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VARIATION OF ARGUMENT AND BERNSTEIN INDEX FOR HOLOMORPHIC FUNCTIONS ON RIEMANN SURFACES

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Abstract. An upper bound of the variation of argument of a holomorphic function along a curve on a Riemann surface is given. This bound is expressed through the Bernstein index of the function multiplied by a geometric constant. The Bernstein index characterizes growth of the function from a smaller domain to a larger one. The geometric constant in the estimate is explicitly given. This result is applied in [1], [3] to the solution of the restricted version of the infinitesimal Hilbert 16th problem, namely, to upper estimates of the number of zeros of abelian integrals in complex domains.

1. Introduction

Consider a holomorphic function f in a topological disk U imbedded in a Riemann surface, and a curve $\Gamma \subset U$. We are interested in an upper bound of the variation of argument of the function f along the curve Γ. This bound may be expressed through the growth rate of f from some intermediate domain U'' to U, and through a geometric factor depending on the relative position of all the three sets: $\Gamma \subset U'' \subset U$. Our result improves the theorem from [6] that provides a similar estimate for a domain $U \subset \mathbb{C}$, and without explicit formula for the geometric factor mentioned above. We provide a formula for this factor, and consider functions on a Riemann surface. Both improvements are crucial for an application of our main result to an upper estimate of the number of zeroes of Abelian integrals in the complex domain. This latter estimate forms a part of the solution of so called *restricted infinitesimal Hilbert 16th problem* [1] [3].

Our main tool is Growth-and-Zeros theorem stated and improved in the next section. The main results of the paper are stated in Section 3.

2. Bernstein index and Growth-and-Zeros theorem

Let W be a Riemann surface, $\pi\colon W \to \mathbb{C}$ be a holomorphic function (called projection) with non-zero derivative. Let ρ be the metric on W lifted from $\mathbb{C}$ by projection π. Let $U \subset W$ be a connected domain, and $K \subset U$ be a compact set. For any $p \in U$ let $\varepsilon(p, \partial U)$ be the supremum of radii of the disks centered at p, located in U and such that π is bijective on these disks. *The π-gap between K and ∂U*, is defined as

$$\pi\text{-gap } (K, \partial U) = \min_{p \in K} \varepsilon(p, \partial U).$$

Growth-and-zeros theorem. *Let W, π, ρ be the same as before. Let $U \subset W$ be a domain conformally equivalent to a disk. Let $K \subset U$ be a path connected compact*

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subset of U *(different from a single point). Suppose that the following two assumptions hold:*

Diameter condition:

$$diam_{\,int} K \leq D;$$

Gap condition:

$$\pi\text{-}gap(K, \partial U) \geq \varepsilon.$$

Let I *be a bounded holomorphic function on* $\bar{U}$*. Then*

$$\#\{z \in K | I(z) = 0\} \leq e^{\frac{2D}{\varepsilon}} \log \frac{\max_U |I|}{\max_K |I|} \tag{1}$$

The definition of the intrinsic diameter is well known; yet we recall it for the sake of completeness.

Definition 1. The *intrinsic distance* between two points of a path connected set in a metric space is the infinum of the length of paths in K that connect these points (if exists). The *intrinsic diameter* of K is a supremum of intrinsic distances between two points taken over all the pairs of points in K.

Definition 2. The second factor in the right hand side of (1) is called *the Bernstein index* of I with respect to U and K and denoted by $B_{K,U}(I)$:

$$B_{K,U}(I) = \log \frac{M}{m}, \ M = \sup_U |I|, \ m = \max_K |I|.$$

Proof of the Growth-and-Zeros theorem. The above theorem is proved in [5] for the case when $W = \mathbb{C}, \pi = id$. In fact, in [4] another version of (1) is proved with (1) replaced by

$$\#\{z \in K | I(z) = 0\} \leq B_{K,U}(I) e^{\rho}, \tag{2}$$

where $\rho \geq \log 2$ is the diameter of K in the Poincaré metric of U. In this case it does not matter whether U belongs to $\mathbb{C}$ or to the Riemann surface.

