A NOTE ON EXISTENCE AND NON-EXISTENCE OF HORIZONS IN SOME ASYMPTOTICALLY FLAT 3-MANIFOLDS

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Abstract. We consider asymptotically flat manifolds of the form \((S^3 \setminus \{P\}, G^4 g)\), where
\(G\) is the Green’s function of the conformal Laplacian of \((S^3, g)\) at a point \(P\). We show
if \(Ric(g) \geq 2g\) and the volume of \((S^3, g)\) is no less than one half of the volume of the
standard unit sphere, then there are no closed minimal surfaces in \((S^3 \setminus \{P\}, G^4 g)\). We
also give an example of \((S^3, g)\) where \(Ric(g) > 0\) but \((S^3 \setminus \{P\}, G^4 g)\) does have closed
minimal surfaces.

1. Introduction

Let \((N^3, g, p)\) be an initial data set satisfying the dominant energy constraint con-
dition in general relativity. It is a fascinating question to ask under what conditions
an apparent horizon (of a back hole) exists in \((N^3, g, p)\). Here an apparent horizon is
a 2-surface \(\Sigma^2 \subset N^3\) satisfying
\[
H_\Sigma = Tr_\Sigma p,
\]
where \(H_\Sigma\) is the mean curvature of \(\Sigma\) in \(N\) and \(Tr_\Sigma p\) is the trace of the restriction of
\(p\) to \(\Sigma\).

A fundamental result of Schoen and Yau states that matter condensation causes
apparent horizons to be formed [11]. Their result is remarkable not only because
it provides a general criteria to the existence question, but also because it leads to
a refined problem – besides matter fields, what is the pure effect of gravity on the
formation of apparent horizons?

To analyze this refined problem, one considers an asymptotically flat initial data
set \((N^3, g, p)\) in a vacuum spacetime. As the first step, one assumes \((N^3, g, p)\) is time-
symmetric (i.e. \(p \equiv 0\)). In this context, an apparent horizon is simply a minimal
surface, and the relevant topological assumption is that \(N^3\) is diffeomorphic to \(\mathbb{R}^3\).
(If \(N^3\) has nontrivial topology, a closed minimal surface always exists by [8].)

There is a geometric construction of such an initial data set. Let \([g]\) be a conformal
class of metrics on the three-sphere \(S^3\). Recall the Yamabe constant of \((S^3, [g])\) is
defined by
\[
Y(S^3, [g]) = \inf_{v \in W^{1,2}(S^3)} \frac{\int_M [8|\nabla v|^2_g + R(g)v^2]dV_g}{(\int_M v^6dV_g)^{1/3}},
\]
where \(R(g)\) is the scalar curvature of \(g\). If \(Y(S^3, [g]) > 0\), there exists a positive
Green’s function \(G\) of the conformal Laplacian \(8\triangle_g - R(g)\) at any fixed point \(P \in S^3\).
Consider the new metric \(G^4 g\) on \(S^3 \setminus \{P\}\), it is easily checked that \((S^3 \setminus \{P\}, G^4 g)\) is

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asymptotically flat with zero scalar curvature. One basic fact about this construction is that the blowing-up manifold \( (S^3 \setminus \{ P \}, G^4 g) \), up to a constant scaling, depends only on the conformal class \([g]\). Precisely, if one replaces \( g \) by another metric \( \tilde{g} \in [g] \) and let \( \tilde{G} \) be the Green’s function associated to \( \tilde{g} \), then the metric \( \tilde{G}^4 \tilde{g} \) differs from \( G^4 g \) only by a constant multiple. Therefore, it is of interest to seek conditions on \([g]\) that determine whether \( (S^3 \setminus \{ P \}, G^4 g) \) has a horizon.

So far, no such a conformal invariant condition has been found. However, there are results where conditions in terms of a single metric are given. In [1], Beig and Ó Murchadha studied the behavior of a critical sequence, i.e. a sequence of metrics \( \{ g_n \} \) on \( S^3 \) converging to a metric \( g_0 \) with zero scalar curvature. They showed the blowing-up manifold \( (S^3 \setminus \{ P \}, G^4 g_n) \) has a horizon for sufficiently large \( n \). Their idea was further explored by Yan [12]. Given a metric \( g \) on \( S^3 \), assuming the diameter of \( (S^3, g) \leq D \), the volume of \( (S^3, g) \geq V \) and the Ricci curvature of \( g \) satisfies 
\[ \text{Ric}(g) \geq \mu g, \] 
Yan showed that, for any \( r > \frac{3}{4} \), there exists a small positive number \( \delta = \delta(\mu, V, D, r) \leq 1 \) such that, if \( R(g) > 0 \) and \( \| R(g) \|_{L^\infty(S^3, g)} < \delta \), then the blowing-up manifold \( (S^3 \setminus \{ P \}, G^4 g) \) has a horizon.

