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# THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES ON COMPLETE MANIFOLDS

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Abstract. We find a new sharp Caffarelli-Kohn-Nirenberg inequality and show that the Euclidean spaces are the only complete non-compact Riemannian manifolds of non-negative Ricci curvature satisfying this inequality. We also show that a complete open manifold with non-negative Ricci curvature in which the optimal Nash inequality holds is isometric to a Euclidean space.

## 1. Introduction

Let $C_0^\infty(I\!R^n)$ be the space of smooth functions with compact support in the $n$-dimensional Euclidean space $I\!R^n$. Among a much more general family of inequalities, Caffarelli, Kohn and Nirenberg proved the following result.

**Theorem 1.1.** ([CKN]) *Let $n \geq 2$, $r > p > 1$, and $\alpha$, $\beta$ be fixed real numbers satisfying*

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{p-1}{p(r-1)} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{1.1}$$

*where*

$$\gamma = \frac{1}{r}(\alpha - 1) + \frac{p-1}{pr}\beta. \tag{1.2}$$

*There exists a positive constant $C$ such that the following inequality holds for all $f \in C_0^\infty(I\!R^n)$*

$$\begin{aligned} &\int_{I\!R^n} |x|^{\gamma r}|f|^r dx \\ &\leq \quad C\left(\int_{I\!R^n} |x|^{\alpha p}|\nabla f|^p dx\right)^{\frac{1}{p}} \left(\int_{I\!R^n} |x|^{\beta}|f|^{\frac{p(r-1)}{p-1}} dx\right)^{\frac{p-1}{p}}, \end{aligned} \tag{1.3}$$

*where $|x|$ is the Euclidean length of $x \in R^n$.*

In this paper, we obtain the exact value of the smallest admissible constant $C$ in (1.3) for some special cases. Namely, we have

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**Theorem 1.2.** *Let $n$, $r$, $p$, $\alpha$, $\beta$, $\gamma$ be as in Theorem 1.1. Then for all $f \in C_0^\infty(I\!R^n)$, we have*

$$\int_{I\!R^n} |x|^{\gamma r}|f|^r dx \leq \frac{r}{n+\gamma r}\left(\int_{I\!R^n} |x|^{\alpha p}|\nabla f|^p dx\right)^{\frac{1}{p}}\left(\int_{I\!R^n} |x|^{\beta}|f|^{\frac{p(r-1)}{p-1}} dx\right)^{\frac{p-1}{p}}. \tag{1.4}$$

*Moreover, when*

$$n+\beta < \left(1-\alpha+\frac{\beta}{p}\right)\frac{(r-1)p}{r-p}, \tag{1.5}$$

*the inequality (1.4) is best possible in the sense that*

$$\inf_{f\in C_0^\infty(I\!R^n)-\{0\}} \frac{\left(\int_{I\!R^n} |x|^{\alpha p}|\nabla f|^p dx\right)^{\frac{1}{p}}\left(\int_{I\!R^n} |x|^{\beta}|f|^{\frac{p(r-1)}{p-1}} dx\right)^{\frac{p-1}{p}}}{\int_{I\!R^n} |x|^{\gamma r}|f|^r dx} = \frac{n+\gamma r}{r} \tag{1.6}$$

*and a family of minimizers of (1.6) is given by*

$$g(x) = \left(\lambda + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{1-p}{r-p}}, \quad \lambda > 0.$$

In the next part of the present paper, we study complete manifolds with nonnegative Ricci curvature in which the Caffarelli-Kohn-Nirenberg type inequality (1.4) is satisfied. For a Riemannian manifold $M$, we let $dv$ be the Riemannian volume element on $M$, denote by $\nabla$ the gradient operator, $C_0^\infty(M)$ the space of smooth functions on $M$ with compact support, $B(x,r)$ the geodesic ball with center $x \in M$ and radius $r$ and $\mathrm{vol}[B(p,r)]$ be the volume of $B(p,r)$.

