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"BOTTOM OF THE WELL" SEMI-CLASSICAL TRACE INVARIANTS

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Abstract. Let $\hat{H}$ be an $\hbar$-admissible pseudodifferential operator whose principal symbol, H, has a unique non-degenerate global minimum. We give a simple proof that the semi-classical asymptotics of the eigenvalues of $\hat{H}$ corresponding to the "bottom of the well" determine the Birkhoff normal form of H at the minimum. We treat both the resonant and the non-resonant cases.

1. Introduction

Let X be an $n-$dimensional manifold and $\hat{H}$ a self-adjoint zeroth order semi-classical ΨDO acting on the space of half-densities, $|\Omega|^{1/2}(X)$. We will assume that the principal symbol, $H(x,\xi)$, of $\hat{H}$ has a unique non-degenerate global minimum, $H = C$, at some point (x_0,ξ_0), and that outside a small neighborhood of (x_0,ξ_0) H is bounded from below by $C+\delta$, for some $\delta > 0$. We will also assume that at (x_0,ξ_0) the subprincipal symbol of $\hat{H}$ vanishes. From these assumptions one can deduce that on an interval

$$C < E < C+\epsilon, \quad \epsilon < \delta,$$

the spectrum of $\hat{H}$ is discrete and consists of eigenvalues:

$$E_i(\hbar), \quad 1 \le i \le N(\hbar), \tag{1.1}$$

where

$$N(\hbar) \sim (2\pi\hbar)^{-n} \operatorname{Vol} \{ (x,\xi) \,;\, H(x,\xi) \le C+\epsilon \}. \tag{1.2}$$

In addition, we will make a non-degeneracy assumption on the Hessian of $H(x,\xi)$ at (x_0,ξ_0). Choose a Darboux coordinate system centered at (x_0,ξ_0) such that

$$H(x,\xi) = C + \sum_{i=1}^{n} \frac{u_i}{2}(x_i^2+\xi_i^2) + \cdots . \tag{1.3}$$

In this paper we present a short proof of the following theorem:

Theorem 1.1. *Assume that the u_i's are linearly independent over the rationals and that the subprincipal symbol of $\hat{H}$ vanishes at (x_0,ξ_0). Then the eigenvalues, (1.1), determine the Taylor series of H at (x_0,ξ_0) up to symplectomorphism or, in other words, determine the* Birkhoff canonical form *of H at (x_0,ξ_0).*

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Our results are closely related to some recent results of [4] on the Schrödinger operator, $\hat{H} = \hbar^2\Delta + V$, which show that the "bottom of the well" spectral asymptotics determines the Taylor series of V at x_0. They are also related to inverse spectral results of [2], [6], [3] and [5]. In these papers it is shown that if

$$\exp tv_H, \qquad v_H = \sum \frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i} \tag{1.4}$$

is the classical dynamical system on T^*M associated with H and T_γ is the period of a periodic trajectory γ of this system, the asymptotic behavior of the wave trace, trace$\left(\exp \frac{-it\hat{H}}{\hbar}\right)$ at NT_γ, $N \in \mathbb{Z}$, determines the Birkhoff canonical form of (1.4) in a formal neighborhood of γ. (In the first two papers we have cited, results of this type are proved for standard ΨDOs, and [3] and [5] are versions of these results in the semiclassical setting.) It turns out that if the trajectory, γ, is replaced by a fixed point of the system (1.4) and, in particular, if this fixed point is a non-degenerate minimum of $H(x,\xi)$, the recovery of the Birkhoff canonical form from the spectral data, (1.1), can be greatly simplified. Our goal, in this short note, is to show why.

We also obtain nearly-optimal results in the resonant case, see §4.

2. The semi-classical Birkhoff canonical form

We quickly review here the construction of the semi-classical Birkhoff canonical form of $\hat{H}$. We follow the exposition in [1] and refer to that paper for details.

