

FIBRATIONS WITH CONSTANT SCALAR CURVATURE KÄHLER METRICS AND THE CM-LINE BUNDLE

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ABSTRACT. Let $\pi : X \rightarrow B$ be a holomorphic submersion between compact Kähler manifolds of any dimensions, whose fibres and base have no non-zero holomorphic vector fields and whose fibres admit constant scalar curvature Kähler metrics. This article gives a sufficient topological condition for the existence of a constant scalar curvature Kähler metric on X . The condition involves the CM-line bundle—a certain natural line bundle on B —which is proved to be nef. Knowing this, the condition is then implied by $c_1(B) < 0$. This provides infinitely many Kähler manifolds of constant scalar curvature in every dimension, each with Kähler class arbitrarily far from the canonical class.

1. Introduction

1.1. Background. In [4] Calabi proposed that, when one exists, a constant scalar curvature Kähler (cscK) metric should provide a canonical representative for a given Kähler class. Since this suggestion, much work has focused on the topic. The general existence theory has been looked at in depth, motivated by a suggestion of Yau [21] relating the existence of a Kähler–Einstein metric on a Fano manifold to the stability of the canonical polarisation in the sense of geometric invariant theory. Later, Tian [18] introduced the notion of K-stability, and suggested that the existence of a cscK metric in the first Chern class of an ample line bundle L should be equivalent to the K-stability of the polarisation defined by L . Finally, Donaldson [6] gave an interpretation of the conjecture in terms of symplectic geometry and moment maps. See [8] for a precise statement of the conjecture, including a definition of K-stability.

The difficulty, from the analytic viewpoint, in determining whether or not a cscK metric exists is that the resulting PDE is fourth order and fully non-linear. Despite this, some examples have been found, but only in situations with prescribed geometry which is then used to get a handle on the PDE. For two recent such examples see [1, 17].

This paper adds to the list of examples, the prescribed geometry here being that of a fibration. CscK metrics are found on the total space of a holomorphic submersion $X \rightarrow B$ between compact Kähler manifolds. Even in this restricted context, however, a clear relationship between the existence of a cscK metric and stability arises. Moreover, the varying moduli of the fibres plays an important rôle. One is led to a natural line bundle on B —the so-called CM-line bundle—which is shown to be nef on the moduli space of K-semi-stable polarised varieties (ignoring the technical issues of whether such a space exists).

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1.2. Overview of the results. Let $\pi: X \rightarrow B$ be a holomorphic submersion between compact Kähler manifolds and let $L \rightarrow X$ be a relatively ample line bundle. There is a natural line bundle, called the CM-line bundle, on B which was introduced by Tian [19]. It is defined as follows.

Let Y be a fibre, $n = \dim Y$ and let s denote the constant

$$(1.1) \quad s = \frac{nc_1(Y)c_1(L)|_Y^{n-1}}{c_1(L)|_Y^n}.$$

(In terms of differential geometry, given a Kähler metric in $c_1(L)$, its restriction to a fibre has average scalar curvature $2\pi s$. In terms of algebraic geometry, s is essentially the second coefficient in the Hilbert polynomial of $L|_Y$.) Let $K_{X/B}$ denote the relative canonical bundle, and let \mathcal{E} denote the virtual bundle

$$\mathcal{E} = (n+1)(K_{X/B}^* - K_{X/B}) \otimes (L - L^*)^n - s(L - L^*)^{n+1}.$$

The CM-line bundle is then $L_{\text{CM}} = [\det(\pi_! \mathcal{E})]^*$. The bundle makes sense when π is a flat proper morphism of varieties. This level of generality is not needed for the result concerning existence of cscK metrics.

In fact, only the first Chern class of L_{CM} is used in this article. It can be defined without recourse to virtual bundles or direct images: by Grothendieck–Riemann–Roch,

$$c_1(L_{\text{CM}}) = 2^{n+1}\pi_* \left[((n+1)c_1(K_{X/B}) + sc_1(L)) c_1(L)^n \right].$$

Notice that $c_1(L_{\text{CM}})$ is unchanged when L is twisted by a line bundle pulled back from B . This is because after pushing forward the factor involving s cancels other unwanted terms.

The relation of L_{CM} to K-stability and the existence of cscK metrics has already been studied by Tian, [18] and Paul–Tian [16, 15]. The fact that $c_1(L_{\text{CM}})$ is well behaved when all fibres of X admit a cscK metric is noted in Fujiki–Schumacher [10].