**Theorem 1.3.** *Let $n$, $r$, $p$, $\alpha$, $\beta$, $\gamma$ be as in Theorem 1.1 and assume that (1.5) holds. Let $M$ be an $n$-dimensional complete open Riemannian manifold with nonnegative Ricci curvature. Fix a point $x_0 \in M$ and denote by $\mu$ the distance function on $M$ from $x_0$. If for any $f \in C_0^\infty(M)$, we have*

$$\int_M \mu^{\gamma r}|f|^r dv \leq \frac{r}{n+\gamma r}\left(\int_M \mu^{\alpha p}|\nabla f|^p dv\right)^{\frac{1}{p}}\left(\int_M \mu^{\beta}|f|^{\frac{p(r-1)}{p-1}} dv\right)^{\frac{p-1}{p}}. \tag{1.7}$$

*Then $M$ is isometric to $I\!R^n$.*

The sharp logarithmic Sobolev inequalities or entropy-energy inequalities in Euclidean space states that for any $f \in C_0^\infty(I\!R^n)$ with $\int_{I\!R^n} f^2 dx = 1$ it holds (cf. [D], [B])

$$\int_{I\!R^n} f^2 \log f^2 dx \leq \frac{n}{2}\log\left(\frac{2}{n\pi e}\int_{I\!R^n} |\nabla f|^2 dx\right).$$

It has been shown by Bakry, Concordet and Ledoux that (cf. [BCL]) a complete $n$-dimensional Riemannian manifold $M$ with non-negative Ricci curvature satisfying the optimal logarithmic Sobolev inequality, i.e.

$$\int_M f^2 \log f^2 dv \leq \frac{n}{2} \log\left(\frac{2}{n\pi e}\int_M |\nabla f|^2 dv\right),$$
$$\forall f \in C_0^\infty(M) \text{ with } \int_M f^2 dv = 1,$$

is isometric to $I\!R^n$.

A similar result for the Sobolev inequality was obtained by Ledoux in [Le] and he showed that a complete $n$-dimensional Riemannian manifold $M$ with nonnegative Ricci curvature in which one of the Sobolev inequalities

$$\left(\int_M |f|^p dv\right)^{1/p} \leq C\left(\int_M |\nabla f|^q dv\right)^{1/q},$$
$$\forall f \in C_0^\infty(M),\ 1 \leq q < n, \frac{1}{p} = \frac{1}{q} - \frac{1}{n},$$

is satisfied with $C$ the optimal constant of this inequality in $I\!R^n$ is isometric to $I\!R^n$. This theorem of Ledoux has been generalized in [X1].

Another important inequality is the so called Nash inequality stating that for any $f \in C_0^\infty(I\!R^n)$ we have

$$\left(\int_{I\!R^n} f^2 dv\right)^{1+2/n} \leq C_n \left(\int_{I\!R^n} |\nabla f|^2 dv\right)\left(\int_{I\!R^n} |f| dv\right)^{4/n} \tag{1.8}$$

for a constant $C_n$ depending only on $n$. This inequality is a particular case of the Gagliardo-Nirenberg inequalities for which numerous applications have been found. It has been shown by Carlen and Loss [CL] that the best possible constant in (1.7) is given by

$$C_n = \frac{2((n+2)/2)^{(n+2)/n}}{n\lambda_1^N(B_n)|B_n|^{2/n}} \tag{1.9}$$

where $|B_n|$ denotes the volume of the unit ball in $I\!R^n$, and where $\lambda_1^n(B_n)$ denotes the first nonzero Neumann eigenvalue of the Laplacian operator on $B_n$.

In [L1], Ledoux proposed the problem that if a complete $n$-manifold of non-negative Ricci curvature in which the Nash inequality (1.7) holds with $C_n$ given by (1.8) is isometric to $I\!R^n$. With respect to this problem, Druet, Hebey and Vaugon showed in [DHV] that a complete $n$-dimensional manifold $M$ with non-negative Ricci curvature in which the following Nash inequality holds:

$$\left(\int_M f^2 dv\right)^{1+\frac{2}{n}} \leq C_n \left(\int_M |f| dv\right)^{\frac{4}{n}} \int_M |\nabla f|^2 dv, \quad \forall f \in C_0^\infty(M),$$

is flat, where $C_n$ is given by (1.9).

In this paper, using the result proved by Druet, Hebey and Vaugon [DVH], we show that the problem by Ledoux has a positive answer.