Performing a preliminary microlocalization and conjugation by an $\hbar$-FIO, we can assume that $\hat{H}$ is an operator on $\mathbb{R}^n$, and that the global minimum (x_0, ξ_0) is the origin $(0,0) \in T^*\mathbb{R}^n$. Let us denote by

$$[\hat{H}] = \sum_{\alpha,\beta,k} x^\alpha \xi^\beta \hbar^k$$

the Taylor series of the full Weyl symbol of $\hat{H}$, where the monomial $x^\alpha\xi^\beta\hbar^j$ has degree $|\alpha| + |\beta| + 2j$. In fact we can assume that

$$[\hat{H}] = \sum_i \frac{u_i}{2}(x_i^2 + \xi_i^2) + \cdots$$

where the dots indicate terms of degree three and higher. Notice that

$$[\hat{H}]|_{\hbar=0} = \text{the Taylor series of the principal symbol of } \hat{H}.$$

Let $\hat{H}_2$ denotes the Weyl quantization of

$$H_2 := \sum_i \frac{u_i}{2}(x_i^2 + \xi_i^2),$$

and set

$$\hat{H} = \hat{H}_2 + \hat{L}.$$

To construct the quantum Birkhoff canonical form of $\hat{H}$, one conjugates $\hat{H}$ by suitable Fourier integral operators in order to successively make higher-order terms in L commute with $\hat{H}_2$. The resulting series is the quantum Birkhoff canonical form, $H_{\rm can}$ of H.

The non-resonance condition implies that we can write H_{can} in the form:

$$\widehat{H_{\text{can}}} = \hat{H}_2 + F(P_1, \ldots, P_n, \hbar), \quad P_i = \hbar^2 D_i^2 + x_i^2 \tag{2.1}$$

with F an $\hbar$-admissible symbol whose Taylor series is of the form

$$[F] = \sum_{|r|\geq 1} c_r(\hbar) p^r, \tag{2.2}$$

where $p_i = \xi_i^2 + x_i^2$, $r = (r_1, \ldots, r_n)$,

$$c_r(\hbar) = \hbar^{|r|-1}\Big(c_{r,0} + \cdots\Big) \tag{2.3}$$

and $c_r(0) = 0$ for $|r| = 1$ (so that all the monomials in $[F] - H_2$ have degree ≥ 3).

Theorem 1.1 is a direct consequence of the following:

Theorem 2.1. *Under the assumptions of Theorem 1.1, the eigenvalues, (1.1), determine the semi-classical Birkhoff canonical form of $\hat{H}$.*

3. The proof of Theorem 2.1

The first step in our argument is more or less identical with that of [4], [3] and [5]. Assume without loss of generality that $C = 0$, and let $\rho \in C_0^\infty(\mathbb{R})$ be equal to one on the interval $[-1/2\,,\,1/2]$ and zero outside the interval $[-1\,,\,1]$.

Proposition 3.1. *There exists $\tau > 0$ such that for all sufficiently small ϵ and for all $t \in (0, \tau)$, the ρ-truncated wave trace*

$$\textit{trace}\ \rho\Big(\frac{\hat{H}}{\epsilon}\Big) \exp \frac{-it\hat{H}}{\hbar} \tag{3.1}$$

is equal modulo $O(\hbar^\infty)$ to the ρ-truncated wave trace for the Birkhoff canonical form,

$$\operatorname{Tr}(t, \hbar) := \textit{trace}\ \rho\Big(\frac{\widehat{H_{can}}}{\epsilon}\Big)\ \exp\Big(\frac{-it\widehat{H_{can}}}{\hbar}\Big). \tag{3.2}$$

Moreover, the truncated wave trace admits an asymptotic expansion as $\hbar \to 0$ of the form

$$\operatorname{Tr}(t, \hbar) \sim a_0(t) + a_1(t)\hbar + \cdots .$$

The coefficients $a_j(t)$ are real-analytic functions of t and are independent of ρ, provided ρ is chosen as above.