One question arising from Yan’s theorem is whether a positive Ricci curvature metric on \( S^3 \) can produce a blowing-up manifold with a horizon, as it is unclear whether Yan’s theorem could be applied when \( \mu > 0 \). Another motivation to this question is, as a positive Ricci curvature metric can be deformed to the standard metric on \( S^3 \) through metrics of positive Ricci curvature, it is of potential interest to study how the horizon disappears in the corresponding deformation of the blowing-up manifold if it exists initially.

In this paper, we focus on conformal classes of metrics with a positive Ricci curvature metric. Our main result is the observation of a volume condition which guarantees non-existence of horizons in the blowing-up manifold. Throughout the paper, \( S^3 \) denotes \( S^3 \) with the standard metric of constant curvature +1.

**Theorem** Let \([g]\) be a conformal class of metrics on \( S^3 \) which has a metric of positive Ricci curvature. Consider 
\[ V_{\text{max}}(S^3, [g]) = \sup_{\tilde{g} \in [g]} \{ \text{Vol}(S^3, \tilde{g}) \mid \text{Ric}(\tilde{g}) \geq 2\tilde{g} \}, \]
where \( \text{Vol}(\cdot) \) is the volume functional. If 
\[ V_{\text{max}}(S^3, [g]) \geq \frac{1}{2} \text{Vol}(S^3), \]
then the blowing-up manifold \( (S^3 \setminus \{ P \}, G^4 g) \) has no horizon.

We also give an example of \( (S^3, g) \) where \( \text{Ric}(g) > 0 \) and \( (S^3 \setminus \{ P \}, G^4 g) \) does have horizons.

2. Positive Ricci curvature and maximum volume

We first explain the volume assumption in the Theorem. Let \( M^n \) be a smooth, connected, closed manifold of dimension \( n \geq 3 \). Assume \([g]\) is a conformal class of metrics on \( M^n \) which has a metric of positive Ricci curvature. One can define 
\[ V_{\text{max}}(M^n, [g]) = \sup_{\tilde{g} \in [g]} \{ \text{Vol}(M^n, \tilde{g}) \mid \text{Ric}(\tilde{g}) \geq (n-1)\tilde{g} \}. \]
The following result relating $V_{\text{max}}(M^n, [g])$ and the Yamabe constant of $(M^n, [g])$ was observed in [5].

**Proposition 1.** Let $[g]$ be a conformal class of metrics on $M^n$ which has a metric of positive Ricci curvature. Then the Yamabe constant of $(M^n, [g])$ satisfies

\begin{equation}
Y(M^n, [g]) \geq n(n-1)V_{\text{max}}(M^n, [g])^{\frac{2}{n}}.
\end{equation}

**Proof.** By definition,

\begin{equation}
Y(M^n, [g]) = \inf_{v \in W^{1,2}(M)} \frac{\int_M [c_n |v|^2 + R(\bar{g})v^2]dV_{\bar{g}}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}}}
\end{equation}

for any $\bar{g} \in [g]$, where $c_n = \frac{4(n-1)}{n-2}$.

Assume $\text{Ric}(\bar{g}) \geq (n-1)\bar{g}$, by a result of Ilias [7], which is based on the isoperimetric inequality of Gromov [9], we have

\begin{equation}
\int_M [c_n |v|^2 + n(n-1)v^2]dV_{\bar{g}} \geq \left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}} n(n-1)\text{Vol}(M^n, \bar{g})^{\frac{2}{n}}
\end{equation}

for any $v \in W^{1,2}(M)$. Note that $R(\bar{g}) \geq n(n-1)$, hence

\begin{equation}
Y(M^n, [g]) \geq \inf_{v \in W^{1,2}(M)} \frac{\int_M [c_n |v|^2 + n(n-1)v^2]dV_{\bar{g}}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}}} \geq n(n-1)\text{Vol}(M^n, \bar{g})^{\frac{2}{n}}.
\end{equation}