For notational convenience, let α denote the following class in $H^{1,1}(B)$:

$$(1.2) \quad \alpha = \frac{c_1(L_{\text{CM}})}{2^{n+1}(n+1)\pi_*(c_1(L)^n)}.$$

The main theorem proved here is the following. (See section 2 for the proof.)

Theorem 1.1. *Let $\pi: X \rightarrow B$ be a holomorphic submersion between compact Kähler manifolds whose fibres and base admit no non-zero holomorphic vector fields. Let L be a relatively ample line bundle on X such that the restriction of $c_1(L)$ to each fibre admits a cscK metric. Suppose, moreover, that $\alpha - c_1(B) \geq 0$.*

Then, for all sufficiently large r , the class

$$\kappa_r = c_1(L) + r\pi^*\kappa_B$$

contains a cscK metric where, if $\alpha - c_1(B) = 0$, κ_B is any Kähler class on the base, whilst if $\alpha - c_1(B) > 0$, then $\kappa_B = \alpha - c_1(B)$.

In checking the condition $\alpha - c_1(B) \geq 0$, it is helpful to know about the positivity of α itself. This is provided by the following result. (See section 3 for the proof.) Here, a line bundle $E \rightarrow Y$ is *asymptotically Hilbert stable* if for all sufficiently large m , E^m gives a Hilbert stable embedding of Y .

Theorem 1.2. *Let $\pi: X \rightarrow B$ be a flat proper morphism of varieties. Let L be a relatively ample line bundle on X whose restriction to a generic fibre is asymptotically Hilbert semi-stable.*

Then L_{CM} is nef; that is, $c_1(L_{\text{CM}})$ evaluates non-negatively over any curve in B .

It is a result due to Donaldson [7] that if $E \rightarrow Y$ is an ample line bundle over a compact Kähler manifold with $\text{Aut}(Y, E)$ discrete (modulo scalars), then the existence of a cscK metric in $c_1(E)$ implies that E is asymptotically Hilbert stable (and hence semi-stable). Combining this with Theorems 1.1 and 1.2 and the fact that nef plus positive is positive gives the following corollary.

Corollary 1.3. *Let $\pi: X \rightarrow B$ be a holomorphic submersion of compact Kähler manifolds whose fibres and base admit no non-zero holomorphic vector fields. Let L be a relatively ample line bundle on X such that the restriction of $c_1(L)$ to each fibre admits a cscK metric. Suppose, moreover, that $c_1(B) < 0$.*

Then, for all sufficiently large r , the class

$$\kappa_r = c_1(L) + r\pi^*(\alpha - c_1(B))$$

contains a cscK metric.

The condition $c_1(B) < 0$ is straight forward to check and general enough to provide a large number of examples of cscK metrics. Some of these are described in section 4, giving infinitely many Kähler manifolds of constant scalar curvature in each dimension, all with Kähler class arbitrarily far from the canonical class.

When $c_1(L_{\text{CM}})$ is actually ample, one can say something in the case when $c_1(B) = 0$. In this direction, Fujiki–Schumacher [10] prove the following.

Theorem 1.4 (Fujiki–Schumacher). *Let $\pi: X \rightarrow B$ be a holomorphic submersion whose fibres admit no non-zero holomorphic vector fields. Let $L \rightarrow X$ be a relatively ample line bundle such that the restriction of $c_1(L)$ to each fibre admits a cscK metrics. Suppose, moreover, that π is not trivial over any curve in B . Then $c_1(L_{\text{CM}})$ is ample on B .*

(Here π is trivial over a curve means that all fibres of π over that curve are biholomorphic as polarised varieties.) Combining this with Theorem 1.1 gives the following.

Corollary 1.5. *Let $\pi: X \rightarrow B$ be a holomorphic submersion of compact Kähler manifolds whose fibres and base admit no non-zero holomorphic vector fields. Let L be a relatively ample line bundle on X such that the restriction of $c_1(L)$ to each fibre admits a cscK metric. Suppose, moreover, that π is not trivial on any curve in B and that $c_1(B) = 0$.*

Then, for all sufficiently large r , the class

$$\kappa_r = c_1(L) + r\pi^*\alpha$$

contains a cscK metric.

Note that neither of Theorems 1.2 and 1.4 imply the other; whilst ample is stronger than nef, Theorem 1.2 makes no mention of cscK metrics and applies both when the fibres are singular and have automorphisms.