Proof. The operator $\rho(\epsilon^{-1}\hat{H})e^{it\hbar^{-1}\hat{H}}$ is an $\hbar$-Fourier integral operator, associated to the graph of the Hamilton flow of H restricted to the portion of phase space $\Sigma_\epsilon = \{H < \epsilon\}$. If ϵ is sufficiently small, for some $\tau > 0$ there are no periodic trajectories in Σ_ϵ with period in $(0, \tau)$, other than the fixed point (x_0, ξ_0). Therefore, writing the truncated trace as an oscillatory integral and applying the method of stationary phase, for each t the phase has a unique critical point, corresponding to the absolute minimum (x_0, ξ_0) which is a fixed point of the classical flow. This proves the existence of the asymptotic expansion of (3.1). Since the cutoff operator $\rho\Big(\frac{\hat{H}}{\epsilon}\Big)$ is microlocally equal to the identity in a neighborhood of (x_0, ξ_0), the asymptotic expansion of the cutoff trace is independent of ρ, provided $\rho \in C_0^\infty$ is identically equal to one near zero.

For each $\hbar$, (3.1) is an analytic function of t (it's a finite weighted sum of exponentials), and the expansion is locally uniform in t. Therefore, the $a_j(t)$ are analytic in t as well.

The operator $\hat{H}$ is unitarily equivalent to an operator of the form $\widehat{H_{\text{can}}} + \hat{R}$, where $\hat{R}$ is an operator whose full symbol vanishes at (x_0, ξ_0) to infinite order. Since the coefficients of the truncated trace of $\widehat{H_{\text{can}}} + \hat{R}$ only depend on the derivatives of the full symbol of this operator evaluated at (x_0, ξ_0), the asymptotic expansions of (3.1) and (3.2) are the same. □

Remark 3.2. The truncated trace of the Birkhoff canonical form equals

$$\text{Tr}(t,\hbar) = \sum_{k\in(\mathbb{Z}_+)^n} \rho\Big(\frac{H_{\text{can}}(\hbar(k+1/2),\hbar)}{\epsilon}\Big)\, e^{it\hbar^{-1}H_{\text{can}}(\hbar(k+1/2),\hbar)}. \tag{3.3}$$

However, since ρ is identically equal to one in a neighborhood of zero, as a power series in $\hbar$

$$\text{Tr}(t,\hbar) \sim \sum_{k\in(\mathbb{Z}_+)^n} e^{it\hbar^{-1}H_{\text{can}}(\hbar(k+1/2),\hbar)}. \tag{3.4}$$

Notice that, although the asymptotic expansion of the truncated trace (3.1) equals the right-hand side of (3.4) only for $t \in (0,\tau)$, by analyticity our spectral data determine the right-hand side of (3.4) *for all* $t \in \mathbb{R}$.

We now rewrite (3.4) in a more amenable form using a variant of the "Zelditch trick" (see [6]).

Proposition 3.3. *For any choice of ρ as above and for ϵ small, as $\hbar \to 0$*

$$\text{Tr}(t,\hbar) \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!}\,\Big(\sum_{|r|\geq 1} \hbar^{|r|-1} c_r(\hbar)\Big(\frac{1}{t}D_\theta\Big)^r\Big)^m\, \frac{e^{it\frac{1}{2}\sum_j \theta_j}}{\Pi_j(1-e^{it\theta_j})}\Big|_{\theta=u}, \tag{3.5}$$

where

$$D_\theta = -i\Big(\frac{\partial}{\partial\theta_1},\ldots,\frac{\partial}{\partial\theta_n}\Big)$$

and the right-hand side of (3.5) is understood as a power series in $\hbar$.

