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STARK'S QUESTION ON SPECIAL VALUES OF L–FUNCTIONS

Cristian D. Popescu

Abstract. In 1980, Stark posed a far reaching question regarding the second derivatives at $s = 0$ of L–functions of order of vanishing two associated to abelian extensions of number fields (global fields of characteristic 0) (see [20]). Over the years, various mathematicians, most notably Grant [7], Sands [16], and Tangedal [22] have provided evidence in favor of an affirmative answer to Stark's question. In [13], we extrapolated Stark's question to the appropriate class of L–functions associated to abelian extensions of function fields (global fields of characteristic $p > 0$) and showed that, in general, it has a negative answer in that context. Unfortunately, the methods developed in [13] are specific to the geometric, characteristic $p > 0$ situation and cannot be carried over to the characteristic 0 case. In the present paper, we develop new methods which permit us to prove that, in general, (even a weak form of) Stark's question has a negative answer in its original, characteristic 0 formulation as well.

1. STARK'S QUESTION IN THE GENERAL CONTEXT OF STARK'S CONJECTURES

Throughout this section, K/k will denote an abelian extension of global fields (of arbitrary characteristic) of Galois group $G := G(K/k)$. For a field F (of characteristic 0 in what follows), we denote by $\widehat{G}(F)$ the set of characters associated to the irreducible F–representations of G. We let μ_K denote the group of roots of unity in K and define $w_K := \mathrm{card}(\mu_K)$. Let S be a finite nonempty set of primes in k, containing at least all the infinite primes and all the primes which ramify in K/k. We denote by U_S the $\mathbf{Z}[G]$–module of S–units in K, by $A_{K,S}$ the S–ideal class group of K (i.e. the ideal–class group of the ring of S–integers O_S of K.) For every prime $v \in S$, we fix a prime w in K sitting above v, and denote by G_v the decomposition group of $w \mid v$ (which is clearly independent of the chosen w.) We let X_S denote the kernel of the usual ($\mathbf{Z}[G]$–equivariant) augmentation map

$$X_S := \ker\Big(\bigoplus_{v\in S}\mathbf{Z}[G/G_v] \xrightarrow{\text{aug.}} \mathbf{Z}\Big). \tag{1}$$

In what follows, if M is a group and R a commutative ring, we let $RM := R \otimes_{\mathbf{Z}} M$. Also, $\widetilde{M}$ denotes the image of the natural map $M \to \mathbf{Q}M$. All the exterior and tensor products considered are viewed in the category of $\mathbf{Z}[G]$–modules, unless otherwise specified. The $\mathbf{Q}[G]$–modules $\mathbf{Q}U_S$ and $\mathbf{Q}X_S$ are (non-canonically) isomorphic (see [23]). A canonical $\mathbf{C}[G]$–module isomorphism

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$$R_S : \mathbf{C}U_S \xrightarrow{\sim} \mathbf{C}X_S$$

is given by the $\mathbf{C}$–linearization of the map $R_S(u) = -\sum_{v\in S}\sum_{\sigma\in G/G_v} \log|u|_{w^\sigma}\cdot\sigma$, for all $u \in U_S$, where $|\cdot|_{w^\sigma}$ denotes the (canonically normalized) metric associated to w^σ. After choosing a canonical $\mathbf{Q}[G]$–basis for the rank 1 free $\mathbf{Q}[G]$–module $\det_{\mathbf{Q}[G]}(\mathbf{Q}X_S)$ (see [10] for the construction of the basis), which is equivalent to fixing a $\mathbf{Q}[G]$–module isomorphism $\rho_S : \det_{\mathbf{Q}[G]}(\mathbf{Q}X_S) \xrightarrow{\sim} \mathbf{Q}[G]$, one obtains a $\mathbf{C}[G]$–module isomorphism (a so-called G–equivariant S–regulator map)

$$\mathcal{R}_S : \det_{\mathbf{C}[G]}(\mathbf{C}U_S) \underset{\sim}{\xrightarrow{\det(R_S)}} \det_{\mathbf{C}[G]}(\mathbf{C}X_S) \underset{\sim}{\xrightarrow{\rho_S\otimes\mathbf{1}_C}} \mathbf{C}[G].$$

For the basic properties of the determinant $\det_R(M)$ of a finitely generated projective module M over a commutative, Noetherian ring R, the reader can consult [14]. A still unanswered fundamental question in number theory is the following.

Question I (integral version). What is the preimage $\mathcal{L}_S := \mathcal{R}_S^{-1}(\mathbf{Z}[G])$, viewed as a $\mathbf{Z}[G]$–submodule of $\det_{\mathbf{C}[G]}(\mathbf{C}U_S)$?
Question R (rational version). What is the preimage $\mathcal{R}_S^{-1}(\mathbf{Q}[G]) = \mathbf{Q}\mathcal{L}_S$, viewed as a $\mathbf{Q}[G]$–submodule of $\det_{\mathbf{C}[G]}(\mathbf{C}U_S)$?

Example. In the simplest case, that where $K = k$ (i.e. $\mathbf{Z}[G] = \mathbf{Z}$), the answer to the integral version of the above question is equivalent to Dirichlet's classical S–class–number formula, up to a sign (i.e. an element in $\mathbf{Z}^\times$). More precisely, the following equivalent equalities hold.

$$\mathcal{L}_S = \mathbf{Z}\left(\frac{h_{k,S}}{w_k\cdot\zeta^*_{k,S}(0)}\, u_1\wedge\cdots\wedge u_r\right),\quad \zeta^*_{k,S}(0)\cdot\mathcal{L}_S = \mathbf{Z}\left(\frac{h_{k,S}}{w_k}\, u_1\wedge\cdots\wedge u_r\right)$$

Above, $\zeta^*_{k,S}(0)$ is the leading Taylor coefficient at $s = 0$ of the S–incomplete zeta function $\zeta_{k,S}(s)$ of k, $r := \mathrm{card}(S) - 1$, $\{u_1,\ldots,u_r\}$ is a $\mathbf{Z}$–basis of $\widetilde{U_S}$, and $h_{k,S} := \mathrm{card}(A_{k,S})$. Note that $\det_{\mathbf{C}[G]}(\mathbf{C}U_S) = \mathbf{C}(u_1\wedge\cdots\wedge u_r)$. Also, if $S = \{v_0, v_1,\ldots,v_r\}$, then the $\mathbf{Q}[G]$–basis of $\mathbf{Q}X_S$ is given by $\{\mathbf{1}_{G/G_{v_i}} - \mathbf{1}_{G/G_{v_0}} | i = 1,\ldots,r\}$.

