*}
error term $CM^{-\alpha-1}$ appears because of replacing $\varphi(\frac{m^\alpha}{M^\alpha})$ by $\varphi(\frac{m^\alpha}{M^\alpha})$ and is easily estimated by Taylor’s formula. We will prove the estimate
\begin{equation}
I \leq \frac{1}{M^2} \sum_{P=0}^{Q-1} \sum_{k=M \frac{m}{T^\alpha}}^{2M \frac{m}{T^\alpha} - 1} \sum_{m=M^1 - \frac{1}{T^\alpha} k}^{M^1 - \frac{1}{T^\alpha} (k+1)} \Psi_{k,P}((x_P + m^\alpha)^{1/\alpha}) \Psi_P(m^\alpha) \varphi \left( \frac{m^\alpha}{M^\alpha} \right) \varphi \left( \frac{x+m^\alpha}{M^\alpha} \right),
\end{equation}
where $A = \{[m^\alpha] : m \in \mathbb{N} \}$, and $\ast$ in the sign of summation above denotes that the first term with $P = 0$ is taken two times: with $x_P = x$ and $x_P = x + 1$. In order to prove (2.10) we need to show that the conditions (here $\|x\| = \min_{k \in \mathbb{Z}} |x-k|$), denotes the distance of $x$ to the nearest integer)
\begin{equation}
\|m^\alpha - \frac{P}{Q}\| \leq \frac{1}{Q}, \quad x + [m^\alpha] = [y^\alpha], \quad M^1 - \frac{1}{T^\alpha} k \leq m \leq M^1 - \frac{1}{T^\alpha} (k+1), \quad M \leq y, m \leq 2M
\end{equation}
imply that
\begin{equation}
y - A_k(1 + \frac{10}{Q}) \leq (x_P + m^\alpha)^{1/\alpha} \leq y + \frac{10A_k}{Q} \quad \text{for } P \neq 0,
\end{equation}
and one of the following estimates for $P = 0$,
\begin{equation}
y - A_k(1 + \frac{10}{Q}) \leq (x_0 + m^\alpha)^{1/\alpha} \leq y + \frac{10A_k}{Q}.\end{equation}
or
\[(2.14)\quad y - A_k(1 + \frac{10}{Q}) \leq (x_1 + m^\alpha)^{1/\alpha} \leq y + \frac{10A_k}{Q},\]
and consequently
\[1_A(x + [m^\alpha]) \leq \begin{cases} \Psi_{k,P}^u((x_1 + m^\alpha)^{1/\alpha}) & \text{for } P \neq 0, \\ \Psi_{k,0}^u((x_0 + m^\alpha)^{1/\alpha}) + \Psi_{k,0}^u((x_0 + 1 + m^\alpha)^{1/\alpha}) & \text{for } P = 0, \end{cases}\]
which implies (2.10). In order to obtain (2.12)–(2.14) we notice that a number \(y \in \mathbb{N}\) satisfies \([y^\alpha] = x + [m^\alpha] =: z\) if and only if there exists \(\theta \in [0,1)\) such that the first of the equalities below holds
\[y = (z + \theta)^{1/\alpha} = z^{1/\alpha} + \frac{\theta}{\alpha z^{1-\alpha}} + \frac{\eta}{M^{2\alpha-1}}, |\eta| < 1.\]
Thus,
\[z^{1/\alpha} \in \left( y - \frac{1}{\alpha z^{\frac{1-\alpha}{\alpha}}} - \frac{1}{M^{2\alpha-1}}, y + \frac{1}{M^{2\alpha-1}} \right).\]