Taking the supremum over $\bar{g} \in [g]$ satisfying $\text{Ric}(\bar{g}) \geq (n-1)\bar{g}$, we have

\begin{equation}
Y(M^n, [g]) \geq n(n-1)V_{\text{max}}(M^n, [g])^{\frac{2}{n}}.
\end{equation}

\[\square\]

As an immediate corollary, we see the assumption

\[V_{\text{max}}(S^3, [g]) \geq \frac{1}{2}\text{Vol}(S^3)\]

in the Theorem implies

\begin{equation}
Y(S^3, [g]) \geq 6 \left(\frac{1}{3}\right)^{\frac{4}{3}} \text{Vol}(S^3)^{\frac{2}{3}}
\end{equation}

(9)

where $RP^3$ is the three dimensional projective space and $g_0$ is the standard metric on $RP^3$ which has constant sectional curvature +1.
3. An upper bound of the Sobolev constant when a horizon is present

One basic fact relating the conformal class \([g]\) on \(S^3\) and the blowing-up metric \(h = G^4 g\) on \(\mathbb{R}^3 = S^3 \setminus \{P\}\) is

\[
Y(S^3, [g]) = 8S(h),
\]

where \(S(h)\) is the Sobolev constant of the asymptotically flat manifold \((\mathbb{R}^3, h)\) \([3]\). Recall \(S(h)\) is defined by

\[
S(h) = \inf_{u \in W^{1,2}(\mathbb{R}^3, h)} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2_h \, dV_h \left/ \left( \int_{\mathbb{R}^3} u^6 \, dV_h \right)^{\frac{1}{3}} \right. \right\}.
\]

The next proposition, which plays a key role in the derivation of the Theorem, was essentially established by Bray and Neves in \([3]\) using the inverse mean curvature flow technique \([6]\). As the statement of Bray and Neves is different from what we need, we include the proof here.

**Proposition 2.** Let \(h\) be a complete metric on \(\mathbb{R}^3\) such that \((\mathbb{R}^3, h)\) is asymptotically flat. If \((\mathbb{R}^3, h)\) has nonnegative scalar curvature and has a closed minimal surface, then

\[
S(h) < \frac{1}{8} Y(\mathbb{R}P^3, [g_0]).
\]

**Proof.** Since \((\mathbb{R}^3, h)\) has a closed minimal surface, the outermost minimal surface \(S\) in \((\mathbb{R}^3, h)\), i.e. the closed minimal surface that is not enclosed by any other minimal surface \([2]\), exists and consists of a finite union of disjoint, embedded minimal two-spheres and projective planes. As our background manifold is \(\mathbb{R}^3\), \(S\) must consist of embedded minimal two-spheres alone, furthermore each component of \(S\) necessarily bounds a three-ball.

We fix a component \(\Sigma\) of \(S\) and denote by \(\Omega\) the three-ball that \(\Sigma\) bounds in \(\mathbb{R}^3\). Let \(\phi\) be the weak solution to the inverse mean curvature flow in \((\mathbb{R}^3 \setminus \bar{\Omega}, h)\) with initial condition \(\Sigma\) \([6]\). \(\phi\) satisfies

\[
\phi \geq 0, \quad \phi|_{\Sigma} = 0, \quad \lim_{x \to \infty} \phi = \infty.
\]

Let \(\Sigma_t\) be the set \(\partial \{u < t\}\) for \(t > 0\) and \(\Sigma_0\) be the starting surface \(\Sigma\), then the family of surfaces \(\{\Sigma_t\}\) satisfies the following properties \([6]\):

1. \(\{\Sigma_t\}\) consists of \(C^{1,\alpha}\) surfaces. For a.e. \(t\), \(\Sigma_t\) has weak mean curvature \(H\) and \(H = |\nabla u|_h\) for a.e. \(x \in \Sigma_t\).
2. \(|\Sigma_t| = e^t|\Sigma_0|\), where \(|\Sigma_t|\) denotes the area of \(\Sigma_t\).
3. Since \((\mathbb{R}^3, h)\) has nonnegative scalar curvature, \(\Sigma\) is connected and \(\mathbb{R}^3 \setminus \bar{\Omega}\) is simply connected, the Hawking quasi-local mass of \(\Sigma_t\),

\[
m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2 \, d\mu \right),
\]

is monotone increasing. Here \(d\mu\) is the induced surface measure.