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2. Proof of Theorem 1.1

Assume throughout this section that $\pi: X \rightarrow B$ is a holomorphic submersion between compact Kähler manifolds, that the fibres and base of π have no non-zero holomorphic vector fields, that L is relatively ample and that the restriction of $c_1(L)$ to each fibre admits a cscK metric. Moreover, assume that $\alpha - c_1(B) \geq 0$. If $\alpha - c_1(B) = 0$, let κ_B denote any Kähler class on B ; if $\alpha - c_1(B) > 0$, let $\kappa_B = \alpha - c_1(B)$.

For sufficiently large r , $\kappa_r = c_1(L) + r\pi^*\kappa_B$ is a Kähler class on X . The first step is to find a Kähler representative whose fibrewise restriction is cscK. Begin with any Kähler form $\omega' \in \kappa_{r_0}$ for some large fixed r_0 .

For $b \in B$, write $Y_b = \pi^{-1}(b)$ and let ω_b be a cscK metric in $c_1(L)|_{Y_b}$. Since $H^0(TY_b) = 0$, a theorem of Donaldson [7] says that ω_b is in fact unique. For each b , there is a unique function $\phi_b \in C^\infty(Y_b)$ with mean-value zero (with respect to ω_b) and such that $\omega'|_{Y_b} + i\bar{\partial}\partial\phi_b = \omega_b$. Provided the ϕ_b are smooth in b , they fit together to give a function $\phi \in C^\infty(X)$; then $\omega'' = \omega' + i\bar{\partial}\partial\phi$ is a $(1, 1)$ -form in κ_{r_0} whose fibrewise restriction is cscK. The smoothness of ω'' is provided by the following lemma.

Lemma 2.1. *The functions ϕ_b depend smoothly on b .*

Proof. This is essentially a standard result in the theory of elliptic PDEs. Choose a local trivialisation for π over some disc $D \subset B$, $X|_D \cong D \times Y$ as smooth manifolds. By restriction in this trivialisation, the complex structure and Kähler form ω' on X give a smooth family of Kähler structures (J_b, ω'_b) on Y parameterised by $b \in D$.

Define a map $S: D \times C^\infty(Y) \rightarrow C^\infty(Y)$ by

$$S(b, \phi) = \text{Scal}(\omega'_b + i(\bar{\partial}\partial)_b\phi)$$

where $(\bar{\partial}\partial)_b$ is defined with respect to J_b . S extends to a smooth map $D \times L^2_{k+4}(Y) \rightarrow L^2_k(Y)$. (S is smooth in the D factor because $\text{Scal}(\omega, J)$ depends smoothly on ω and J , see, for example, section 2.2 in [9].)

By definition, $S(b, \phi_b)$ is a constant. The linearisation of S with respect to ϕ at such a point is given by

$$\mathcal{D}_b^*\mathcal{D}_b: C^\infty(Y) \rightarrow C^\infty(Y).$$

Here \mathcal{D}_b is defined by

$$\mathcal{D}_b(\psi) = \bar{\partial}_b(h_b(\psi))$$

where $h_b(\psi)$ is the Hamiltonian vector field of ψ with respect to ω'_b and $\bar{\partial}_b$ is the $\bar{\partial}$ -operator on TY determined by J_b . The map \mathcal{D}_b^* is the L^2 -adjoint of \mathcal{D} with respect to the metric ω'_b . (The linearisation of the scalar curvature map is computed in several places in the literature; see, for example, [9].)

Now $\mathcal{D}_b^*\mathcal{D}_b$ is elliptic (again, see [9]), self adjoint and so has index zero. Since the fibres of π have no non-zero holomorphic vector fields, $\ker \mathcal{D}_b^*\mathcal{D}_b$ is the constant functions. Hence $\mathcal{D}_b^*\mathcal{D}_b$ is an isomorphism modulo the constants. By the implicit function theorem, the map $b \mapsto \phi_b$ is a smooth map $D \rightarrow L^2_k(Y)$ for any k . By

Sobolev embedding, it is a smooth map $D \rightarrow C^r(Y)$ for any r . Hence ϕ_b is smooth in b . \square

Now $\omega'' = \omega' + i\bar{\partial}\partial\phi$ needn't be Kähler as it may be degenerate transverse to the fibres. To fix this, take a Kähler form $\omega_B \in \kappa_B$; for sufficiently large r , the form $\omega = \omega'' + (r - r_0)\pi^*\omega_B$ is a Kähler metric in κ_r whose fibrewise restriction is cscK.