Proof. Recalling that $[\hat{H}_2,\hat{F}] = 0$,

$$\text{Tr}(t,\hbar) \sim \sum_{k\in(\mathbb{Z}_+)^n} e^{itu\cdot(k+1/2)}\ \langle k|e^{it\hbar^{-1}\hat{F}}|k\rangle,$$

where $\{|k\rangle\}$ is an orthonormal basis of eigenvectors of the canonical n-torus representation on $L^2(\mathbb{R}^n)$, and $u\cdot(k+1/2) = \sum_{j=1}^n u_j(k_j+1/2)$. For each k, the Taylor expansion, [F], gives us an asymptotic expansion

$$\begin{aligned} \langle k|e^{it\hbar^{-1}\hat{F}}|k\rangle &= \sum_{m=0}^{\infty} \frac{(it)^m}{\hbar^m m!} F(\hbar(k+1/2),\hbar)^m \\ &\sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!}\Big(\sum_r \hbar^{|r|-1}\, c_r(\hbar)(k+1/2)^r\Big)^m. \end{aligned} \tag{3.6}$$

Let us introduce the variables $\theta = (\theta_1, \dots, \theta_n)$ and write:

$$(k+1/2)^r e^{itu\cdot(k+1/2)} = \Big(\frac{1}{t}D_\theta\Big)^r e^{it\theta\cdot(k+1/2)}\Big|_{\theta=u}.$$

Then $\mathrm{Tr}(t,\hbar) \sim$

$$\sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{k\in\mathbb{Z}_+} \rho\Big(\frac{F(\hbar(k+1/2),\hbar)}{\epsilon}\Big) \Big(\sum_r \hbar^{|r|-1}\, c_r(\hbar)\Big(\frac{1}{t}D_\theta\Big)^r\Big)^m e^{it\theta\cdot(k+1/2)}\Big|_{\theta=u}.$$

Finally, for each m (summing a geometric series)

$$\sum_{k\in\mathbb{Z}_+} \Big(\sum_r \hbar^{|r|-1}\, c_r(\hbar)\Big(\frac{1}{t}D_\theta\Big)^r\Big)^m e^{it\theta\cdot(k+1/2)}\Big|_{\theta=u} =$$

$$= \Big(\sum_r \hbar^{|r|-1}\, c_r(\hbar)\Big(\frac{1}{t}D_\theta\Big)^r\Big)^m \frac{e^{it\frac{1}{2}\sum_j \theta_j}}{\Pi_i(1-e^{it\theta_j})}\Big|_{\theta=u},$$

and the result follows. □

We will show that the $m = 0$ term in the series on the right-hand side of (3.5) suffices to determine the $c_r(\hbar)$. More precisely:

Theorem 3.4. *From the coefficients of $\hbar^s$, $s \leq \ell$, in the series in $\hbar$*

$$V(t,\hbar) = \sum_{|r|\geq 1} \hbar^{|r|-1}\, c_r(\hbar)\Big(\frac{1}{t}D_\theta\Big)^r \frac{e^{it\frac{1}{2}\sum_j \theta_j}}{\Pi_j(1-e^{it\theta_j})}\Big|_{\theta=u} \tag{3.7}$$

one can determine the coefficients of $\hbar^s$, $s \leq \ell$, in $c_r(\hbar)$ for all r.

Proof. Let ρ be a cutoff function as before, and $\hat{\varphi} = \rho$. Integrating (3.7) against $\epsilon^n\varphi(\epsilon t)$ and essentially reversing the previous calculation, we find:

$$\tilde{V}(\epsilon,\hbar) = \sum_k \Big(\sum_r \hbar^{|r|-1} c_r(\hbar)(k+1/2)^r\Big)\, \rho\Big(\epsilon^{-1}u\cdot(k+1/2)\Big) = \tag{3.8}$$

$$= \sum_k \hbar^{-1}F(\hbar(k+1/2))\, \rho\Big(\epsilon^{-1}u\cdot(k+1/2)\Big).$$