In the 1970s, inspired by this fundamental example, Stark had the clear vision to formulate a conjectural answer to the rational version of the question above in terms of G–equivariant L–functions. The G–equivariant L–function in question is

$$\Theta_S : \mathbf{C}\setminus\{1\} \to \mathbf{C}[G],\qquad \Theta_S(s) = \sum_{\chi\in\widehat{G}(\mathbf{C})} L_S(\chi, s)\cdot e_{\chi^{-1}},$$

where $L_S(\chi, s)$ denotes the complex valued Dirichlet L–function with Euler factors at primes in S removed and $e_\chi \in \mathbf{C}[G]$ is the usual idempotent, for all $\chi \in \widehat{G}(\mathbf{C})$. The function $\Theta_S(s)$ is holomorphic for $s \neq 1$. For compact subsets of the half plane $\Re e(s) > 1$ it admits the usual uniformly and absolutely convergent infinite product representation

$$\Theta_S(s) = \prod_{v\notin S}(1 - \sigma_v^{-1}\cdot (Nv)^{-s}), \tag{2}$$

where σ_v is the Frobenius morphism and Nv is the cardinality of the residue field associated to v. The leading (first non–vanishing) Taylor coefficients $\Theta_S^*(0)$ and $L_S^*(\chi, 0)$ at $s = 0$ obviously satisfy $\Theta_S^*(0) = \sum_\chi L^*(\chi, 0)\cdot e_{\chi^{-1}}$. Obviously, $\Theta_S^*(0)$ is a non–zero divisor in $\mathbf{C}[G]$. A theorem of Deligne-Ribet ([5], [4]) states that

$$\mathrm{Ann}_{\mathbf{Z}[G]}(\mu_K)\cdot \Theta_S(0) \in \mathbf{Z}[G]. \tag{3}$$

Stark's *conjectural answer* to Question R (rational version) formulated above is easily seen to be equivalent to the following (compare [17],[18],[19],[21],[23].)

Conjecture A (Stark, 1970s). *The following equivalent equalities hold.*

$$\mathbf{Q}\mathcal{L}_S = \Theta_S^*(0)^{-1}\cdot\ \det_{\mathbf{Q}[G]}(\mathbf{Q}U_S), \qquad \Theta_S^*(0)\cdot\mathbf{Q}\mathcal{L}_S =\ \det_{\mathbf{Q}[G]}(\mathbf{Q}U_S).$$

Obviously, the integral version of the above question is much more difficult and at the present moment we have only *partial conjectural answers* to it, due to work of Stark [21], Burns [2]–[3], Rubin [15], the present author [11], [10]. However, one can try (as Stark, Rubin and the present author have done in loc. cit.) to answer (at least conjecturally) a question which lies somewhere in between the rational and integral versions above. We describe this semi-integral question below. For every $\mathbf{Q}[G]$–module M and $\psi \in \widehat{G}(\mathbf{Q})$, let M^ψ denote the ψ–eigenspace of M. Obviously, $M = \oplus_\psi M^\psi$ and M^ψ is a $\mathbf{Q}(\chi)$–vector space, for all $\chi \in \widehat{G}(\overline{\mathbf{Q}})$, such that $\psi = \sum_{\sigma\in G(\mathbf{Q}(\chi)/\mathbf{Q})}\chi^\sigma$. As Tate shows in [23], we have the following equalities

$$r_{S,\psi} := \dim_{\mathbf{Q}(\chi)}(\mathbf{Q}U_S)^\psi = \dim_{\mathbf{Q}(\chi)}(\mathbf{Q}X_S)^\psi = \mathrm{ord}_{s=0}L_S(\chi, s) =: r_{S,\chi}, \tag{4}$$

for all $\psi \in \widehat{G}(\mathbf{Q})$ and all $\chi \in \widehat{G}(\overline{\mathbf{Q}})$, such that $\psi = \sum_{\sigma\in G(\mathbf{Q}(\chi)/\mathbf{Q})}\chi^\sigma$.

Question SI (semi-integral version). *For all $n \in \mathbf{Z}_{\geq 0}$, let $e_{S,n} := \sum_{\chi, r_{S,\chi}=n} e_\chi$. What is the preimage $\mathcal{L}_S^{(n)} := \mathcal{R}_S^{-1}(e_{S,n}\cdot \mathbf{Z}[G]) = e_{S,n}\mathcal{L}_S$? Equivalently, describe $\Theta_S^*(0)\mathcal{L}_S^{(n)} = \mathcal{R}_S^{-1}(e_{S,n}\Theta_S^*(0)\mathbf{Z}[G])$.*

Note that $\mathbf{Q}\mathcal{L}_S = \bigoplus_{n\geq 0}\mathbf{Q}\mathcal{L}_S^{(n)}$, whereas in general $\mathcal{L}_S \subsetneq \bigoplus_{n\geq 0}\mathcal{L}_S^{(n)}$. Consequently, answering the semi-integral version of the above question for all $n \in \mathbf{Z}_{\geq 0}$ will lead to an answer of the rational but not the integral version, in general. Also, note that the definition of the determinant leads to the following equalities

$$\det_{\mathbf{Q}[G]}(\mathbf{Q}U_S) = \bigoplus_{\psi\in\widehat{G}(\mathbf{Q})} \bigwedge_{\mathbf{Q}[G]}^{r_{S,\psi}}(\mathbf{Q}U_S)^\psi = \bigoplus_{n\in\mathbf{Z}_{\geq 0}} e_{S,n}(\mathbf{Q}\overset{n}{\wedge} U_S).$$

Since $\det_{\mathbf{C}[G]}(\mathbf{C}U_S) = \mathbf{C}\otimes_{\mathbf{Q}}\det_{\mathbf{Q}[G]}(\mathbf{Q}U_S)$, we can view $\mathcal{L}_S^{(n)}$ naturally as a $\mathbf{Z}[G]$–submodule of $e_{S,n}(\mathbf{C}\wedge^n U_S) \subseteq \mathbf{C}\wedge^n U_S$, for all $n \in \mathbf{Z}_{\geq 0}$.

Now, let us fix an integer $r \in \mathbf{Z}_{\geq 0}$ and make the following additional hypotheses.

Hypotheses (H_r)**.** *We have* $\operatorname{card}(S) \geq r+1$ *and* S *contains* r *distinct primes* $\{v_1, v_2, \dots, v_r\}$ *which split completely in* K/k*.*

Let us fix a set of primes $W := \{w_1, \dots, w_r\}$ in K, with w_i sitting above v_i, for all $i = 1, \dots, r$. Under hypotheses (H_r), (1) and (4) above imply right away that $r_{S,\chi} \geq r$, for all $\chi \in \widehat{G}(\overline{\mathbf{Q}})$. This shows that $e_{S,n} = 0$ and $\mathcal{L}_S^{(n)} = 0$, for all $n \leq r-1$. Also, in this case, (4) implies that we have equalities

$$e_{S,r}\Theta_S^*(0) = \Theta_S^{(r)}(0) := \lim_{s \to 0} \frac{\Theta_S(s)}{s^r} = \lim_{s \to 0} \sum_{\chi, r_{\chi,S}=r} \frac{L_S(\chi, s)}{s^r} \cdot e_{\chi^{-1}}.$$