Proposition 1. *Let* K, U *be two sets in the Riemann surface* W *from Definition 2, and let the Diameter and Gap conditions from the Growth-and-Zeros theorem hold. Then the diameter of* K *in the Poincaré metric of* U *admits the following upper estimate:*

$$\rho \leq \frac{2D}{\varepsilon}.$$

Growth-and-Zeros theorem now follows from (2) and Proposition 1. □

Proof of Proposition 1. By the gap condition, for any point $p \in K$, the domain U contains a disk D of radius ε centered at p. In more detail, $D \subset U$ is a set mapped by π bijectively onto a disk of radius ε in $\mathbb{C}$ centered at $\pi(p)$ and such that $p \in D$. *In what follows, for any topological disk* V *the Poincaré metric of* V *is denoted by* PV. The length of a vector v attached at p in the metric PD equals to the Eucleadian length $|v|$ of v divided by ε and multiplied by 2. By the monotonicity property of the Poincaré metric, the length of v in sense of PU is no greater then the previous one. This implies Proposition 1. □

3. Upper estimate of the variation of argument

A definition of a *variation of argument of a complex valued function along a curve* is contained in this, rather lengthy, name; this variation is denoted $V_\Gamma(f)$ for a function f and a curve Γ. In more detail, let U be a domain on a Riemann surface, $\Gamma : [0,1] \to U$ be a curve, $f : U \to \mathbb{C}$, $f|_\Gamma \neq 0$ a function. Fix an arbitrary branch of $\arg f$ on Γ and let $\phi = \arg f \circ \Gamma$. By definition, $V_\Gamma(f)$ is the total variation of ϕ on [0,1]. An equivalent definition of variation of argument of a function f along a curve Γ is:

$$V_\Gamma(f) = \int_\Gamma |Im(\log f)'| ds, \tag{3}$$

where s is the arc length parameter on Γ.

A first step in establishing a relation between variation of argument and the Benstein index is done by the following KYa (Khovansky-Yakovenko) theorem. Let U, V and f be the same as in the previous paragraph.

KYa theorem, [6]**.** *For any tuple U, K, Γ, where $U \subset \mathbb{C}$ is a connected open set, $K \subset U$ is a compact set, $\Gamma \subset U$ is a curve, there exists a geometric constant $\alpha = \alpha(U, K, \Gamma)$, such that*

$$V_\Gamma(f) \leq \alpha B_{K,U}(f).$$

In [6] an upper estimate of the Bernstein index through the variation of the argument along $\Gamma = \partial U$ is given; we do not use this estimate. On the contrary, we need an improved version of the previous theorem with α explicitly written and U being a domain on a Riemann surface. These two improvements are achieved in the following two theorems.

Let $|\Gamma|$ be the length, and $\kappa(\Gamma)$ be the total curvature of a curve on a surface endowed with a Riemann metric.

Theorem 1. *Let $\Gamma \subset U'' \subset U' \subset U \subset \mathbb{C}$ be respectively a piecewise smooth curve, and three connected open sets in $\mathbb{C}$, $\bar{U}$ is compact and simply connected. Let $f : \bar{U} \to \mathbb{C}$ be a holomorphic function, $f|_\Gamma \neq 0$. Let $\frac{D}{\varepsilon} > 3$, and the following two conditions hold:*

Gap condition:

$$\rho(\Gamma, \partial U'') \geq \varepsilon, \;\; \rho(U'', \partial U') \geq \varepsilon, \;\; \rho(U', \partial U) \geq \varepsilon; \tag{4}$$

Diameter condition:

$$diam\ _{int}U'' \leq D, \quad diam\ _{int}U' \leq D. \tag{5}$$

Then

$$V_\Gamma(f) \leq B_{U'',U}(f)(\frac{|\,\Gamma\,|}{\varepsilon} + \kappa(\Gamma) + 1)e^{\frac{5D}{\varepsilon}}. \tag{6}$$

Theorem 2. *Let $\Gamma \subset U'' \subset U' \subset U \subset W$ be respectively a piecewise smooth curve, and three connected open sets in a Riemann surface W, $\bar{U}$ is compact and simply connected. Let $f : \bar{U} \to \mathbb{C}$ be a holomorphic function, $f|_\Gamma \neq 0$. Let $\pi : W \to \mathbb{C}$ be a projection which is locally biholomorphic, and the metric on W be a pullback of that on $\mathbb{C}$. Let $\frac{D}{\varepsilon} > 3$, and the following two conditions hold:*

Gap condition:

$$\pi\text{-}gap\ (\Gamma, U'') \geq \varepsilon, \;\; \pi\text{-}gap\ (U'', U') \geq \varepsilon, \;\; \pi\text{-}gap\ (U', U) \geq \varepsilon; \tag{7}$$

Diameter condition:

$$diam\ _{int}U'' \le D,\ \ diam\ _{int}U' \le D. \tag{8}$$

Then inequality (6) holds.