Letting

$$c_r(\hbar) = \sum_{i=0}^{\infty} c_{r,i}\, \hbar^{|r|-1+i},$$

we can rearrange (3.8) in increasing powers of $\hbar$ (using the variable $\ell = 2|r| - 2 + i$ for the exponent of $\hbar$):

$$\tilde{V}(\epsilon,\hbar) = \sum_{\ell=0}^{\infty} \hbar^\ell \sum_{j=0}^{[\frac{\ell}{2}]} \sum_{|r|=j+1} c_{r,\ell-2j} \Big(\sum_k (k+1/2)^r \rho\Big(\epsilon^{-1}u\cdot(k+1/2)\Big)\Big). \tag{3.9}$$

Now arrange the numbers

$$u_k = u\cdot(k+1/2), \qquad k \in (\mathbb{Z}_+)^n$$

in strictly increasing order (which is possible because there are no repetitions among them):

$$0 < \nu_1 = u_k|_{k=0} < \nu_2 < \cdots . \tag{3.10}$$

Let us write: $\nu_s = u_{k^{(s)}}$. Now vary ϵ in (3.9), starting with a very small value. Gradually increasing ϵ, we can arrange that the coefficient of $\hbar^\ell$ in (3.9) is

$$\sum_{j=0}^{[\frac{\ell}{2}]} \sum_{|r|=j+1} c_{r,\ell-2j} \sum_{s=1}^{m} (k^{(s)}+1/2)^r \, \rho\Big(\epsilon^{-1}\nu_s\Big)$$

for any given m. Therefore, by an inductive argument on m we can recover the values of the polynomial

$$\mathfrak{p}_\ell(x) = \sum_{j=0}^{[\frac{\ell}{2}]} \sum_{|r|=j+1} c_{r,\ell-2j} \, (x+1/2)^r$$

at all $k \in (\mathbb{Z}_+)^n$. But these values determine the polynomial, and therefore its coefficients. □

Now we show that the asymptotic expansion of the trace, $\mathrm{Tr}(t,\hbar)$, determines V:

Theorem 3.5. *From the coefficients of $\hbar^s$, $s \leq \ell$, in the expansion (3.5) one can determine the coefficients of $\hbar^s$, $s \leq \ell$ in the series $V(t,\hbar)$.*

Proof. We proceed by induction on ℓ.

The coefficient of $\hbar$ in V coincides with the coefficient of $\hbar$ in (3.5), since all terms in the sum (3.5) except the first are of order $O(\hbar^m)$, $m > 1$.

By theorem 3.4 the coefficient of $\hbar$ in V enables us to determine the coefficient of $\hbar$ in $c_r(\hbar)$, and hence the coefficient of $\hbar^2$ in the second summand of (3.5). But the coefficient of $\hbar^2$ in the first summand coincides with the coefficient of $\hbar^2$ in V, so the coefficients of $\hbar$ and $\hbar^2$ in (3.5) determine the coefficient of $\hbar^2$ in V. It is clear that this procedure can be continued indefinitely. □

Theorem 2.1 is an immediate consequence of theorems 3.4 and 3.5.

4. The resonant case

We now consider the case when the frequencies $u_1,\dots,u_n$ are not linearly independent over $\mathbb{Q}$. Following [1], let us introduce the number

$$d = \min\{|\alpha|, \alpha \in \mathbb{Z}^n \setminus \{0\} \mid \alpha \cdot u = 0\}$$

which is a measure of the rational relations among the frequencies (here $|\alpha| = \sum_{j=1}^n |\alpha_j|$). We will make use below of the following observation:

Lemma 4.1. *Among the eigenvalues of $\hat{H}_2$ of the form:*

$$\lambda_k = k\cdot(u+1/2) \qquad \text{with} \qquad |k| < d/2 \tag{4.1}$$

there are no repetitions (i. e. the mapping $k \mapsto \lambda_k$ is 1-1 in the range $|k| < d/2$).