Now, one can try to say something arithmetically meaningful about the first (potentially) non-vanishing lattice $\mathcal{L}_S^{(r)}$ in the sequence of lattices $(\mathcal{L}_S^{(n)})_{n \geq 0}$. The main reason why it is potentially easier to handle $\mathcal{L}_S^{(r)}$ is because under the current hypotheses we have a simple canonical choice of an $(e_{S,r} \cdot \mathbf{Q}[G])$–basis for $(e_{S,r} \cdot \mathbf{Q}X_S)$ given by $\{e_{S,r} \cdot (\mathbf{1}_{G/G_{v_i}} - \mathbf{1}_{G/G_{v_0}}) | i = 1, \dots, r\}$, for any $v_0 \in S \setminus \{v_1, ..., v_r\}$. This leads to a simple expression of the restriction $\mathcal{R}_S^{(r)}$ of the regulator $\mathcal{R}_S$ to the eigenspace $e_{S,r} \cdot \det_{\mathbf{C}[G]}(\mathbf{C}U_S) \subseteq \mathbf{C} \overset{r}{\wedge} U_S$ which contains the lattice $\mathcal{L}_S^{(r)}$. Namely

$$\mathcal{R}_S^{(r)} : e_{S,r} \cdot \ \det_{\mathbf{C}[G]}(\mathbf{C}U_S) \hookrightarrow \mathbf{C} \overset{r}{\wedge} U_S \xrightarrow{R_W} \mathbf{C}[G],$$

where the left–most map is the inclusion and, for all $u_1, \dots, u_r \in U_S$, we have

$$R_W(u_1 \wedge \dots \wedge u_r) = \det(-\sum_{\sigma \in G} \log |u_i|_{w_j^\sigma} \cdot \sigma).$$

In the case $r = 1$, Stark formulated in 1980 a far reaching conjecture regarding $\mathcal{L}_S^{(1)}$, a weak form of which we state below (see [21] and [23] for the original statement.)

Conjecture B (Stark, weak form). *Under hypotheses* (H_1)*, we have*

$$\Theta_S^{(1)}(0) \cdot \mathcal{L}_S^{(1)} \subseteq \mathbf{Z}[1/w_K]\widetilde{U_S}.$$

Equivalently, there exists $\varepsilon \in \mathbf{Z}[1/w_K]\widetilde{U_S}$*, such that* $R_W(\varepsilon) = \Theta_S^{(1)}(0)$*.*

At this point, there is overwhelming evidence in favor of Conjecture B even in its original, strong form. For example, it is known to hold true if $k = \mathbf{Q}$, or $k = \mathbf{Q}(\sqrt{-d})$ with $d \in \mathbf{Z}_{>0}$, or if $\operatorname{char}(k) = p > 0$ (see [23]). Its implications and applications to number theory are simply staggering (see [12], for example.) During the same year (1980), Stark started exploring the possibility of tackling the lattice $\mathcal{L}_S^{(2)}$, under hypotheses (H_2). Due to lack of evidence, he came up with a question rather than conjecture, now known among experts as "Stark's Question"(see [20]). We state a weak form of it below. For the original (stronger) formulation, the reader may consult [16], [13], for example.

Stark's Question (1980 - weak form). *Assume that* (H_2) *hold. Is it true that*

$$\Theta_S^{(2)}(0)\cdot\mathcal{L}_S^{(2)}\subseteq\mathbf{Z}[1/w_K]\widetilde{\overset{2}{\wedge}U_S}?$$

Equivalently, does there exist $\varepsilon\in\mathbf{Z}[1/w_K]\widetilde{\overset{2}{\wedge}U_S}$, *such that* $R_W(\varepsilon)=\Theta_S^{(2)}(0)$?

Originally, Stark formulated his conjectures and questions in the case of number fields (i.e. characteristic 0 global fields) only. The extension to arbitrary global fields is natural. In the case of characteristic $p>0$ global fields (function fields), we showed in [13] that the answer to the above question is negative, in general. In the case of characteristic 0 global fields (number fields), various mathematicians, most notably Grant [7], Sands [16], and Tangedal [22] have provided evidence in favor of an affirmative answer to the above question even in its original, much stronger form. In what follows, we will show that even in the case of number fields the above question has a negative answer, in general.

We should mention that in recent years, Rubin [15] and the present author [11] have given conjectural descriptions of the lattice $\mathcal{L}_S^{(r)}$, under hypotheses (H_r), for arbitrary values of r (including $r=2$.) At this point, there is strong evidence in support of these conjectures (see [12].) Conjectural descriptions of the family of lattices $(\mathcal{L}_S^{(n)})_{n\neq 0}$, under no additional assumptions on S, have also been given recently by Burns [2, 3] and the present author [10] (see also [6]). All these approaches are much more technical than Stark's original approach to these problems, but lead to the same type of far reaching applications in number theory.

2. STARK'S QUESTION AND FITTING IDEALS

Let R be an arbitrary commutative, unital, Noetherian ring and let M be a finitely generated R–module. In what follows, we remind the reader of the definition of the (first) Fitting ideal of the R–module M. We pick a set of say r generators for M and construct an exact sequence of R–modules

$$\mathcal{K}\xrightarrow{\phi}R^r\xrightarrow{\pi}M\to 0,$$

where π sends the standard basis of R^r into the chosen set of generators of M. Also, let $\det_R^{(r)}:\overset{r}{\underset{R}{\wedge}}R^r\xrightarrow{\sim}R$ be the canonical R–module isomorphism given by the determinant associated to the standard basis of R^r.

Definition 2.1. The (first) Fitting ideal $\mathrm{Fit}_R(M)$ of the R–module M is defined to be the image of the R–module morphism $(\det_R^{(r)}\circ\overset{r}{\underset{R}{\wedge}}\phi):\overset{r}{\underset{R}{\wedge}}\mathcal{K}\longrightarrow R$.

For the basic properties of Fitting ideals (including the independence on the various choices we have made when giving the above definition) the reader can consult the appendix of [9], for example. The main property we will use here is the extension of scalars property

$$\mathrm{Fit}_{R'}(M \otimes_R R') = \mathrm{Fit}_R(M)R',$$

for all R–modules M and all commutative R–algebras R' (see appendix of [9]).

Throughout the rest of the paper we work under the hypotheses that K/k is an abelian extension of *number fields* and the set of data $(K/k, S)$ satisfies hypotheses (H_2). Also, the notations will be the same as in the previous section. In addition, we assume that the two primes v_1, v_2 in S which split completely in K/k are finite primes. Let $S_0 := S \setminus \{v_1, v_2\}$ and let A_{S,S_0} be the $\mathbf{Z}[G]$–submodule of A_{K,S_0} generated by the ideal–classes $\widehat{w_1}$ and $\widehat{w_2}$ associated to w_1 and w_2, respectively. Then, we have an exact sequence of $\mathbf{Z}[G]$–modules (see [15], for example)

$$0 \to U_{S_0} \to U_S \xrightarrow{\lambda_W} \mathbf{Z}[G] \oplus \mathbf{Z}[G] \xrightarrow{j_W} A_{K,S_0}\,, \tag{5}$$

where j_W is the unique $\mathbf{Z}[G]$–module morphism satisfying $j_W(1,0) = \widehat{w_1}$, $j_W(0,1) = \widehat{w_2}$ (obviously, $\mathrm{Im}(j_W) = A_{S,S_0}$), and for all $u \in U_S$ one defines

$$\begin{aligned}\lambda_W(u) : &= (-\frac{1}{Nw_1}\sum_{\sigma\in G}\log|u|_{w_1^\sigma}\cdot\sigma, \quad -\frac{1}{Nw_2}\sum_{\sigma\in G}\log|u|_{w_2^\sigma}\cdot\sigma) = \\ &= (-\sum_{\sigma\in G}\mathrm{ord}_{w_1}(u^{\sigma^{-1}})\cdot\sigma, \qquad -\sum_{\sigma\in G}\mathrm{ord}_{w_2}(u^{\sigma^{-1}})\cdot\sigma).\end{aligned}$$