The idea is that for large r , the geometry is dominated by that of the fibres which are cscK. One might hope then to be able to perturb the metric ω to a genuine cscK metric. A previous paper [9] considers this problem in detail. In the case where X is a surface it solves it completely; in higher dimensions it reduces it to solving a certain PDE for a metric on B . To describe this PDE, some notation is required.

The fibrewise cscK metrics define a Hermitian structure in the vertical tangent bundle V and hence in the relative canonical bundle $K_{X/B} = \Lambda^{\max}V^*$; denote its curvature by F . Notice that the restriction to a fibre of $-iF$ is the Ricci form of that fibre with its cscK metric.

The metric ω defines a vertical-horizontal decomposition of TX . Let F_H denote the horizontal-horizontal component of F with respect to this splitting. Define a form $a \in \Omega^{1,1}(B)$ by taking the fibrewise mean value of iF_H with respect to ω . More precisely, the differential $D\pi$ identifies the horizontal distribution restricted to Y_b with $Y_b \times T_bB$; using this identification, on Y_b , $F_H: Y_b \rightarrow \Lambda^{1,1}T_b^*B$ is a vector valued function; then $a_b \in \Lambda^{1,1}T_b^*B$ is given by

$$a_b = \frac{\int_{Y_b} iF_H \omega_b^n}{\int_{Y_b} \omega_b^n}.$$

Theorem 2.2 ([9]). *Let $\pi: X \rightarrow B$ be a fibration and $L \rightarrow X$ a relatively ample line bundle as above. Suppose, moreover, that there is a Kähler metric ω_B on the base solving*

$$(2.1) \quad \text{Scal}(\omega_B) - \text{tr}_{\omega_B} a = \lambda,$$

for some constant λ , and that there are no other cohomologous solutions to this equation. Then, for all sufficiently large r , the class $\kappa_r = \kappa + r\pi^*[\omega_B]$ contains a cscK metric.

The difficulty with applying Theorem 2.2 is that the PDE for ω_B is as awkward to solve as the cscK equation. However, the *un-traced* version,

$$(2.2) \quad \rho(\omega_B) - a = \lambda\omega_B,$$

(where $\rho(\omega)$ denotes the Ricci form of ω) is essentially the complex Monge–Ampere equation. When $\lambda \leq 0$ this has been solved by Aubin [3] and Yau [20]. Before describing this, it is first necessary to give a cleaner description of a . (See (1.1) and (1.2) for the definitions of s and α .)

Lemma 2.3.

$$a = \frac{1}{\pi_*(\omega^n)} \pi_* \left[\left(iF + \frac{2\pi s}{n+1} \omega \right) \wedge \omega^n \right]$$

In particular, a is closed and $[a] = 2\pi\alpha$.

Proof. Write F_V and F_H for the purely vertical and purely horizontal components of F respectively. Since $-iF_V$ is the Ricci-form of ω_b which is cscK, $-niF_V \wedge \omega_b^{n-1} = 2\pi s \omega_b^n$. So,

$$\begin{aligned} \pi_*(iF_V \wedge \omega^n) &= \pi_*(niF_V \wedge \omega_b^{n-1} \wedge \omega_H) \\ &= -2\pi s \pi_*(\omega_b^n \wedge \omega_H) \\ &= -\frac{2\pi s}{n+1} \pi_*(\omega^{n+1}) \end{aligned}$$

Hence,

$$\begin{aligned} \pi_*(iF \wedge \omega^n) &= \pi_*(iF_H \wedge \omega^n) + \pi_*(iF_V \wedge \omega^n) \\ &= \pi_*(\omega^n) a - \frac{2\pi s}{n+1} \pi_*(\omega^{n+1}) \end{aligned}$$

which gives the formula for a .

To deduce that $[a] = 2\pi\alpha$, observe that $[iF] = 2\pi c_1(K_{X/B})$, whilst $[\omega] = \kappa_r = c_1(L) + \pi^* \kappa_B$. Twisting L by bundles pulled back from B leaves α unchanged, so α can be computed using κ_r in place of $c_1(L)$. \square

Everything is now in place to complete the proof.

Proof of Theorem 1.1. Let $\omega'_B \in \kappa_B$ be a Kähler metric on B . Given $\phi \in C^\infty(B)$, let $\omega_B = \omega'_B + i\bar{\partial}\partial\phi$. The aim is to find ϕ such that ω_B solves equation (2.2). The reduction to the complex Monge–Ampere equation is standard; brief details are given below, for more information see, for example, the account in [12].