Proof. If $k\cdot(u+1/2) = k'\cdot(u+1/2)$, then $(k-k')\cdot u = 0$ and therefore $|k-k'| \geq d$. The conclusion now follows from the triangle inequality. □

Continuing to assume that the subprincipal symbol vanishes at the absolute minimum, the semi-classical Birkhoff canonical form in the resonant case has the following structure (see [1]):

$$H_{can} = H_2 + F + K,$$

where

(1) $F = F(p_1, \dots, p_n, \hbar)$ where F is a polynomial in all variables of degree at most $[\frac{d-1}{2}]$.
(2) $[K]$ is a power series with monomials $\hbar^j x^\alpha \xi^\beta$ where $|\alpha| + |\beta| + 2j > d$ and $[\hat{H}_2, \hat{K}] = 0$.

In this section we prove the following:

Theorem 4.2. *If d is even the eigenvalues (1.1) determine the entire semi-classical canonical form, $F(x, \hbar)$. If d is odd, those eigenvalues determine the semi-classical canonical form except for the monomials of maximal degree, $[\frac{d-1}{2}]$.*

Except for a few additional complications, the method of proof is the same as in the non-resonant case. We begin by checking that the asymptotic expansion of the truncated trace can be treated by the same methods as before, up to a sufficiently high order in $\hbar$:

Proposition 4.3. *In the resonant case, the expansion (3.5) is valid modulo $O(\hbar^{[\frac{d}{2}]})$.*

Proof. Once again we write the trace as a sum of diagonal matrix elements over a normalized basis $\{|k\rangle\}$ of eigenfunctions of the standard representation of the n-torus, splitting off the H_2 part (which is possible since $\hat{F} + \hat{K}$ commutes with H_2):

$$\mathrm{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{itu\cdot(k+1/2)} \, \langle k | e^{it\hbar^{-1}(\hat{F}+\hat{K})} | k \rangle.$$

We next expand the exponential in its Taylor series. We want to show that every term involving $\hat{K}$ is $O(\hbar^{[\frac{d}{2}]})$.

A term involving

$$\langle k | (\hat{F} + \hat{K})^m | k \rangle$$

is a sum of terms of the form

$$\langle k | \hat{F}_1 \hat{K}_1 \cdots \hat{F}_s \hat{K}_s | k \rangle \tag{4.2}$$

where the F_j are powers of F and the K_j are powers of K. Therefore, the K_j are sums of monomials $\hbar^j x^\alpha \xi^\beta$ where $2j + |\alpha| + |\beta| > d$, just as is K. Let us express those monomials in terms of raising and lowering operators,

$$A_{\alpha\beta} = z^\alpha \overline{z}^\beta, \qquad z = x + i\xi.$$

Then (4.2) is a linear combination of terms of the form

$$\hbar^{\sum_{i=1}^s j_i} \langle k | \hat{F}_1 \widehat{A_{\alpha^1\beta^1}} \cdots \hat{F}_s \widehat{A_{\alpha^s\beta^s}} | k \rangle \tag{4.3}$$

where, for each i,

$$2j_i + |\alpha^i| + |\beta^i| > d.$$

Now recall that (i) the $\hat{F}_j$ are diagonal in the basis $\{|k\rangle\}$ and (ii) the $\widehat{A_{\alpha\beta}}$ act on the basis vectors by:

$$\widehat{A_{\alpha\beta}} |k\rangle = \hbar^{|\beta|} c_{\alpha\beta} |k + \alpha - \beta\rangle$$

where $c_{\alpha\beta}$ is a constant whose value we won't need. Therefore, a diagonal matrix element of the sort (4.3) is zero unless

$$\sum_{i=1}^{s} \alpha^i - \beta^i = 0,$$

in which case (4.3) is $O(\hbar^{j+\sum_i |\beta^i|})$ where $j = \sum_i j_i$. However, $|\alpha^i| + |\beta^i| > d - 2j_i$ for each i and

$$\sum_{i=1}^{s} \alpha^i - \beta^i = 0 \quad \Rightarrow \quad \Big|\sum_{i=1}^{s} \alpha^i\Big| = \Big|\sum_{i=1}^{s} \beta^i\Big|.$$