Proposition 2.2. *Let $R := \mathbf{Z}[1/w_K][G]$. If the split primes v_1 and v_2 are finite primes, then Stark's Question has an affirmative answer if and only if*

$$\Theta_{S_0}(0) \in \mathrm{Fit}_R(A_{S,S_0} \otimes R).$$

Proof. Since R is a flat $\mathbf{Z}[G]$–algebra, the exact sequence above remains exact after tensoring with R over $\mathbf{Z}[G]$. Definition 2.1 implies that we have an equality

$$\mathrm{Fit}_R(A_{S,S_0} \otimes R) = (\det{}_R^{(2)} \circ (\overset{2}{\wedge}\lambda_W \otimes \mathbf{1}_R))(\widetilde{\mathbf{Z}[1/w_K]\overset{2}{\wedge}U_S})\,.$$

However, the definitions of λ_W and R_W lead to the equality

$$R_W(\varepsilon) = (\log Nw_1 \cdot \log Nw_2) \cdot (\det{}_R^{(2)} \circ (\overset{2}{\wedge}\lambda_W \otimes \mathbf{1}_R))(\varepsilon),$$

for all $\varepsilon \in \widetilde{\mathbf{Z}[1/w_K]\overset{2}{\wedge}U_S}$. On the other hand, since v_1 and v_2 split completely in K/k, (2) gives the following equality at the level of G–equivariant L–values.

$$\Theta_S^{(2)}(0) = (\log Nw_1 \cdot \log Nw_2) \cdot \Theta_{S_0}(0).$$

Combining the last three equalities concludes the proof of the proposition. □

The main strategy employed in producing examples for which Stark's Question has a negative answer will consist of constructing examples where v_1 and v_2 are finite primes, but where $\Theta_{S_0}(0) \notin \mathrm{Fit}_R(A_{S,S_0} \otimes R)$.

3. GREITHER'S "NICE "EXTENSIONS

In what follows, we describe a class of abelian extensions K/k of number fields, first introduced by Greither in [8]. Assume that k is a totally real number field and K is a CM–field, with K/k abelian of Galois group $G := G(K/k)$. As usual, K^+ denotes the maximal totally real subfield of K. Since K is CM, $G(K/K^+)$ has order two and is generated by the unique complex conjugation automorphism of K, denoted by j in what follows. Let K^{cl} denote the Galois closure of K over $\mathbf{Q}$. It is not difficult to check that K^{cl} is a CM–field as well.

Definition 3.1. Let $\mathfrak{p}$ be a finite prime in k of residual characteristic p. Then $\mathfrak{p}$ is called **critical** for K/k if at least one of the following conditions is satisfied.
(1) $\mathfrak{p}$ is ramified in K/k.
(2) $K^{cl} \subseteq (K^{cl})^+(\zeta_p)$, where $\zeta_p := e^{\frac{2\pi i}{p}}$.

Definition 3.2. An extension K/k as above is called **nice** if the following conditions are simultaneously satisfied.
(1) For all critical primes $\mathfrak{p}$ in k, the decomposition group $G_\mathfrak{p}$ of $\mathfrak{p}$ in K/k contains j.
(2) The $\mathbf{Z}[G]$–module $\mu_K \otimes \mathbf{Z}[1/2]$ is G–cohomologically trivial.

In what follows, if $\mathcal{R}$ is a commutative ring containing $\mathbf{Z}[1/2][G]$ and M is an $\mathcal{R}$–module, we let $M^- := (1-j)M$ and $M^+ = (1+j)M$ denote the usual "minus"and "plus "eigenspaces of M relative to the action of j. Obviously, M^- and M^+ are $\mathcal{R}$–submodules of M and one has a direct sum decomposition $M = M^- \oplus M^+$. Obviously, M^- inherits a natural module structure over the ring $\mathcal{R}^-$.

Remark. Note that the rings $\mathcal{R}^-$ and $\mathcal{R}_- := \mathcal{R}/(1+j)$ are isomorphic via the obvious projection modulo $(1+j)$ map. Also, note that under the current assumptions we have $\Theta_{S_0}(0) \in \mathbf{Z}[1/w_K][G]^-$. Indeed, $\Theta_{S_0}(0) \in \mathbf{Z}[1/w_K][G]$ (see (3) in §1 above), and $(1+j)\Theta_{S_0}(0) = 0$ because for all $\chi \in \widehat{G}(\mathbf{C})$ we have $\chi(1+j) = 0$ if $\chi(j) = -1$ and $\chi(\Theta_{S_0}(0)) = 0$ if $\chi(j) = +1$ (apply (4) for the last equality.)

Let A_K denote the ideal–class group of K. In [8], Greither proves the following.

Theorem 3.3 (Greither). *Let K/k be a nice extension of Galois group G and let S_0 be the set consisting of all the primes in k which ramify in K/k. Let $\mathcal{R} := \mathbf{Z}[1/2][G]$ and let $\mathcal{A} := \mathrm{Ann}_\mathcal{R}(\mu_K \otimes \mathbf{Z}[1/2])$. Then one has an $\mathcal{R}^-$–module isomorphism*

$$(A_K \otimes \mathcal{R})^- \xrightarrow{\sim} \mathcal{R}^-/(\mathcal{A} \cdot \Theta_{S_0}(0)).$$

An immediate consequence of Greither's theorem is the following.

Corollary 3.4. *Under the assumptions of Theorem 3.3, let* $R := \mathbf{Z}[1/w_K][G]$. *Then, one has an isomorphism of* R^-*–modules*

$$(A_{K,S_0} \otimes R)^- \xrightarrow{\sim} R^-/(\Theta_{S_0}(0)) .$$

In particular, $\mathrm{Fit}_{R^-}((A_{K,S_0} \otimes R)^-) = \Theta_{S_0}(0) \cdot R^-$.

Proof. First, let's observe that since μ_K is cyclic, we have an $\mathcal{R}$–module isomorphism $\mu_K \otimes \mathbf{Z}[1/2] \xrightarrow{\sim} \mathcal{R}/\mathcal{A}$. If we tensor this isomorphism with $\mathbf{Z}[1/w_K]$ we conclude that $\mathcal{A} \cdot R = R$. Second, we have an exact sequence of $\mathbf{Z}[G]$–modules

$$\bigoplus_{v \in S_0} \mathbf{Z}[G/G_v] \to A_K \to A_{K,S_0} \to 0,$$

where the left–most map sends $1 \in \mathbf{Z}[G/G_v]$ into the class $\widehat{w}$ of w in A_K for the fixed w sitting above v in K if $v \in S_0$ is finite, and into 0 if $v \in S_0$ is infinite. Since R is a flat $\mathbf{Z}[G]$–algebra, the sequence above remains exact after tensoring with R and taking (-1)–eigenspaces. However, since $j \in G_v$, we have $(\mathbf{Z}[G/G_v] \otimes R)^- = 0$, for all $v \in S_0$. Therefore, we obtain an isomorphism of R^-–modules

$$(A_K \otimes R)^- \xrightarrow{\sim} (A_{K,S_0} \otimes R)^-.$$

Now, the corollary follows directly from Greither's Theorem 3.3. □

Lemma 3.5. *Under the assumptions of Theorem 3.3, if* v_1 *and* v_2 *are two distinct primes in* k *which split completely in* K/k*, then the answer to Stark's Question for* $(K/k, S := S_0 \cup \{v_1, v_2\})$ *is affirmative if and only if*

$$\Theta_{S_0}(0) \in \mathrm{Fit}_{R^-}((A_{S,S_0} \otimes R)^-).$$

Proof. This is a direct consequence of Proposition 2.2 and the observation that, under the current hypotheses, $\Theta_{S_0}(0) \in R^-$ (see the Remark above). □

4. A SPECIAL CLASS OF "NICE "EXTENSIONS

In what follows, we will restrict our search to a class of nice extensions satisfying a set of additional properties and show that for this class the answer to Stark's Question is negative. For the moment, let K/k be a nice extension of Galois group $G := G(K/k)$, such that the following hold.