The proofs of these theorems are partly based on the methods of [6]. Yet our presentation is self contained. Theorem 1 improves the parallel result of [6] by giving an explicit expression for the geometric factor in the KYa Theorem. Theorem 2 has no analoge in [6]. Note that in [6] the variation of argument (modulo a factor $(2\pi)^{-1}$) is called a *Voorhoeve index.*

4. Variation of argument of nowhere zero functions on Riemann surfaces

Lemma 1. *Let $\Gamma, U'', U' \subset W$ be the same as in Theorem 2, and $F : U' \to \mathbb{C}$ be a nowhere zero holomorphic function. Then*

$$V_\Gamma(F) \le B_{U'',U'}(F)\frac{|\Gamma|}{\varepsilon}e^{\frac{2D}{\varepsilon}}. \tag{9}$$

Proof. Definition (3) implies:

$$V_\Gamma(F) \le \int_\Gamma \mid (\log F)' \mid ds. \tag{10}$$

Without loss of generality, we may assume that

$$\max_{\bar{U}''} \mid F \mid = 1,$$

and $F(a) = 1$ for some $a \in \bar{U}''$. Then, for $B' = B_{U'',U'}(F)$ we have: $(\log F)(U') \subset \mathbb{C}^-_{B'} := \{w \mid \ \text{Re}\ w \le B'\}$. Let $g = \frac{1}{B'}\log F$. Then, by (10),

$$V_\Gamma(F) \le \mid \Gamma \mid \cdot B' \cdot \max_\Gamma \mid g' \mid. \tag{11}$$

Note that

$$g(U') \subset \mathbb{C}^-_1. \tag{12}$$

We want to get an upper bound for $\mid g' \mid$ on Γ making use of the Cauchy estimates. For this we need to estimate from above $\max_{z\in U''} \mid g(z) \mid$. To do that we will use an upper estimate of the diameter of U'' in the Poincaré metric PU' of U', and the fact that g does not increase the Poincaré metric. By Proposition 1, we have:

$$\text{diam}\ _{PU'}U'' \le \frac{2D}{\varepsilon}.$$

Note that for the above chosen $a \in U''$, $g(a) = 0$. Then, by (12) and monotonicity property of the Poincaré metric, for any $z \in U''$,

$$\rho_{P\mathbb{C}^-_1}(0, g(z)) \le \rho_{PU'}(a, z) \le \frac{2D}{\varepsilon}.$$

This estimate for the Poincaré distance between 0 and $g(z)$ in sense of $P\mathbb{C}^-_1$ implies an estimate on the Euclidean distance between 0 and $g(z)$:

$$\mid g(z) \mid \le e^{\frac{2D}{\varepsilon}}.$$

By the Cauchy estimate and gap condition (4) for Γ and U'',

$$\max_{\Gamma} \mid g' \mid \leq \varepsilon^{-1} e^{\frac{2D}{\varepsilon}}.$$

Together with (11), this implies the lemma. □

5. Variation of argument of holomorphic functions on $\mathbb{C}$

Proof of the Theorem 1. Lemma 1 implies the theorem in the case when $f \neq 0$ in U'. The lemma should be applied to $F = f \mid_{U'}$ keeping in mind that $B_{U'',U'}(F) \leq B_{U'',U}(F)$. In general, f may have zeros in U'. Let $d = \#\{f(z) = 0 \mid z \in U'\}$, zeros of f are counted with multiplicities. By the Growth-and-Zeros theorem,

$$d \leq B_{U',U}(f) e^{\frac{2D}{\varepsilon}}.$$

Let $B = B_{U'',U}(f)$. Note that $B_{U',U}(f) \leq B$. Hence,

$$d \leq B e^{\frac{2D}{\varepsilon}}. \tag{13}$$

Let p be a monic polynomial of degree d that has the same zeros as f in U', with the same multiplicities. Then the function

$$F = \frac{f}{p}$$

is holomorphic and nowhere zero in U'. Equality $f = Fp$ implies

$$V_\Gamma(f) \leq V_\Gamma(F) + V_\Gamma(p). \tag{14}$$

Inequality (14) follows from the triangle inequality applied to moduli of derivatives of $\arg F$ and $\arg p$ along Γ.