Since $\rho(\omega'_B) - a \in 2\pi(c_1(B) - \alpha)$, there exists f such that

$$\rho(\omega'_B) - a = \lambda \omega'_B + i\bar{\partial}\partial f$$

where $\lambda = 0$ if $a - c_1(B) = 0$ and $\lambda = -2\pi$ if $a - c_1(B) > 0$. Define $M(\phi) = \omega_B^r / \omega_B'^r$ where $r = \dim B$. Then $\rho(\omega_B) = \rho(\omega'_B) + i\bar{\partial}\partial \log M(\phi)$. Hence ω_B solves (2.2) if and only if ϕ solves $\log M(\phi) = \lambda\phi - f$. By the famous theorems of Aubin [3] ($\lambda < 0$) and Yau [20] ($\lambda = 0$), this has a unique solution. Hence there is a unique solution ω_B to (2.2) in $[\omega'_B]$.

It remains to show that ω_B is the unique solution in $[\omega'_B]$ to the *traced* equation (2.1) appearing in Theorem 2.2. Suppose ω_B solves $\text{Scal} - \text{tr} a = \lambda$, with $\lambda \omega_B \in a - c_1(B)$. Now $\rho(\omega_B) - a$ is $\bar{\partial}$ -closed and it follows from the Kähler identity $[\text{tr}, \partial] = i\bar{\partial}^*$ and the fact that $\text{Scal} - \text{tr} a$ is constant that it is also $\bar{\partial}^*$ -closed. Hence $\rho(\omega_B) - a$ and $\lambda \omega_B$ are harmonic representatives for the same class and so, by Hodge theory, are equal. That is, ω_B solves (2.2) also.

Theorem 1.1 now follows from Theorem 2.2. \square

3. Non-negativity of the CM-line bundle

This section gives the proof of Theorem 1.2. The key step is provided by the following theorem of Cornalba–Harris [5].

Theorem 3.1 (Cornalba–Harris). *Let $\pi: X \rightarrow B$ be a flat proper morphism of varieties where B is one-dimensional and X is $(n+1)$ -dimensional. Let $E \rightarrow X$ be a relatively very ample line bundle whose restriction to at least one fibre gives a Hilbert semi-stable embedding.*

Then

$$hc_1(E)^{n+1} - (n + 1)c_1(E)^n \pi^* c_1(\pi_* E) \geq 0$$

where h is the rank of $\pi_* E$.

Theorem 1.2 is proved by looking at the asymptotics of this result.

Proof of Theorem 1.2. Recall that $\pi: X \rightarrow B$ is a flat proper morphism of varieties, L is a relatively ample line bundle whose restriction to a generic fibre is asymptotically semi-stable. The aim is to show that α (defined in (1.2)) is nef. It suffices to consider the case when B is a curve.

For sufficiently large m , L^m is relatively very ample and makes at least one of the fibres Hilbert semi-stable. By Theorem 3.1, with $E = L^m$,

$$(3.1) \quad h(m)m^{n+1}c_1(L)^{n+1} - m^n(n + 1)c_1(L)^n \pi^* c_1(\pi_*(L^m)) \geq 0,$$

where $h(m) = \text{rank } \pi_*(L^m)$ is the Hilbert polynomial of the restriction of L to a fibre.

By Grothendieck–Riemann–Roch,

$$c_1(\pi_* L^m) = \pi_* \left(\frac{c_1(L)^{n+1}}{(n + 1)!} m^{n+1} - \frac{c_1(L)^n c_1(K_{X/B})}{2n!} m^n + \dots \right).$$

On the other hand,

$$h(m) = \pi_*(c_1(L)^n) \left(\frac{1}{n!} m^n + \frac{s}{2n!} m^{n-1} + \dots \right).$$

Apply π_* to (3.1) and collect terms. The m^{2n+1} -coefficient is zero and so the leading term is m^{2n} . The coefficient of m^{2n} is $\alpha/(2(n + 1)!)$. Since π_* of (3.1) is positive for all large m , its leading coefficient must be positive. \square

One situation in which the positivity of $c_1(L_{CM})$ is completely understood is that of curves of genus at least two. In this case, all polarisations are multiples of the canonical polarisation and $c_1(CM)$ is essentially $\pi_*(c_1(K_{X/B})^2)$, the first tautological class, whose positivity properties are completely known. It is not ample on the Hilbert scheme, even over the stable locus ([11], page 313). On the other hand, it *is* ample when considered over the smooth locus ([11], page 312).