Now we restrict attention to functions \(u \in W^{1,2}(\mathbb{R}^3, h)\) that have the form

\[
u(x) = \begin{cases} f(0) & x \in \Omega \\ f(\phi(x)) & x \in \mathbb{R}^3 \setminus \Omega \end{cases}
\]
for some $C^1$ functions $f(t)$ defined on $[0, \infty)$. By the coarea formula and Property 1 above, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dv_h = \int_0^\infty f'(t)^2 \left( \int_{\Sigma_t} H \, d\mu \right) \, dt$$

\(\leq \int_0^\infty f'(t)^2 \sqrt{16\pi |\Sigma|(e^t - e^{\frac{t}{2}})} \, dt\),

(14)

where the inequality follows from Property 2, 3 and Hölder’s inequality. Similarly, we have

$$\int_{\mathbb{R}^3} u^6 \, dv_h \geq \int_0^\infty f(t)^6 \left( \int_{\Sigma_t} H^{-1} \, d\mu \right) \, dt$$

\(\geq \int_0^\infty f(t)^6 e^{2t}|\Sigma|^2(16\pi |\Sigma|(e^t - e^{\frac{t}{2}}))^{-\frac{1}{2}} \, dt\).

(15)

Therefore,

$$\frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dv_h}{(\int_{\mathbb{R}^3} u^6 \, dv_h)^\frac{1}{3}} \leq \frac{(16\pi)^{\frac{2}{3}} \int_0^\infty f'(t)^2(e^t - e^{\frac{t}{2}})^{\frac{1}{2}} \, dt}{(\int_0^\infty f(t)^6 e^{2t}(e^t - e^{\frac{t}{2}})^{-\frac{1}{2}} \, dt)^{\frac{1}{3}}}.$$  

(16)

To pick an optimal $f(t)$ that minimizes the right side of (16), we consider the half spatial Schwarzschild manifold

$$(M^3, g_S) = (\mathbb{R}^3 \setminus B_1(0), (1 + \frac{1}{|x|})^4 \delta_{ij})$$

and the quotient manifold $(\tilde{M}^3, \tilde{g}_S)$ obtained from $(M^3, g_S)$ by identifying the antipodal points of $\{|x| = 1\}$. Up to scaling, $(\tilde{M}^3, \tilde{g}_S)$ is isometric to $(\mathbb{R}P^3 \setminus \{Q\}, \tilde{g}_0)$, the blowing-up manifold of $(\mathbb{R}P^3, g_0)$ by its Green function at a point $Q$. Hence, the Sobolev constant $S(\tilde{g}_S)$ of $(\tilde{M}^3, \tilde{g}_S)$ equals $\frac{1}{8} Y(\mathbb{R}P^3, [g_0])$. On the other hand, $S(\tilde{g}_S)$ is achieved by a function $u_0$ that is a constant on each coordinate sphere $\{|x| = t\}$ in $\tilde{M}$, and the level set of the solution $\phi_0$ to the inverse mean curvature flow starting at $\{|x| = 1\}$ in $(M, g_S)$ is also given by coordinate spheres. Therefore, lifted as a function on $(M^3, g_S)$, $u_0$ has the form of

$$u_0 = f_0 \circ \phi_0,$$

for some explicitly determined function $f_0(t)$, and

$$S(\tilde{g}_S) = \frac{\int_{\mathbb{R}^3} |\nabla u_0|^2 \, dv_{g_S}}{(\int_{\mathbb{R}^3} u_0^6 \, dv_{g_S})^{\frac{1}{3}}} = \frac{(16\pi)^{\frac{2}{3}} \int_0^\infty f_0'(t)^2(e^t - e^{\frac{t}{2}})^{\frac{1}{2}} \, dt}{(\int_0^\infty f_0(t)^6 e^{2t}(e^t - e^{\frac{t}{2}})^{-\frac{1}{2}} \, dt)^{\frac{1}{3}}},$$

(17)

where the second equality holds because the Hawking quasi-local mass remains unchanged along the level sets of $\phi_0$. Now consider $u = f_0 \circ \phi$ on $(\mathbb{R}^3, h)$. It was verified in [3] that $u \in W^{1, 2}(\mathbb{R}^3, h)$. Therefore, we have

$$S(h) \leq \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dv_h}{(\int_{\mathbb{R}^3} u^6 \, dv_h)^{\frac{1}{3}}} \leq \frac{(16\pi)^{\frac{2}{3}} \int_0^\infty f_0'(t)^2(e^t - e^{\frac{t}{2}})^{\frac{1}{2}} \, dt}{(\int_0^\infty f_0(t)^6 e^{2t}(e^t - e^{\frac{t}{2}})^{-\frac{1}{2}} \, dt)^{\frac{1}{3}}} \leq S(\tilde{g}_S) = \frac{1}{8} Y(\mathbb{R}P^3, [g_0]).$$