Since by (2.11), \(\Psi_P(m^\alpha) \neq 0\), we can write
\[x + [m^\alpha] = x - \frac{P}{Q} + m^\alpha + \frac{\eta_0}{Q}\]
for some \(|\eta_0| < 1\). Then there exists \(\eta_1, |\eta_1| \leq 1\) such that
\[z^{1/\alpha} = (x + m^\alpha + \frac{\eta_0}{Q})^{1/\alpha} = (x + m^\alpha)^{1/\alpha} + \frac{\eta_1}{QM^{\alpha-1}}.\]
Similar statement holds for \(P = 0\), possibly with \(x\) replaced by \(x_1\). Hence,
\[(x_P + m^\alpha)^{1/\alpha} \in \left( y - \frac{1}{\alpha z^{\frac{1-\alpha}{\alpha}}} - \frac{2}{QM^{\alpha-1}}, y + \frac{2}{QM^{\alpha-1}} \right).\]
In particular, for \(M^{1-\frac{1}{2\alpha}}k < m < M^{1-\frac{1}{2\alpha}}(k + 1)\) we have,
\[(2.15)\quad (x_P + m^\alpha)^{1/\alpha} \in \left( y - \frac{1}{\alpha (x + m^\alpha)^{\frac{1-\alpha}{\alpha}}} - \frac{2}{QM^{\alpha-1}}, y + \frac{2}{QM^{\alpha-1}} \right).\]
Since for some \(\eta_2\) with \(|\eta_2| \leq 1\),
\[(2.16)\quad \frac{1}{z^{\frac{1-\alpha}{\alpha}}} = \frac{1}{(x + m^\alpha)^{\frac{1-\alpha}{\alpha}}} + \frac{\eta_2}{M^{2\alpha-1}},\]
it follows from (2.15) and (2.16) that
\[(x_P + m^\alpha)^{1/\alpha} \in \left( y - \frac{1}{\alpha (x + m^\alpha)^{\frac{1-\alpha}{\alpha}}} - \frac{2}{QM^{\alpha-1}}, y + \frac{2}{QM^{\alpha-1}} \right) \subset \left( y - \frac{1}{\alpha (x + u_k^{\alpha})^{\frac{1-\alpha}{\alpha}}} - \frac{3}{QM^{\alpha-1}}, y + \frac{3}{QM^{\alpha-1}} \right).\]
Thus we get (2.12)–(2.14).
Expanding $\Psi^u_{k,P}$ and $\Psi_P$ in (2.10) into the Fourier series we obtain,

$$
\mu_M \ast \tilde{\mu}_M(x)
\leq \frac{1}{M^2} \sum_{P=0}^{Q-1} \sum_{k=M \triangleq (j_1,j_2)\neq (0,0)}^{2M \frac{\alpha}{\pi M}} \sum_{m=M \frac{\alpha}{\pi M}}^{M^1 \frac{\alpha}{\pi M}} (k+1) \sum_{m=M \frac{\alpha}{\pi M}}^{M^1 \frac{\alpha}{\pi M}} \varphi \left( \frac{m^\alpha}{M^\alpha} \right) \varphi \left( \frac{x+m^\alpha}{M^\alpha} \right) e^{2\pi i f_P(m)}
$$

$$
+ \frac{1}{M^2} \sum_{P=0}^{Q-1} \sum_{k=M \triangleq (j_1,j_2)\neq (0,0)}^{2M \frac{\alpha}{\pi M}} \sum_{m=M \frac{\alpha}{\pi M}}^{M^1 \frac{\alpha}{\pi M}} A_k(1+\frac{20}{Q^2}) \frac{1}{Q} \varphi \left( \frac{m^\alpha}{M^\alpha} \right) \varphi \left( \frac{x+m^\alpha}{M^\alpha} \right) =: I_1 + I_2,
$$

where $c_{j_1}^{(k,P)}$ and $c_{j_2}^{(P)}$ are Fourier coefficients of $\Psi^u_{k,P}$ and $\Psi_P$, and moreover, we have used the fact that for $kM \frac{\alpha}{\pi M} \leq m < (k+1)M \frac{\alpha}{\pi M}$,

$$
0 < c_0^{(k,P)} < A_k \left( 1 + \frac{22}{Q^2} \right) \leq \frac{1}{\alpha(x+m^\alpha)\frac{\alpha}{\pi M}} + CM^{1-\frac{1}{\pi M}}
$$

and $c_0^{(P)} = Q^{-1}$.

Let

$$
\rho_M(x) = \frac{1}{M^2} \sum_{m=1}^{\infty} \varphi \left( \frac{m^\alpha}{M^\alpha} \right) \varphi \left( \frac{x+m^\alpha}{M^\alpha} \right) e^{2\pi i f_P(m)}
$$

It is easy to see that

1. $|\rho_M(x) - I_2| \leq CM^{-\alpha}Q^{-1}$.
2. $\rho_M$ satisfies conditions (2.6).

Therefore, we have to show that for $1 < \alpha < 1 + \frac{1}{1000}$ we have $|I_1| \leq M^{-\alpha - \frac{1}{\pi M}}$.