(N1) There exists a prime number p which does not divide w_K and a p–subgroup P of G, such that G is the internal direct product $G = \langle j \rangle \times P$ of P and its subgroup $\langle j \rangle$ generated by the complex conjugation morphism j. Note that this condition is satisfied if and only if $\mathrm{card}(G) = 2 \cdot p^n$, for some prime number $p \nmid w_K$ and some natural number n.

(N2) The subgroup P in (1) above is bicyclic, i.e. $P = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$, with $\sigma_1, \sigma_2 \in P$, both nontrivial.

Under these assumptions, we have obvious ring isomorphisms

$$\mathbf{Z}[G]/(1+j) \xrightarrow{\sim} \mathbf{Z}[P], \quad R^- := \mathbf{Z}[1/w_K][G]^- \xrightarrow{\sim} \mathbf{Z}[1/w_K][P], \quad R_p^- \xrightarrow{\sim} \mathbf{Z}_p[P],$$

where $\mathbf{Z}_p$ denotes the ring of p–adic integers and $R_p^- := (R \otimes \mathbf{Z}_p)^- = \mathbf{Z}_p[G]^-$. Since P is a finite, abelian p–group, the ring $\mathbf{Z}_p[P]$ is a local ring of maximal ideal $\mathcal{M} := p\mathbf{Z}_p[P] + I_P$, where I_P is the usual augmentation ideal in $\mathbf{Z}_p[P]$. Under the current assumptions, I_P is generated as an ideal by the set $\{\sigma_1 - 1,\, \sigma_2 - 1\}$.

Proposition 4.1. *Let θ be a fixed element of $\mathcal{M}$. Assume that θ is not a zero-divisor in $\mathbf{Z}_p[P]$. Let $I_{P,\theta}$ denote the $\mathbf{Z}_p[P]$–module $(I_P + \theta\mathbf{Z}_p[P])/\theta\mathbf{Z}_p[P]$. Then*

$$\mathrm{Fit}_{\mathbf{Z}_p[P]}(I_{P,\theta}) \subseteq \mathcal{M}^2\,.$$

Proof. Obviously, the $\mathbf{Z}_p[P]$–module $I_{P,\theta}$ is generated by the classes $\widehat{\sigma_i - 1}$ of $\sigma_i - 1$ modulo $\theta\mathbf{Z}_p[P]$, for $i = 1, 2$. Therefore, we have an exact sequence of $\mathbf{Z}_p[P]$–modules

$$0 \to \mathcal{K} \to \mathbf{Z}_p[P] \oplus \mathbf{Z}_p[P] \xrightarrow{\pi} I_{P,\theta} \to 0,$$

where π is the $\mathbf{Z}_p[P]$–linear map satisfying $\pi(1,0) = \widehat{\sigma_1 - 1}$ and $\pi(0,1) = \widehat{\sigma_2 - 1}$ and $\mathcal{K} := \ker(\pi)$. According to Definition 2.1, the statement in the Proposition would be a direct consequence of the inclusion $\mathcal{K} \subseteq \mathcal{M} \oplus \mathcal{M}$. We proceed to proving this inclusion. Let $\chi \in \widehat{P}(\mathbf{C}_p)$, $\chi \neq \mathbf{1}_P$. We extend χ in the obvious manner to a (surjective) ring morphism $\widetilde{\chi} : \mathbf{Z}_p[P] \twoheadrightarrow \mathbf{Z}_p[\chi]$, where $\mathbf{Z}_p[\chi]$ is the (finite, totally ramified) extension of $\mathbf{Z}_p$ generated by the values of χ. Let $\mathcal{M}_\chi$ denote the maximal ideal of the local ring $\mathbf{Z}_p[\chi]$. Then, since $\mathbf{Z}_p[P]$ is local of Krull dimension one and $\mathbf{Z}_p[P]/\mathcal{M} \xrightarrow{\sim} \mathbf{Z}_p[\chi]/\mathcal{M}_\chi \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}$, we have

$$\widetilde{\chi}^{-1}(\mathcal{M}_\chi) = \mathcal{M}, \qquad x \in \mathbf{Z}_p[P]^\times \iff \widetilde{\chi}(x) \in \mathbf{Z}_p[\chi]^\times, \tag{6}$$

for all χ as above and all $x \in \mathbf{Z}_p[P]$. Let χ_1 denote a generator of the $\mathbf{C}_p$–valued character group of (the cyclic p–group) $P/\langle\sigma_2\rangle \xrightarrow{\sim} \langle\sigma_1\rangle$, and similarly define χ_2. Then $\mathcal{M}_{\chi_i}$ is generated by $(1 - \chi_i(\sigma_i))$, for $i = 1, 2$. Let $\alpha := (\alpha_1, \alpha_2)$ be an arbitrary element in $\mathbf{Z}_p[G] \oplus \mathbf{Z}_p[G]$. Assume that $\alpha \in \mathcal{K}$. Then, by the definition of $I_{P,\theta}$, there exists $\beta \in \mathbf{Z}_p[P]$, such that

$$\alpha_1 \cdot (\sigma_1 - 1) + \alpha_2 \cdot (\sigma_2 - 1) = \beta \cdot \theta.$$

However, since θ is not a zero divisor in $\mathbf{Z}_p[P]$, in particular $\theta \notin I_P$ and therefore $\beta \notin \mathbf{Z}_p[P]^\times$. Consequently, $\beta \in \mathcal{M}$. If we apply $\widetilde{\chi}_1$ to the equality above and take into account that $\widetilde{\chi}_1(\theta) \in \widetilde{\chi}_1(\mathcal{M}) = \mathcal{M}_{\chi_1}$ and $\mathcal{M}_{\chi_1} = (1 - \chi_1(\sigma_1))\mathbf{Z}_p[\chi_1]$, we get

$$\widetilde{\chi}_1(\alpha_1) \in \widetilde{\chi}_1(\beta)\mathbf{Z}_p[\chi_1] \subseteq \mathcal{M}_{\chi_1}.$$

According to (6) above, this implies that $\alpha_1 \in \mathcal{M}$. Similarly, if one uses $\widetilde{\chi}_2$ instead of $\widetilde{\chi}_1$, one proves that $\alpha_2 \in \mathcal{M}$. Consequently, $\alpha \in \mathcal{M} \oplus \mathcal{M}$. This proves the desired inclusion $\mathcal{K} \subseteq \mathcal{M} \oplus \mathcal{M}$, concluding the proof of Proposition 4.1. □

In what follows, we identify the rings $R_p^- := \mathbf{Z}_p[G]^-$ and $\mathbf{Z}_p[P]$ via the canonical ring isomorphism $\mathbf{Z}_p[G]^- \xrightarrow{\sim} \mathbf{Z}_p[G]/(1+j) \xrightarrow{\sim} \mathbf{Z}_p[P]$. Therefore elements of $\mathbf{Z}_p[G]^-$ (e.g. $\Theta_{S_0}(0)$) will be viewed inside $\mathbf{Z}_p[P]$ via this identification. Next, we will assume that the nice extension K/k satisfies the extra-hypotheses **(N1)-(N2)** above and in addition the following hypothesis.