Lemma 2. *Let $U'' \subset U' \subset U \subset \mathbb{C}$ be the same as in Theorem 1. Let F be a holomorphic function in $\bar{U}$, p be a monic polynomial of degree d with zeros located in U'. Let $f = Fp$, $B = B_{U'',U}(f)$, as before. Then*

$$B_{U'',U}(F) \leq B + d \log \frac{D}{\varepsilon}. \tag{15}$$

This lemma is very close to a more complicated statement proved in [6]. The proof of the lemma itself follows; it is straightforward. Before the proof, note that the choice of the domains in Lemma 2 is important. On one hand, the same Bernstein index B is used in (13) and (15). On the other hand, the right hand side of (15) estimates from above the Bernstein index B' used in Lemma 1.

Proof of Lemma 2. Without loss of generality we may assume that

$$\max_{\bar{U}''} |f| = 1.$$

Then

$$\log \max_{\bar{U}} |f| = B.$$

Hence,

$$\log \max_{\partial U} |F| \leq B - \log \min_{\partial U} |p|.$$

On the other hand, p is a product of d binomials of the form $w - w_j$ with $w_j \in U'$. Then, by the gap condition (4) for U' and U,

$$\log \min_{\partial U} |p| \geq d \log \varepsilon.$$

Hence,

$$\log \max_{\bar{U}} |F| \leq B - d \log \varepsilon$$

by the maximum modulus principle.

On U'', $|p| \leq D^d$. Then $\log \max_{U''} |F| \geq -d \log D$. Hence,

$$B_{U'',U}(F) \leq B + d(\log D - \log \varepsilon).$$

This proves the lemma. □

Now let us complete the proof of Theorem 1. To do that, let us estimate the terms in the right hand side of (14). First, by [6],

$$V_\Gamma(p) \leq (\kappa(\Gamma) + 2\pi)d.$$

The proof of this inequality is based on the decomposition of a polynomial into a product of linear factors. For q linear, an elementary estimate $V_\Gamma(q) \leq \kappa(\Gamma) + 2\pi$ holds. By (13),

$$V_\Gamma(p) \leq (\kappa(\Gamma) + 2\pi)Be^{\frac{2D}{\varepsilon}} \qquad 16$$

Second, $V_\Gamma(F)$ is estimated in Lemma 1, see (9). But

$$B_{U'',U'}(F) \leq B_{U'',U}(F).$$

The Bernstein index in the right hand side is already estimated from above in (15). Namely, let $\alpha = \log \frac{D}{\varepsilon}$. Then, by Lemma 2,

$$B_{U'',U}(F) \leq B + \alpha d.$$

¿From this, by (9) and (13) we get:

$$V_\Gamma(F) \leq B(1 + \alpha e^{\frac{2D}{\varepsilon}})\frac{|\Gamma|}{\varepsilon}e^{\frac{2D}{\varepsilon}}.$$

By assumption of Theorem 1, $\frac{D}{\varepsilon} > 3$. Elementary estimates yield:

$$1 + \alpha e^{\frac{2D}{\varepsilon}} \leq e^{\frac{3D}{\varepsilon}}.$$

Hence,

$$V_\Gamma(F) \leq B\frac{|\,\Gamma\,|}{\varepsilon}e^{\frac{5D}{\varepsilon}}. \qquad (17)$$

By (14), (16) and (17) we get now

$$V_\Gamma(f) \leq B(\frac{|\,\Gamma\,|}{\varepsilon} + \kappa(\Gamma) + 2\pi)e^{\frac{5D}{\varepsilon}}.$$

This proves Theorem 1. □

6. Variation of argument of holomorphic functions on Riemann surfaces

Proof of Theorem 2. Theorem 2 is proved in this and the next sections.

Let $\varphi\colon U \to D_1$ be a conformal mapping of U onto the unit disk D_1 that takes zero value at some point $b \in U''$. Let

$$g = f \circ \varphi^{-1}\colon D_1 \to \mathbb{C}.$$

The number of zeros, the Bernstein index and the variation of the argument of a holomorphic function are invariant under biholomorphic maps of the domain of a function. In particular, the function g has the same number d of zeros, counted with multiplicities, in $\varphi(U')$, as f has in U'. Note that estimate (13) for d holds by Growth-and-Zeros theorem because the theorem is stated for Riemann surfaces.