Notice that independently of $(k, P)$ and $(P)$, (see [10, chapter 1], for example)

$$
\sum_{j_1 \geq M^{\frac{1}{\alpha}+\frac{1}{\pi M}}} |c_{j_1}^{(k,P)}| \leq M^{-4} \text{ and } \sum_{j_2 \geq M^{\frac{1}{\pi M}}} |c_{j_2}^{(P)}| \leq M^{-4}.
$$

Therefore, it suffices to take $j_1, j_2$ satisfying assumptions of Corollary 2.3 in the sum defining $I_1$. Since also $\sum_{j_1, j_2} |c_{j_1}^{(k,P)}||c_{j_2}^{(P)}| \leq C \log M$ (the proof is an easy exercise, [10, chapter 1]) we have

$$
|I_1| \leq \frac{QM^{\pi M} \log M}{M^2} \sup_{(j_1, j_2) \neq (0,0)} |S(j_1, j_2)|
$$

$$
\leq QM^{-\frac{1}{\pi M}} M^{-1-\frac{1}{\pi M}} M^{\frac{1}{\pi M}} \log M \leq M^{-\alpha} M^{-\frac{1}{\pi M}} \text{ for } 1 < \alpha \leq 1 + \frac{1}{1000},
$$

where the indices $j_1, j_2$ in the sup $|S(j_1, j_2)|$ are as in Corollary 2.3. Hence, we obtain the upper bound in

$$
-CM^{-\alpha} M^{-\frac{1}{\pi M}} \leq \mu_M \ast \tilde{\mu}_M(x) - \rho_M(x) \leq CM^{-\alpha} M^{-\frac{1}{\pi M}}.
$$

To obtain the lower bound in (2.17) we repeat the proof with the following changes.

- We replace $\psi^u_{k,P}$ in (2.8) by the function $\psi^1_{k,P}$, defined below,

$$
\psi^1_{k,P}(x) = \begin{cases} F \left( \frac{Q}{A_{k+1}} (-A_{k+1} (1 - \frac{20}{Q}) - x) \right) & \text{for } -\frac{1}{2} \leq x \leq -\frac{20A_{k+1}}{Q}, \\
F \left( \frac{Q}{A_{k+1}} (x + \frac{20A_{k+1}}{Q}) \right) & \text{for } -\frac{20A_{k+1}}{Q} \leq x \leq 1/2,
\end{cases}
$$
where $F$ is defined in (2.7).

* Observe that similarly to (2.12), for $kM^{1-rac{10}{Q}} \leq m \leq (k+1)kM^{1-rac{1}{Q}}$, the inequalities
  
  \[ y - A_{k+1}(1 - \frac{10}{Q}) \leq (x_P + m^\alpha)^{1/\alpha} \leq y - \frac{10A_{k+1}}{Q} \text{ for } P \neq 0 \]

  and $\|m^\alpha - P\| \leq \frac{1}{Q}$ imply $x + [m^\alpha] = [y^\alpha]$. Hence, we have that for $P \neq 0$,

  \[ \Psi_{k,P}((x_P + m^\alpha)^{1/\alpha})\Psi_P(m^\alpha) \leq 1_A(x + [m^\alpha])\Psi_P(m^\alpha). \]

* We do not take into account $P = 0$ in the summation in (2.10).

We leave the details for the reader.

In order to prove the estimate for $\rho_M(x)$ for $0 < |x| \leq CM$ we argue as follows. Assume that the following equality holds

\[ x + [m^\alpha] = [(m + s)^\alpha] \text{ where } s > 0, x > 0. \]

Then we have

\[ x - 1 \leq (m + s)^\alpha - m^\alpha \leq x + 1 \]

and hence

\[ cM^{\alpha - 1} \leq csM^{\alpha - 1} \leq x \leq CsM^{\alpha - 1}. \]

Observe that for the increasing function $g(m) = (m + s)^\alpha - m^\alpha$ we have

\[ g(m + 1) - g(m) \approx sM^{\alpha - 2} \lesssim 1, \text{ for } |x| \leq CM, \]

and thus for a fixed $s$ there are at most $1 + s^{-1}M^{2-\alpha} \approx MX^{-1}$ different consecutive values of $m$ for which (2.18) can hold. Since moreover, by (2.19), $s$ has to satisfy $s \approx xM^{1-\alpha}$, the total number of solutions of (2.18) is bounded from above by

\[ CMx^{-1}(xM^{1-\alpha}) = CM^{2-\alpha}. \]