(18)
To show the strict inequality, we assume \( S(h) = \frac{1}{8} Y(RP^3, [g_0]) \). Then, \( S(h) \) is achieved by \( u = f_0 \circ \phi \). It follows from the Euler-Lagrange equation of the Sobolev functional (11) that \( u \) satisfies
\[
\Delta_h u + Cu = 0 \quad \text{on } \mathbb{R}^3,
\]
where \( C = S(h) \frac{|u|_{L^4(\mathbb{R}^3,H)}}{||u||} \). However, \( u \equiv f_0(0) \) on \( \Omega \) and \( f_0(0) \neq 0 \) (Indeed, up to a constant multiple, \( f_0(t) = (2e^t - e^{-2t})^{-\frac{1}{2}} \)). Hence, \( C = 0 \), which contradicts to the fact that \( u \) is not a constant. Therefore, the strict inequality \( S(h) < \frac{1}{8} Y(RP^3, [g_0]) \) holds.

**Proof of the Theorem:** Suppose \((S^3 \setminus \{P\}, G^4 g)\) has a horizon, then it follows from (10) and Proposition 2 that
\[
Y(S^3, [g]) < Y(RP^3, [g_0]).
\]
On the other hand, the assumption \( V_{max}(S^3, [g]) \geq \frac{1}{2} Vol(S^3) \) implies
\[
Y(S^3, [g]) \geq Y(RP^3, [g_0])
\]
by (9), which is a contradiction. Hence, there are no horizons in \((S^3 \setminus \{P\}, G^4 g)\). □

### 4. An example with horizons

In this section, we provide an example to show that there exist metrics on \( S^3 \) with positive Ricci curvature such that the blowing-up manifolds do have horizons.

Our example comes from a 1-parameter family of left-invariant metrics \( \{g_\epsilon\} \) on \( S^3 \), commonly known as the **Berger metrics**. Precisely, we think \( S^3 \) as the Lie Group \( SU(2) = \{ (z \bar{w}, -w \bar{z}) : |z|^2 + |w|^2 = 1 \} \), where the Lie algebra of \( SU(2) \) is spanned by
\[
X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

Then \( \{g_\epsilon\} \) is defined by declaring \( X_1, X_2, X_3 \) to be orthogonal, \( X_1 \) to have length \( \epsilon \) and \( X_2, X_3 \) to be unit vectors. Note that scalar multiplication on \( S^3 \subset \mathbb{C}^2 \) corresponds to multiplication on the left by matrices \( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \) on \( SU(2) \), hence \( X_1 \) is exactly tangent to the circle fiber of the **Hopf fibration**
\[
\pi : S^3 \to S^2 = S^3/S^1
\]
and \( g_\epsilon \) shrinks the circle fiber as \( \epsilon \to 0 \). One fact of \( g_\epsilon \), for small \( \epsilon \), is that all sectional curvature of \((S^3, g_\epsilon)\) lies in the interval \([\epsilon^2, 4 - 3\epsilon^2]\) (see [10]), in particular \( g_\epsilon \) has positive Ricci curvature.

**Proposition 3.** Let \( P \in S^3 \) be a fixed point and \( G_\epsilon \) be the Green’s function of the conformal Laplacian of \( g_\epsilon \) at \( P \). Then \((S^3 \setminus \{P\}, G_\epsilon^4 g_\epsilon)\) has a horizon for \( \epsilon \) sufficiently small.
Proof. For each \( \epsilon \in (0, 1) \), we consider the rescaled metric \( \bar{g}_\epsilon = \epsilon^{-2}g_\epsilon \) and the Green’s function \( \bar{G}_\epsilon \) associated to \( \bar{g}_\epsilon \) at \( P \). Then, with respect to \( \bar{g}_\epsilon \), \( X_1 \) becomes a unit vector and \( X_2, X_3 \) have large length \( \epsilon^{-1} \) as \( \epsilon \to 0 \). Let \( U \subset S^3 \) be a fixed neighborhood of \( P \) such that \( \pi|_U \) is a trivial fibration. Let \( O \) be a fixed point in the product manifold \( S^1 \times \mathbb{R}^2 \). By a scaling argument, there exists a family of diffeomorphisms