(N3) We have $\Theta_{S_0}(0) \in \mathcal{M} \setminus \mathcal{M}^2$, where $\mathcal{M}$ is the (unique) maximal ideal of the local ring $\mathbf{Z}_p[P]$.

Remark. Note that, under the current hypotheses, we have equivalences

$$\Theta_{S_0}(0) \in \mathcal{M} \iff (A_K \otimes \mathbf{Z}_p)^- \neq \{0\} \iff (A_{K,S_0} \otimes \mathbf{Z}_p)^- \neq \{0\}.$$

Indeed, this follows immediately if one tensors the isomorphisms in Theorem 3.3 and Corollary 3.4 with $\mathbf{Z}_p$ and takes into account that $\mathbf{Z}_p[P]^\times = \mathbf{Z}_p[P] \setminus \mathcal{M}$.

Theorem 4.2. *Let K/k be a nice extension satisfying hypotheses* **(N1)–(N3)** *above. Then there exist infinitely many sets $\{v_1, v_2\}$ of distinct primes in k which split completely in K/k, such that the answer to Stark's Question for the set of data $(K/k, S := S_0 \cup \{v_1, v_2\})$ is negative.*

Proof. According to Lemma 3.5, it is sufficient to construct sets $\{v_1, v_2\}$ of distinct primes in k, totally split in K/k, such that if we let $S := S_0 \cup \{v_1, v_2\}$, then $\Theta_{S_0}(0) \notin \mathrm{Fit}_{R_p^-}((A_{S,S_0} \otimes \mathbf{Z}_p)^-)$, where p is the (odd) prime number produced by hypothesis **(N1)**. Let $\theta := \Theta_{S_0}(0)$. We tensor the isomorphism in Theorem 3.3 with $\mathbf{Z}_p$. On one hand, we conclude that $\theta \in \mathcal{M}$ is not a zero-divisor of $\mathbf{Z}_p[P]$, because $(A_{K,S_0} \otimes \mathbf{Z}_p)^-$ is finite and therefore the ideal $\mathbf{Z}_p[P]\theta$ has finite index in the group ring $\mathbf{Z}_p[P]$. On the other hand, we obtain a $\mathbf{Z}_p[P]$–module isomorphism

$$\xi_p : \mathbf{Z}_p[P]/\theta\mathbf{Z}_p[P] \xrightarrow{\sim} (A_{K,S_0} \otimes \mathbf{Z}_p)^-.$$

Now, the module on the left has a submodule $I_{P,\theta}$ defined in Proposition 4.1 and generated by the classes $\widehat{(\sigma_i - 1)}$ of $(\sigma_i - 1)$ modulo $\theta\mathbf{Z}_p[P]$, for $i = 1, 2$. Chebotarev's Density Theorem implies that there are infinitely many sets $\{w_1, w_2\}$ of primes in K, satisfying the following properties.

(1) w_i sits above a prime v_i in k which splits completely in K/k, for all $i = 1, 2$.
(2) The primes v_1 and v_2 are distinct (i.e. w_1 and w_2 are not G–conjugate.)
(3) The class $\widehat{w_i}$ of w_i in $(A_{K,S_0} \otimes \mathbf{Z}_p)^-$ is equal to $\xi_p(\widehat{(\sigma_i - 1)})$, for all $i = 1, 2$.

Choose $\{w_1, w_2\}$ satisfying (1)–(3) above and let $\{v_1, v_2\}$ be as in (2) above. We claim that the answer of Stark's Question for the set of data $(K/k, S := S_0 \cup \{v_1, v_2\})$ is negative. Indeed, by (2) above ξ_p establishes a $\mathbf{Z}_p[P]$–module isomorphism between $I_{P,\theta}$ and the submodule of $(A_{K,S_0} \otimes \mathbf{Z}_p)^-$ generated by $\{\widehat{w_1}, \widehat{w_2}\}$. Since this submodule is by definition $(A_{S,S_0} \otimes \mathbf{Z}_p)^-$ (a direct consequence of tensoring exact sequence (5) with R_p^- over $\mathbf{Z}[G]$), ξ_p induces a $\mathbf{Z}_p[P]$–module isomorphism

$$I_{P,\theta} \xrightarrow{\sim} (A_{S,S_0} \otimes \mathbf{Z}_p)^-.$$

Consequently, Proposition 4.1 implies that $\mathrm{Fit}_{\mathbf{Z}_p[P]}((A_{S,S_0}\otimes\mathbf{Z}_p)^-)\subseteq\mathcal{M}^2$. Since by hypothesis $\Theta_{S_0}(0)\notin\mathcal{M}^2$, we have

$$\Theta_{S_0}(0)\notin\mathrm{Fit}_{\mathbf{Z}_p[P]}((A_{S,S_0}\otimes\mathbf{Z}_p)^-)\,.$$

This concludes the proof of the Theorem. □

Remark. If in our proof of Theorem 4.2 we apply the full strength Chebotarev's Density Theorem, we can even show that the set of sets $\{v_1,v_2\}$ for which the conclusion of Theorem 4.2 holds true has density at least $2/\mathrm{card}(G)^2$.

Obviously, Theorem 4.2 only shows that *if we can find* nice extensions K/k satisfying properties **(N1)–(N3)**, then we can produce infinitely many examples in which Stark's Question has a negative answer. Our next task consists of showing that we can construct such extensions indeed. This will be fulfilled in the next two sections.

5. A CLOSER LOOK AT PROPERTIES (N1)–(N3)

In this section we take a closer look at the special class of nice extensions characterized by properties **(N1)–(N3)** above. The point is that condition **(N3)** is very difficult to verify in practice, which makes the construction of nice extensions satisfying **(N3)** difficult. In this section we will replace **(N3)** by a condition which is slightly stronger but much easier to verify in practice. This will help us construct concrete examples of extensions for which Stark's Question has a negative answer.

Let K/k be a nice extension satisfying the additional properties **(N1)–(N2)**. We use the same notations as in the previous section. Let $K':=K^P$ be the maximal subfield of K fixed by P and let $G':=\mathrm{Gal}(K'/k)$. Since $G'\xrightarrow{\sim}G/P\xrightarrow{\sim}\langle j\rangle$, in order to avoid additional notation, we identify the generator of G' with j. It is an easy exercise to show that K'/k is also a nice extension. Let

$$h_K^-:=\mathrm{card}((A_K\otimes\mathbf{Z}[1/2])^-),\quad h_{K'}^-:=\mathrm{card}((A_{K'}\otimes\mathbf{Z}[1/2])^-).$$

Proposition 5.1. *Assume that the nice extension K/k satisfies properties* **(N1)–(N2)**. *Then K/k satisfies property* **(N3)** *if the following hold true.*
(1) p divides h_K^-.
(2) p^2 does not divide $h_{K'}^-$.