Hence we easily obtain $\rho_M(x) \leq CM^{-\alpha}$ for $0 < |x| < CM$ and the lemma follows. \hfill \Box

3. Proof of Theorem 1.1

Let

\[ M^*f(x) = \sup_{n>0} |\mu_{2n} * f(x)|. \]

With no loss of generality it suffices to show the weak type $(1,1)$ of $M^*$. We will use the argument of [5] and [11] adapted to our setting. Let $\lambda > 0$ and $f \in \ell^1(\mathbb{Z})$. Let $N = 2^n, n \in \mathbb{N}$. We consider the Calderón-Zygmund decomposition

\[ f = g + \sum f_{s,j} = \sum b_s + g, \]

where $|g| < \lambda$ and $b_s$ contains terms $f_{s,j}$ supported by $Q_{s,j}$ with $|Q_{s,j}| \approx 2^{os}$, $\sum_{s,j} |Q_{s,j}| \leq \lambda^{-1}\|f\|_{\ell^1}$ and $\|f_{s,j}\|_{\ell^1} \approx \lambda |Q_{s,j}|$. We do not assume $\int f_{s,j} = 0$, instead we decompose further each $b_s$ writing $b_s(x) = b_s^{(N)}(x) + B_s^{(N)}(x) + g_s^{(N)}(x)$, where $b_s^{(N)}(x) = \chi_{\{|b_s| > \lambda N\}}(x)b_s(x)$, and for $h_s^{(N)}(x) = b_s(x) - b_s^{(N)}(x)$ we have $B_s^{(N)}(x) = h_s^{(N)}(x) - g_s^{(N)}(x)$ and $g_s^{(N)}(x) = \sum_{j} \frac{\chi_{Q_{s,j}}(x)}{2^{as}} \int_{Q_{s,j}} h_s^{(N)}$. Consequently,

\[ f = g + \sum g_s^{(N)} + \sum B_s^{(N)} + \sum h_s^{(N)}. \]
Observe that \( \frac{1}{2\pi} \int_{Q_{s,j}} |b^{(N)}_s| \leq C \lambda \) and, since \( Q_{s,j} \) are mutually disjoint, we get
\[
(3.2) \quad |g(x)| + \sum_s |g^{(N)}_s(x)| \leq C \lambda.
\]
We have
\[
\{ x : \mu_N * |b^{(N)}_s|(x) > 0 \} = \bigcup_{m \approx N} ([m^2] + \{ x : |b^{(N)}_s|(x) > 0 \}).
\]
Thus,
\[
\{|x : \mu_N * |b^{(N)}_s|(x) > 0 \}| = \sum_{m \approx N} \{|x : |b^{(N)}_s|(x) > 0 \} = N \{|x : |b^{(N)}_s|(x) > \lambda N \}.
\]
Consequently (remember that \( N = 2^n \)),
\[
\sum_s \sum_{N - \text{dyadic}} \{|x : \mu_N * |b^{(N)}_s|(x) > 0 \}| \leq \sum_s \sum_{N - \text{dyadic}} N \{|x : |b_s|(x) > \lambda N \} \leq \sum_s \frac{1}{\lambda} \|b_s\|_{\ell^1} \leq \frac{1}{\lambda} \|f\|_{\ell^1}.
\]
Moreover, since for the fixed dyadic \( N \) the supports of \( B^{(N)}_s(x) \) are mutually disjoint, it is easy to see, for a fixed \( x \in Q_{s_0,j_0} \),
\[
\sum_{N \text{dyadic}} \sum_s N^{-1} B^{(N)}_{n-s}(x) \leq \left( \sum_{\{N - \text{dyadic} : N \geq |b_{s_0}(x)|\}} N^{-1} |b_{s_0}(x)|^2 \right) + \lambda^2 \sum_{N - \text{dyadic}} N^{-1} \chi_{\{\text{supp } b_{s_0}\}}(x) \leq C \lambda |b_{s_0}(x)| + \lambda^2 \chi_{\{\text{supp } b_{s_0}\}}(x) \leq C \lambda \sum_s (|b_s(x)| + C \lambda \chi_{\{\text{supp } b_s\}}(x)).
\]
Hence by (3.4), we have
\[
(3.5) \quad \lambda^{-2} \sum_s \sum_{N - \text{dyadic}} N^{-1} \|B^{(N)}_{n-s}\|_{\ell^2}^2 \leq \frac{C}{\lambda} \|f\|_{\ell^1}.
\]
We will use the following lemma,