\[
\Psi_\epsilon : U \longrightarrow \Psi_\epsilon(U) \subset S^1 \times \mathbb{R}^2,
\]

such that \( \Psi_\epsilon(P) = O \in \Psi_\epsilon(U) \), \( \{ \Psi_\epsilon(U) \}_{\epsilon > 0} \) forms an exhaustion family of \( S^1 \times \mathbb{R}^2 \) as \( \epsilon \to 0 \), and the push forward metrics \( \hat{g}_\epsilon = \Psi^{-1}_\epsilon(\bar{g}_\epsilon)|_U \) on \( \Psi_\epsilon(U) \) converge in \( C^2 \) norm on compact sets to a flat metric \( \hat{g} \) on \( S^1 \times \mathbb{R}^2 \). Now fix another point \( Q \in \Psi_1(U) \) that is different from \( O \) and consider the normalized function

\[
\hat{G}_\epsilon(x) = \frac{\hat{G}_\epsilon \circ \Psi^{-1}_\epsilon(x)}{\hat{G}_\epsilon \circ \Psi^{-1}_\epsilon(Q)}
\]

for \( x \in \Psi_\epsilon(U) \setminus \{ O \} \). Then \( \hat{G}_\epsilon \) satisfies

\[
\begin{cases}
8\Delta_{\hat{g}_\epsilon} \hat{G}_\epsilon - R(\hat{g}_\epsilon)\hat{G}_\epsilon = 0 & \text{on } \Psi_\epsilon(U) \setminus \{ O \} \\
\hat{G}_\epsilon = 1 & \text{at } Q
\end{cases}
\]

Since \( \hat{G}_\epsilon \) is positive and \( \hat{g}_\epsilon \) converges to \( \hat{g} \) as \( \epsilon \to 0 \), it follows from the Harnack inequality that \( \hat{G}_\epsilon \) converges to a positive function \( \hat{G} \) on \( (S^1 \times \mathbb{R}^2) \setminus \{ O \} \) in \( C^2 \) norm on any compact set away from \( \{ O \} \). Furthermore, \( \hat{G} \) satisfies

\[
\begin{cases}
\Delta_{\hat{g}} \hat{G} = 0 & \text{on } (S^1 \times \mathbb{R}^2) \setminus \{ O \} \\
\hat{G} = 1 & \text{at } Q
\end{cases}
\]

On the other hand, the fact that the geodesic ball in \( (S^1 \times \mathbb{R}^2, \hat{g}) \) only has quadratic volume growth implies \( (S^1 \times \mathbb{R}^2, \hat{g}) \) does not have a positive Green’s function for the usual Laplacian \( \Delta_{\hat{g}} \) [4]. Therefore, \( \hat{G} \equiv 1 \) on \( (S^1 \times \mathbb{R}^2) \setminus \{ O \} \). Hence, the metrics \( \hat{G}_\epsilon^2\hat{g}_\epsilon \) converge to \( \hat{g} \) in \( C^2 \) norm on any compact set away from \( \{ O \} \). Now let \( V \subset S^1 \times \mathbb{R}^2 \) be a small open ball containing \( O \) such that \( \partial V \) is an embedded two sphere whose mean curvature vector computed with respect to \( \hat{g} \) points towards \( O \). Then, for sufficiently small \( \epsilon \), the mean curvature vector of \( \partial V \) computed with respect to \( \hat{G}_\epsilon^2\hat{g}_\epsilon \) still points towards \( O \). As \( (\Psi_\epsilon(U), \hat{G}_\epsilon^2\hat{g}_\epsilon) \) is isometric to \( (U, \hat{G}_\epsilon^2\hat{g}_\epsilon) \), the mean curvature vector of the boundary of \( \Psi_\epsilon^{-1}(V) \) in \( (S^3 \setminus \{ P \}, \hat{G}_\epsilon^2\hat{g}_\epsilon) \) must point towards the blowing-up point \( P \). On the other hand, as \( (S^3 \setminus \{ P \}, \hat{G}_\epsilon^2\hat{g}_\epsilon) \) is asymptotically flat, its infinity is foliated by two spheres whose mean curvature vector points away from \( P \). Therefore, it follows from standard geometric measure theory that there exists an embedded minimal two sphere in \( \Psi_\epsilon(V) \), hence \( (S^3 \setminus \{ P \}, \hat{G}_\epsilon^2\hat{g}) \) has a horizon. \( \square \)

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