This tallies with the higher dimensional result of Fujiki–Schumacher [10], which shows that, provided the fibres have no holomorphic vector fields, $c_1(L_{CM})$ is ample over the part of the smooth locus which admits a cscK metric. With this in mind, it seems reasonable to guess that $c_1(L_{CM})$ is ample on the smooth, asymptotically semi-stable locus of the Hilbert scheme.

Another question is whether or not the stability hypothesis in Theorem 1.2 is necessary. Certainly *some* stability hypothesis is required. In [5] Cornalba and Harris discuss a family of Hilbert unstable varieties for which their inequality is violated. Is it possible, however, to replace asymptotic Hilbert semi-stability with K-semi-stability?

4. Examples

A large class of submersions satisfying the hypotheses of Corollary 1.3 are provided by the iterated surface bundles of Morita. Their construction is given in detail in chapter 4 of [14].

In what follows, a Σ_g -bundle is a holomorphic submersion between complex manifolds whose fibres are curves of genus g . For each positive integer k , define the set \mathcal{C}_k of connected compact complex k -folds recursively as follows: the only element of \mathcal{C}_0 is a single point; in general, \mathcal{C}_{k+1} is defined to be the set consisting of any finite covering of the total space of a Σ_g -bundle with $g \geq 2$ and whose base belongs to \mathcal{C}_k . Let $\mathcal{C} = \bigcup \mathcal{C}_k$ and call its members *iterated surface bundles*.

Given a Σ_g -bundle $\pi: X \rightarrow B$ with $g \geq 2$ whose total space is an iterated surface bundle, Morita produces a finite covering $B' \rightarrow B$, and a $\Sigma_{g'}$ -bundle $X' \rightarrow B'$ where $g' = m^2g - \frac{1}{2}m(m+1) + 1$ is again at least 2. Moreover, the moduli of the fibres of X' are not constant. (In [14], Morita shows that certain characteristic classes of surface bundles are non-zero on X' ; these classes vanish on isotrivial families.) It remains to check that these fibrations satisfy the hypotheses of Corollary 1.3.

If $\pi: X \rightarrow B$ is a non-constant family of smooth curves of genus at least 2, then the relative canonical bundle $K_{X/B}$ is ample (see [11], page 309). If the canonical bundle of B is also ample then so is the canonical bundle of X . That is, if $c_1(B) < 0$ then $c_1(X) < 0$.

The short exact sequence $0 \rightarrow V \rightarrow TX \rightarrow \pi^*TB \rightarrow 0$, where V denotes the vertical tangent bundle, gives a long exact sequence in cohomology $0 \rightarrow H^0(X, V) \rightarrow H^0(X, TX) \rightarrow H^0(X, \pi^*TB) \rightarrow \dots$. Now $H^0(X, V) = 0$ as the fibres admit no non-zero holomorphic vector fields, whilst $H^0(X, \pi^*TB) = H^0(B, \pi_*\pi^*TB) = H^0(B, TB)$, so $H^0(B, TB) = 0$ implies that $H^0(X, TX) = 0$. Similarly, finite covers of B have no holomorphic vector fields.

Therefore repeatedly applying Morita's construction for various choices of m and g to Σ_g -bundles over a point produces infinitely many Kähler manifolds X of arbitrary dimension all of which satisfy the hypotheses of Corollary 1.3.

It should be pointed out that there are other ways of producing infinitely many cscK manifolds in each dimension, the most obvious being products. Kähler–Einstein metrics give non-trivial examples. For examples which are cscK but not Kähler–Einstein, the Kähler–Einstein metrics can be deformed via a theorem of LeBrun and Simanca [13] which says that, in the absence of holomorphic vector fields, if a Kähler class admits a cscK representative, then nearby Kähler classes also admit a cscK representative. Alternatively, a result of Arezzo and Pacard [2] states that, in the absence of holomorphic vector fields, a cscK metric on X gives a cscK metric on the blow-up of X at a point, providing yet more examples. All these approaches, however, ultimately rely on deforming Kähler–Einstein metrics. By contrast, the examples produced via Theorem 1.1 lie in classes arbitrarily far from the canonical class and are unrelated to any Kähler–Einstein metric on X or B .

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