**Lemma 3.6.** Let \( N = 2^n \), \( n \in \mathbb{N} \). For sufficiently small \( \delta > 0 \) we have the following estimates, see [5],
\[
(3.7) \quad \|\mu_N * B^{(N)}_{n-s}\|_{\ell^2}^2 \leq C \lambda \|B^{(N)}_{n-s}\|_{\ell^1} 2^{-\delta s} + 2^{-n} \|B^{(N)}_{n-s}\|_{\ell^2}^2
\]
and for \( s_1 > s_2 \),
\[
(3.8) \quad |\langle \mu_N * B^{(N)}_{n-s_1}, \mu_N * B^{(N)}_{n-s_2} \rangle| \leq C \lambda \|B^{(N)}_{n-s_2}\|_{\ell^1} 2^{-\delta s_1}.
\]
Thus we estimate the second summand in (3.10) by
\[ \mu_N \ast \tilde{\mu}_N(x) = C \rho_N(x) + 2^{-n} \delta_0(x) + O(2^{-(n(\alpha + \frac{1}{2}))}), \]
where \( \rho_N \) satisfies
\[ |\rho_N(x)| \leq \frac{C}{N^\alpha} = \frac{C}{2^n} \text{ and } \rho_N(0) = 0. \]
Moreover, for \( |x| > CM \) and \( |x + h| > CM \),
\[ |\rho_N(x + h) - \rho_N(x)| \leq \frac{C}{2^n} \frac{|h|}{2^{n \alpha}}. \]

Let for \( s \leq n \), \( \text{supp} B_s \subset [0, 4 \cdot 2^n] \) and \( \text{supp} \rho \subset [0, 4 \cdot 2^n] \). Denote by \( F_{s,j} \) the restriction of \( B_s \) to \( Q_s,j \). By the estimate \( \|B_s\|_1 \leq 2^{n \alpha} \) we have,
\[ A := |\langle \mu_N \ast B_{s_1}, \mu_N \ast B_{s_2} \rangle| = |\langle \mu_N \ast B_{s_1}, \mu_N \ast B_{s_2} \rangle| \]
\[ \leq 2^{-n \alpha} 2^{-\frac{2n}{n+1}} \|B_{s_1}\|_1 \|B_{s_2}\|_1 + \sum_{j} |\rho_N \ast F_{s_1,j} \ast B_{s_2}| \]
\[ + \sum_{j} |\rho_N \ast F_{s_2,j} \ast B_{s_1}|. \]

Observe that for \( s_1 \neq s_2 \) the supports of \( B_{s_1} \) and \( B_{s_2} \) are disjoint and consequently the third summand is equal to zero. Consider the second summand in (3.10). By the regularity estimate (3.9) and the fact that \( \int F_{n-s,j} = 0 \) we get in a standard way
\[ |\rho_N \ast F_{n-s,j}(x)| \leq \frac{2^{(n-s)\alpha}}{2^{n \alpha}} \|F_{n-s,j}\|_{\ell^1} \]
for \( |x - x_{s,j}| > Cn + 2(n-s)\alpha \), where \( x_{s,j} \) denotes the center of \( Q_{s,j} \). Moreover, for any \( x, |\rho_N \ast F_{n-s,j}(x)| \leq N^{-\alpha} \|F_{n-s,j}\|_{\ell^1} \leq \lambda N^{-\alpha} 2^{(n-s)\alpha} \). Consequently, we have
\[ \sum_j |\rho_N \ast F_{n-s,j}(x)| \leq \sum_{\{j:|x - x_{s,j}| \leq Cn + 2(n-s)\alpha\}} 2^{-n \alpha} \|F_{n-s,j}\|_{\ell^1} \]
\[ + 2^{-s} 2^{n \alpha} \|F_{n-s,j}\|_{\ell^1} \sum_{\{j:|x - x_{s,j}| > Cn + 2(n-s)\alpha\}} 2^{-n \alpha} \|F_{n-s,j}\|_{\ell^1} \]
\[ \leq C \lambda \max\{N^{-\alpha} 2^{-s} \alpha \} + C \lambda 2^{-s} \alpha \leq C \lambda 2^{-s} \alpha. \]