Proof. If one tensors the isomorphism in Theorem 3.3 with $\mathbf{Z}_p$, one obtains an isomorphism of $\mathbf{Z}_p[P]$–modules

$$(A_K\otimes\mathbf{Z}_p)^-\xrightarrow{\sim}\mathbf{Z}_p[P]/\Theta_{S_0}(0)\mathbf{Z}_p[P].$$

This implies that condition (1) above is equivalent to $\Theta_{S_0}(0)\notin\mathbf{Z}_p[P]^\times$, or equivalently $\Theta_{S_0}(0)\in\mathcal{M}$. It remains for us to show that condition (2) above implies that $\Theta_{S_0}(0)\notin\mathcal{M}^2$. This is an immediate consequence of the following.

Lemma 5.2. *Assume that K/k is a nice extension satisfying* **(N1)–(N2)**. *Let $n\in\mathbf{Z}_{\geq 0}$ and assume that $\Theta_{S_0}(0)\in\mathcal{M}^n$. Then $p^n\mid h_{K'}^-$.*

Proof of Lemma 5.2. Let χ be the non-trivial character of G'. We identify χ with the character of $G = P \times \langle j \rangle$ which is trivial on P and sends j to (-1). We extend χ in the usual manner to a ring morphism $\chi : \mathbf{Z}_p[G] \longrightarrow \mathbf{Z}_p$. Since $\chi(\mathbf{Z}_p[G]^+) = \{0\}$, the character χ induces a surjective ring morphism (also denoted by χ)

$$\chi : \mathbf{Z}_p[P] \xrightarrow{\sim} \mathbf{Z}_p[G]^- \twoheadrightarrow \mathbf{Z}_p,$$

which sends the augmentation ideal I_P of $\mathbf{Z}_p[P]$ to 0. Since $\mathcal{M} := I_P + p\mathbf{Z}_p[P]$, we have $\chi(\mathcal{M}) = p\mathbf{Z}_p$. Consequently, for n as in the hypotheses of our lemma, we have $\chi(\Theta_{S_0}(0)) \in p^n\mathbf{Z}_p$. On the other hand, $\chi(\Theta_{S_0}(0)) = L_{S_0}(\chi, 0)$ (see §1 above). However, in an open neighborhood of $s = 0$, we also have an equality of holomorphic functions $\zeta_{K',S_0}(s) = L_{S_0}(\chi, s) \cdot \zeta_{k,S_0}(s)$. Consequently, we have

$$\lim_{s\to 0} \frac{\zeta_{K',S_0}(s)}{\zeta_{k,S_0}(s)} \in p^n\mathbf{Z}_p. \tag{7}$$

Dirichlet's class-number formula shows that the leading term in the power series expansion of $\zeta_{K',S_0}(s)$ at $s = 0$ is given by,

$$\zeta^*_{K',S_0}(s) = -\frac{h_{K',S_0} \cdot R_{K',S_0}}{w_{K'}} \cdot s^{r_{K',S_0}}, \tag{8}$$

where $h_{K',S_0} = \mathrm{card}(A_{K',S_0})$, R_{K',S_0} is the usual Dirichlet regulator of the group U_{K',S_0} of S_0–units in K' and $r_{K',S_0} := \dim_{\mathbf{Q}} \mathbf{Q}U_{K',S_0}$. A similar formula holds for $\zeta^*_{k,S_0}(s)$. There is an exact sequence of $\mathbf{Q}[G']$–modules (see §1 above)

$$0 \to \mathbf{Q}U_{K',S_0} \to \bigoplus_{v\in S_0} \mathbf{Q}[G'/G'_v] \to \mathbf{Q} \to 0,$$

where the rightmost nontrivial term is endowed with the trivial G'–action. If we tensor the above sequence with $\mathbf{Q}[G']^-$ over $\mathbf{Q}[G']$ and take into account that $\mathbf{Q}[G'/G'_v]^- = 0$, for all $v \in S_0$ (recall that $j \in G'_v$, for all $v \in S_0$), we conclude that $(\mathbf{Q}U_{K',S_0})^- = 0$. Consequently, we have $\mathbf{Q}U_{K',S_0} = (\mathbf{Q}U_{K',S_0})^+ = \mathbf{Q}U_{k,S_0}$. This equality, combined with the fact that p is odd (and therefore coprime to $\mathrm{card}(G') = 2$) and coprime to $w_{K'}$, implies right away that

$$r_{K',S_0} = r_{k,S_0}, \quad \mathbf{Z}_pU_{K',S_0} = \mathbf{Z}_pU_{k,S_0}, \quad \frac{R_{K',S_0}}{R_{k,S_0}} \in \mathbf{Z}_p^\times. \tag{9}$$

Since $j \in G'_v$, for all $v \in S_0$, and p is odd (therefore coprime to $\mathrm{card}(G') = 2$), a standard application of class–field theory implies that the usual ideal–norm map $N_{K'/k} : A_{K',S_0} \to A_{k,S_0}$ induces an isomorphism

$$N_{K'/K} : (A_{K',S_0} \otimes \mathbf{Z}_p)^+ \xrightarrow{\sim} (A_{k,S_0} \otimes \mathbf{Z}_p).$$

This implies that $(h_{K',S_0}/h_{k,S_0})\mathbf{Z}_p = h^-_{K',S_0}\mathbf{Z}_p$, where $h^-_{K',S_0} := \mathrm{card}((A_{K',S_0} \otimes \mathbf{Z}[1/2])^-)$. If we combine this equality with (7–9) above, we conclude that indeed $p^n \mid h^-_{K',S_0}$. However, as in the proof of Corollary 3.4, we have an isomorphism

$$(A_{K',S_0} \otimes \mathbf{Z}[1/2])^- \xrightarrow{\sim} (A_{K'} \otimes \mathbf{Z}[1/2])^-.$$

Therefore $h^-_{K',S_0} = h^-_{K'}$ and $p^n \mid h^-_{K'}$. This concludes the proofs of Lemma 5.2 and Proposition 5.1. □

□

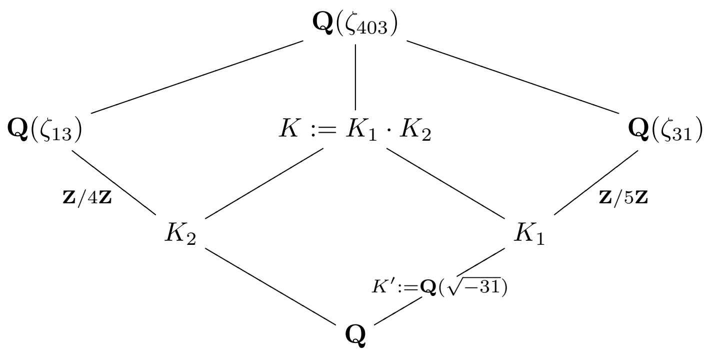
Corollary 5.3. *Let K/k be a nice extension satisfying properties* **(N1)**–**(N2)** *and such that $p \mid h^-_K$ and $p^2 \nmid h^-_{K'}$. Then there are infinitely many sets $\{v_1, v_2\}$ of distinct primes in k which split completely in K/k, such that Stark's Question for the set of data $(K/k, S := S_0 \cup \{v_1, v_2\})$ has a negative answer*