Let us take a polynomial p that has the same zeros with the same multiplicities as g in $\varphi(U')$. Then $\deg p = d$, and (13) holds. Take $P = p \circ \varphi\colon U \to \mathbb{C}$ and $F = \frac{f}{P}$. Then F is nowhere zero in U'. By the same argument as above,

$$V_\Gamma(f) \le V_\Gamma(F) + V_\Gamma(P). \tag{18}$$

In this section the first term in the right hand side of (18) is estimated from above. The next section deals with the second term.

The variation $V_\Gamma(F)$ is estimated from above by Lemmas 1 and 2, with domains U'', U', U replaced by $\varphi(U''), \varphi(U'), \varphi(U) = D_1$. The main part of the proof deals with the diameter and gap conditions for this new triple of domains.

By an obvious inequality for Bernstein indexes, and by the invariance of the Bernstein index mentioned above,

$$B_{U'',U'}(F) \le B_{U'',U}(F) = B_{\varphi(U''),D_1}\left(\frac{g}{p}\right).$$

Once more, by the invariance of the Bernstein index,

$$B_{\varphi(U''),D_1} = B.$$

Hence, by Lemma 2,

$$B_{U'',U'}(F) \le B + d\alpha'$$

where $\alpha' = \log \frac{D'}{\varepsilon'}$, $D' = \mathrm{diam}_{\mathrm{int}}\varphi(U')$, $\varepsilon' = \mathrm{gap}(\varphi(U'), D_1)$. The factor α' is estimated from above in Proposition 2 below. Hence, by (13),

$$B_{U'',U'}(F) \le B + \alpha' B e^{\frac{2D}{\varepsilon}}.$$

By Lemma 1

$$V_\Gamma(F) \le B(1 + \alpha' e^{\frac{2D}{\varepsilon}})|\Gamma|\varepsilon^{-1} e^{\frac{2D}{\varepsilon}}. \tag{19}$$

Proposition 2. *In the assumption above,*

$$D' = diam\ _{int}\varphi(U') \le \frac{D}{\varepsilon}$$

$$\varepsilon' = gap\ (\varphi(U'), D_1) \ge e^{-\frac{2D}{\varepsilon}}$$

Proof. By the gap condition (7),

$$\pi\text{-gap }(U', U) \geq \varepsilon.$$

Hence, we may apply Cauchy inequality to φ and get: for any $a \in U'$,

$$\mid \varphi'(a) \mid \leq \varepsilon^{-1}.$$

Here and below the derivatives are taken with respect to the local parameter z on W lifted by π from $\mathbb{C}$. Together with Diameter condition (8), this proves the first statement of the proposition.

By Proposition 1, diam $_{PU}(U') \leq \frac{2D}{\varepsilon}$. By the invariance of the Poincaré metric under φ, we get that for any $w \in \varphi(U')$, $\rho_{PD_1}(0, w) \leq \frac{2D}{\varepsilon}$. Hence,

$$1 - |w| \geq e^{-\frac{2D}{\varepsilon}}.$$

This proves the proposition. □

Corollary 1. *In (19),*

$$\alpha' \leq \frac{3D}{\varepsilon}. \tag{20}$$

Finally, by (19) and (20),

$$V_\Gamma(F) \leq B(1 + \frac{3D}{\varepsilon} e^{\frac{2D}{\varepsilon}})|\Gamma|\varepsilon^{-1} e^{\frac{2D}{\varepsilon}}. \tag{21}$$

□

7. Total curvature and conformal mappings

Let us now estimate the second term in (18). By the invariance of the variation of argument,

$$V_\Gamma(P) = V_{\varphi(\Gamma)}(p) \leq (\kappa(\varphi(\Gamma)) + 2\pi)d. \tag{22}$$

Once more, the only thing to do is to estimate from above the total curvature of the new curve $\varphi(\Gamma)$.