Thus we estimate the second summand in (3.10) by
\[ \|B_{s_1}\|_1 \| \|F_{n-s,j}\|_{\ell^1} \leq C \lambda 2^{-s} \alpha \|B_{s_2}\|_{\ell^1}. \]

Finally, we get
\[ A \leq C \lambda 2^{-s} \alpha \|B_{n-s}\|_1 + C \lambda \|B_{n-s}\|_1 2^{-\frac{n\alpha}{n+1}} \leq C \lambda \|B_{n-s}\|_1 2^{-s} \alpha. \]

The assumption \( \text{supp} B_{n-s} \subset [0, 4 \cdot 2^n] \) can be removed in a standard way. The estimates (3.7), (3.8) follow. \[ \square \]
Now we are ready to give

**Proof of Theorem 1.1.** Using Lemma 3.6 we have,

\[
\lambda^2 |\{ x : \max_{N} | \sum_{s \geq 0} B_{n-s}^{(N)} \ast \mu_N(x) \} \geq c \lambda \} | \leq \sum_{x} \max_{N} \left| \sum_{s \geq 0} B_{n-s}^{(N)} \ast \mu_N(x) \right|^2
\]

\[
\leq \sum_{x} \sum_{N} \left| \sum_{s \geq 0} B_{n-s}^{(N)} \ast \mu_N(x) \right|^2
\]

(3.11)\[
\leq \sum_{N,s \geq 0} \| \mu_N \ast B_{n-s}^{(N)} \|_{l^2} + 2 \sum_{N,s_1 > s_2} |\langle \mu_N \ast B_{n-s_1}^{(N)}, \mu_N \ast B_{n-s_2}^{(N)} \rangle | \leq \sum_{N,s \geq 0} C \lambda \| B_{n-s}^{(N)} \|_{l^2} \leq 2 - n \| B_{n-s}^{(N)} \|_{l^2} + 2 \sum_{N,s_1 > s_2} C \lambda \| B_{n-s_2}^{(N)} \|_{l^2} \leq \sum_{N,s \geq 0} C \lambda \| f \| _{l^1},
\]

where the second summand in the last inequality is estimated by (3.5).

By (3.1) we have,

\[
\{ \sup_{N} |\mu_N \ast f(x) | > 4 C \lambda \} \subset \{ \sup_{N} |\mu_N \ast \left( | g | + \sum_{s} | g_s^{(N)} | \right)(x) | > C \lambda \}
\]

\[
\cup \{ \sup_{N} |\mu_N \ast \left( \sum_{s > 0} | B_{n-s}^{(N)} |(x) \right) | > C \lambda \} \cup \{ \sup_{N} |\mu_N \ast \left( \sum_{s > 0} | B_{n+s}^{(N)} |(x) \right) | > C \lambda \}
\]

\[
\cup \{ \sup_{N} \sum_{s} |\mu_N \ast | b_s^{(N)} | | > C \lambda \} =: S_1 \cup S_2 \cup S_3 \cup S_4.
\]

In the above sum, by (3.2), the first set is empty if the constant C is sufficiently large. Since \( \sup \| \mu_N \ast B_{n-s}^{(N)} \| \subset \cup_{s,j} Q_{s,j}^{**} \) for \( s \geq n \), the set \( S_1 \) is a subset of \( \cup_{s,j} Q_{s,j}^{**} \), and consequently \( |S_1| \leq \| \cup_{s,j} Q_{s,j}^{**} \| \leq C \sum_{s,j} |Q_{s,j}| \leq \frac{C}{n} \| f \| _{l^1} \). Moreover by (3.11) we have \( |S_2| \leq \frac{C}{n} \| f \| _{l^1} \). The set \( S_4 \subset \cup_{N,s} \{ \mu_N \ast | b_s^{(N)} |(x) > 0 \} \). Hence by (3.3) \( |S_4| \leq \sum_{N,s} \{ |\mu_N \ast | b_s^{(N)} |(x) > 0 \} \} \leq \frac{C}{n} \| f \| _{l^1} \). The theorem follows. \( \square \)

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**References**


Institute of Mathematics, Wroclaw University, Plac Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: urban@math.uni.wroc.pl

Same address in Wroclaw
E-mail address: zenek@math.uni.wroc.pl