Proof. Combine Theorem 4.2 with Proposition 5.1. □

6. CONSTRUCTING CONCRETE EXAMPLES

In this section, we apply Corollary 5.3 to construct a concrete example in which Stark's Question has a negative answer. In what follows, we let $\zeta_n := e^{2\pi i/n}$, for all $n \in \mathbf{Z}_{>0}$. Let K_1 be the unique subfield of $\mathbf{Q}(\zeta_{31})$ with the property that $G(K_1/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}/6\mathbf{Z}$. Let K_2 be the unique subfield of $\mathbf{Q}(\zeta_{13})$ with the property that $G(K_2/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}/3\mathbf{Z}$. We define K to be the compositum $K_1 \cdot K_2$ of K_1 and K_2 inside $\mathbf{Q}(\zeta_{403})$. For obvious ramification related reasons, we have

$$G := G(K/\mathbf{Q}) \xrightarrow{\sim} G(K_1/\mathbf{Q}) \times G(K_2/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}.$$



Proposition 6.1. *For K defined above, the extension $K/\mathbf{Q}$ is "nice"and it satisfies the hypotheses of Corollary 6.2 for the prime number $p := 3$.*

Proof. **Step 1.** First, we will show that the extension $K/\mathbf{Q}$ is a nice extension à la Greither (see Definition 3.2 above). Since $[\mathbf{Q}(\zeta_{31}) : K_1] = 5$, the field K_1 and consequently $K := K_1 \cdot K_2$ are imaginary abelian extensions of $\mathbf{Q}$ and therefore CM–fields. Greither shows in [8] that if $K/\mathbf{Q}$ is an imaginary abelian extension of $\mathbf{Q}$ such that $[K^+ : \mathbf{Q}]$ is odd, then the critical primes for $K/\mathbf{Q}$ are precisely the primes which ramify in $K/\mathbf{Q}$. In our case, $[K^+ : \mathbf{Q}] = 9$ and consequently the critical primes are $\mathfrak{p} = 13$ and $\mathfrak{p} = 31$. Now, it is an elementary exercise to show that we have $G_{13} = G_{31} = G$, where $G_\mathfrak{p}$ denotes the decomposition group of $\mathfrak{p}$ in $K/\mathbf{Q}$, as usual.

Consequently the complex conjugation morphism j belongs to $G_{\mathfrak{p}}$, for all critical primes $\mathfrak{p}$. Obviously, we have $\mu_K = \{+1, -1\}$ in this case. Consequently $\mu_K \otimes \mathbf{Z}[1/2]$ is (G–cohomologically) trivial. This concludes the proof of the fact that $K/\mathbf{Q}$ is a nice extension.

Step 2. Obviously, the extension $K/\mathbf{Q}$ satisfies properties **(N1)-(N2)** with $p := 3$ and $P := G(K/K') \xrightarrow{\sim} \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$, where $K' := \mathbf{Q}(\sqrt{-31}) \subseteq K_1$.

Step 3. Finally, we have to show that for K and K' defined above and $p := 3$, we have $p \mid h_K^-$ and $p^2 \nmid h_{K'}^-$. Indeed, the tables in [1] show that $h_{K'} = 3 = p$. However, since $h_{K'}^+ := \operatorname{card}(A_{K'} \otimes \mathbf{Z}[1/2])^+ = \operatorname{card}(A_{\mathbf{Q}} \otimes \mathbf{Z}[1/2]) = 1$, this implies that $h_{K'}^- = p$. Also, since the extension K_1/K' is totally ramified at $\mathfrak{p} = 31$ and K/K_1 is totally ramified at $\mathfrak{p} = 13$, a standard application of class-field theory implies that the ideal-norm maps induce surjective morphisms of $\mathbf{Z}[G]$-modules

$$A_K \overset{N_{K/K_1}}{\twoheadrightarrow} A_{K_1} \overset{N_{K_1/K'}}{\twoheadrightarrow} A_{K'}\,.$$

If we tensor the above diagram with $\mathbf{Z}[1/2][G]^-$ over $\mathbf{Z}[G]$, then surjectivity is preserved and we obtain a divisibility $h_{K'}^- \mid h_K^-$. Since $h_{K'}^- = p$, this implies that $p \mid h_K^-$ as well. This concludes the proof of Proposition 6.1. □

Corollary 6.2. *For the extension $K/\mathbf{Q}$ defined above, there exist infinitely many sets $\{v_1, v_2\}$ of distinct primes in $\mathbf{Q}$ which split completely in $K/\mathbf{Q}$, such that Stark's Question for the data $(K/\mathbf{Q}, S := \{\infty, 13, 31, v_1, v_2\})$ has a negative answer.*

Proof. Combine Proposition 6.1 and Corollary 5.3 above. □

Final Comment. We conclude with a comment emphasizing the difference between the class of examples we constructed in [13] and the one constructed in the present paper. In the characteristic $p > 0$ setting, we constructed in [13] examples of abelian extensions K/k for which the p–part of the so-called Strong Brumer Conjecture (roughly stating that $\Theta_{S_0}(0) \in \operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{K,S_0} \otimes \mathbf{Z}_p)$) is false. For these examples we showed that (the p-part of) Stark's Question has a negative answer. In the characteristic 0 setting we work from the beginning with "nice"extensions K/k, for which the odd part of the Strong Brumer Conjecture holds true (see Theorem 3.3 above). Despite this fact, we show that for a special class of such "nice"extensions Stark's Question has a negative answer still. This situation might seem mystifying at first. However, as Proposition 2.2 above shows, Stark's Question is equivalent to $\Theta_{S_0} \in \operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{S,S_0} \otimes \mathbf{Z}_p)$, for all primes $p \nmid w_K$. Now, since the groups $G = \operatorname{Gal}(K/k)$ we are considering have *a non-cyclic p-Sylow subgroup P,* for some prime p as above, although $A_{S,S_0} \otimes \mathbf{Z}_p \subseteq A_{K,S_0} \otimes \mathbf{Z}_p$, there is in general no link between $\operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{S,S_0} \otimes \mathbf{Z}_p)$ and $\operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{K,S_0} \otimes \mathbf{Z}_p)$ (see appendix of [9]). That is one of the main reasons why although the Strong Brumer Conjecture is true, Stark's Question has a negative answer in general, in the cases considered in this paper. Let's note that for a "nice"extension K/k with the property that the p–Sylow subgroups P of its Galois group are cyclic, for all primes $p \nmid w_K$, Stark's Question has an affirmative answer. Indeed, under these hypotheses we have an inclusion $\operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{K,S_0} \otimes \mathbf{Z}_p) \subseteq \operatorname{Fit}_{\mathbf{Z}_p[G]}(A_{S,S_0} \otimes \mathbf{Z}_p)$ (see appendix of [9]), for all $p \nmid w_K$, and therefore Stark's Question has an affirmative answer, as a direct consequence of Proposition 2.2 and Greither's Theorem 3.3.

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Mathematics Department, 0112, Univ. of California, San Diego, La Jolla CA 92093-0112

E-mail address: cpopescu@math.ucsd.edu