Lemma 3. *In the assumption above,*

$$\kappa(\varphi(\Gamma)) \leq \kappa(\Gamma) + 2\varepsilon^{-1} \mid \Gamma \mid. \tag{23}$$

Proof. We will treat the case when Γ is not only piecewise smooth but smooth. The contribution to $\kappa(\Gamma)$ given by the vertexes of the piecewise smooth curve remains unchanged under φ, because φ is a conformal mapping. Let $\Gamma = \{\gamma(s) \mid s \in [0, |\Gamma|]\}$, s be a natural parameter on Γ. Let dot denote derivative in s. Then $|\dot\gamma(s)| \equiv 1$. Let s_1 be a natural parameter on $\varphi(\Gamma)$. Then

$$ds_1 = \mid \varphi' \circ \gamma(s) \mid ds.$$

Let $\varphi \circ \gamma(s) = (x(s), y(s))$, $\kappa_\Gamma(s)$ be the curvature of the curve Γ at a point s. Then

$$\kappa_{\varphi(\Gamma)}(s) = \frac{|\ddot x \dot y - \dot x \ddot y|}{(\dot x^2 + \dot y^2)^{3/2}}.$$

Hence,

$$\kappa(\varphi(\Gamma)) = \int_0^{|\varphi(\Gamma)|} \kappa_{\varphi(\Gamma)}(s(s_1)) ds_1 = \int_0^{|\Gamma|} \kappa_{\varphi(\Gamma)}(s) \mid \varphi' \circ \gamma(s) \mid ds.$$

On the other hand,

$$|\ddot{x}\dot{y}-\dot{x}\ddot{y}| \le |(\ddot{x},\ddot{y})||(\dot{x},\dot{y})|,\ \dot{x}^2+\dot{y}^2 = |(\dot{x},\dot{y})|^2,$$

$$|\ (\dot{x},\dot{y})\ |\ (s) = |\ \varphi'\circ\gamma(s)\ |\cdot|\dot{\gamma}(s)| = |\ \varphi'\circ\gamma(s)\ |\ .$$

$$|(\ddot{x},\ddot{y})|(s) = |\ \varphi''\circ\gamma\cdot\dot{\gamma}^2+\varphi'\circ\gamma\cdot\ddot{\gamma}\ |\ (s) \le |\ \varphi''\circ\gamma\ |\ (s)+\kappa_\Gamma(s)\cdot|\varphi'\circ\gamma|.$$

Hence,

$$\kappa(\varphi(\Gamma)) = \int_0^{|\Gamma|}\frac{|\ddot{x}\dot{y}-\dot{x}\ddot{y}|}{\dot{x}^2+\dot{y}^2}ds \le \int_0^{|\Gamma|}\left(\frac{|\ (\varphi''\circ\gamma)\ |}{|\ (\varphi'\circ\gamma)\ |}(s)+\kappa_\Gamma(s)\right)ds \le$$

$$\max_\Gamma\left|\frac{\varphi''}{\varphi'}\right|\cdot|\Gamma|+\kappa(\Gamma). \tag{24}$$

For any $a\in\Gamma$, the function

$$\psi(z) = \frac{\varphi(\varepsilon z+a)}{\varepsilon\varphi'(a)}$$

maps the unit disc conformally into $\mathbb{C}$, and $\psi'(0)=1$. By the Koebe theorem, [2, §8 Ch 6], we have: $|\ \psi''(0)\ |\le 2$. But

$$\psi''(0) = \frac{\varphi''(a)\varepsilon}{\varphi'(a)}.$$

Hence,

$$\left|\frac{\varphi''(a)}{\varphi'(a)}\right| \le 2\varepsilon^{-1}.$$

This estimate substituted to (24) implies (23). Lemma 3 is proved. □

We can now complete the proof of Theorem 2. By (22) and (13),

$$V_\Gamma(P) \le B(\kappa(\varphi(\Gamma))+2\pi)e^{\frac{2D}{\varepsilon}}.$$

By Lemma 3,

$$V_\Gamma(P) \le B(\kappa(\Gamma)+2\frac{|\Gamma|}{\varepsilon}+2\pi)e^{\frac{2D}{\varepsilon}}.$$

By (18), (21), assumption $D/\varepsilon>3$, and the previous inequality,

$$V_\Gamma(f) \le Be^{\frac{2D}{\varepsilon}}\left[2\pi+\kappa(\Gamma)+|\Gamma|\left(\frac{3}{\varepsilon}+\frac{3D}{\varepsilon^2}e^{\frac{2D}{\varepsilon}}\right)\right] \le Be^{\frac{5D}{\varepsilon}}\left(1+\kappa(\Gamma)+\frac{|\Gamma|}{\varepsilon}\right).$$

This proves Theorem 2.

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