Therefore, $\sum_i |\beta^i| \geq [sd/2] - j$ and so (4.3) is $O(\hbar^{[sd/2]})$. It follows that all diagonal matrix elements to which $\hat{K}$ contributes are at least $O(\hbar^{[\frac{d}{2}]})$. □

Lemma 4.4. *$F(\hbar(x+1/2), \hbar)$ is a polynomial in $\hbar$ of degree at most $[\frac{d-1}{2}]$, and if we write*

$$F(\hbar(x+1/2), \hbar) = \sum_{j=0}^{[\frac{d-1}{2}]} \hbar^j F_j(x)$$

the power series expansion of Tr$(t, \hbar)$ *determines the values $F_j(k)$ for all $k \in (\mathbb{Z}_+)^n$ such that $|k| < d/2$, for all $j \leq [\frac{d-1}{2}]$ if d is even and for all $j < [\frac{d-1}{2}]$ if d is odd.*

Proof. The first statement follows from the general form of F.

By theorems 3.5 and 3.4, for any ℓ the first ℓ terms of the expansion of Tr$(t, \hbar)$ determine the first ℓ terms of (3.8), provided we replace F by $F + K$. But, by the previous proposition, the expansion of Tr$(t, \hbar)$ mod $O(\hbar^{[\frac{d}{2}]})$ is insensitive to what K is. Therefore, (3.8) remains valid mod $O(\hbar^{[\frac{d}{2}]})$, where F now stands for the part of the canonical form we are determining from the spectrum.

If d is even

$$[\frac{d-1}{2}] < [\frac{d}{2}],$$

and so it follows that the expansion of Tr determines the sums

$$\sum_k \hbar^{-1} F(\hbar(k+1/2))\ \rho\Big(\epsilon^{-1} u \cdot (k+1/2)\Big).$$

Now we proceed as before, letting ϵ grow starting at a very small value. Since the eigenvalues (4.1) are all different, we can determine the polynomial in $\hbar$, $F(\hbar(x+1/2))$, evaluated at each k with $|k| < d/2$. If d is odd we must discard the term F_j with $j = [\frac{d-1}{2}]$. □

Since F_j is a polynomial of degree at most $[\frac{d-1}{2}]$, the proof of theorem 4.2 is completed by the following result:

Lemma 4.5. *Let $f(x_1, \ldots, x_n)$ be a polynomial of degree N. Then f is completely determined by its values at the points*

$$(k_1 + 1/2, \ldots, k_n + 1/2),$$

for all k such that $|k| \leq N$ and $k_j \geq 0$.

Proof. The proof is by induction on the number of variables. The case $n = 1$ is trivial. Assume the result is true for polynomials of $n-1$ variables, and let

$$f = f_N(x_2, \dots, x_n) + f_{N-1}(x_2, \dots, x_n)x_1 + \cdots + f_0\, x_1^N.$$

Note that degree $f_i = i$.

Evaluating f at $(k+1/2, 1/2, \dots, 1/2)$, $0 \leq k \leq N$ determines $f_i(1/2, 1/2, \dots, 1/2)$, $i = 0, \dots N$, and in particular determines f_0.

Evaluating $f - f_0\, x_1^N$ at $(k+1/2, k_2+1/2, \dots, k_n+1/2)$, $0 \leq k \leq N-1$, $k_2 + \cdots + k_n \leq 1$ determines $f_i(k_2 + 1/2, \dots, k_n + 1/2)$ at all $k_2 + \cdots + k_n \leq 1$ and in particular determines f_1.

Evaluating $f - f_1\, x_1^{N-1} - f_0\, x^N$ at $(k + 1/2, k_2 + 1/2, \dots, k_n + 1/2)$, $0 \leq k \leq N-2$, $k_2 + \cdots + k_n \leq 2$ determines $f_i(k_2 + 1/2, \dots, k_n + 1/2)$ at all $k_2 + \cdots + k_n \leq 2$ and in particular determines f_2. Etc.

□

When d is odd our methods recover the classical Birkhoff normal form of H except for its monomials of top degree, $\frac{d-1}{2}$.

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