**Theorem 1.4.** *Let $M$ be an complete $n$-dimensional manifold $M$ with non-negative Ricci curvature. If for any $f \in C_0^\infty(M)$, it holds*

$$\left(\int_M f^2 dv\right)^{1+\frac{2}{n}} \le C_n \left(\int_M |f| dv\right)^{\frac{4}{n}} \int_M |\nabla f|^2 dv, \tag{1.10}$$

*where $C_n$ is given by (1.9), then $M$ is isometric to $\mathbb{R}^n$.*

Complete manifolds with non-negative Ricci curvature in which some other type Caffarelli-Kohn-Nirenberg ineqaulity (cf. [CKN], [CCh], [L], [CW]) holds were studied in [CX1]. The structure of complete manifolds with non-negative Ricci curvature in which some Gagliardo-Nirenberg type ineqaulity (cf. [DPD1], [DPD2]) holds has been studied recently in [X3]. For some interesting results about complete manifolds with non-negative Ricci curvature, we refer to [AG], [A], [CX1], [CC], [Li], [SS], [S1], [S2], [SSO], [SO1], [SO2], [W], [X2].

## 2. A Proof of Theorem 1.2

Let $\Delta$ be the Laplacian operator on $\mathbb{R}^n$; then for the position vector $x$ in $\mathbb{R}^n$, it holds $\Delta|x|^2 = 2n$. For any $f \in C_0^\infty(\mathbb{R}^n)$, since $|\nabla|x|| = 1$ almost everywhere, it follows from the divergence theorem that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r dx \\
={}& \frac{1}{2n}\int_{\mathbb{R}^n} |f|^r|x|^{\gamma r}\Delta|x|^2 dx \\
={}& -\frac{1}{2n}\int_{\mathbb{R}^n} \langle\nabla(|x|^2), \nabla(|f|^r|x|^{\gamma r})\rangle dx \\
={}& -\frac{\gamma r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r|\nabla|x||^2 dx + \frac{r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r+1}\langle\nabla|x|, \nabla|f|\rangle|f|^{r-1}dx \\
={}& -\frac{\gamma r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r dx + \frac{r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r+1}\langle\nabla|x|, \nabla|f|\rangle|f|^{r-1}dx
\end{aligned} \tag{2.1}$$

The Schwarz inequality implies that $\langle\nabla|x|, \nabla|f|\rangle \le |\nabla|x|| \cdot |\nabla|f|| = |\nabla|f||$ almost everywhere. Since $|\nabla|f|| = |\nabla f|$ almost everywhere, we conclude from (2.1) that

$$\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r dx \le -\frac{\gamma r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r dx + \frac{r}{n}\int_{\mathbb{R}^n} |x|^{\gamma r+1}|\nabla f||f|^{r-1}dx,$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\gamma r}|f|^r dx \\
\le{}& \frac{r}{n+\gamma r}\int_{\mathbb{R}^n} |x|^{\gamma r+1}|\nabla f||f|^{r-1}dx \\
={}& \frac{r}{n+\gamma r}\int_{\mathbb{R}^n} |x|^{\alpha}|\nabla f||x|^{\frac{(p-1)\beta}{p}}|f|^{r-1}dx \\
\le{}& \frac{r}{n+\gamma r}\left(\int_{\mathbb{R}^n} |x|^{\alpha p}|\nabla f|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n} |x|^{\beta}|f|^{\frac{p(r-1)}{p-1}}dx\right)^{\frac{p-1}{p}},
\end{aligned} \tag{2.2}$$

where in the last inequality, we used the Hölder's inequality. Thus (1.4) holds.

To conclude the proof of Theorem 1.2, it suffices to check that when

$$g(x) = \left(\lambda + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{1-p}{r-p}}, \quad \lambda > 0,$$

then

$$\frac{\left(\int_{\mathbb{R}^n} |x|^{\alpha p} |\nabla g|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|^{\beta} |g|^{\frac{p(r-1)}{p-1}} dx\right)^{\frac{p-1}{p}}}{\int_{\mathbb{R}^n} |x|^{\gamma r} |g|^r dx} = \frac{n+\gamma r}{r}. \tag{2.3}$$

For $\lambda \in (0, +\infty)$, set

$$A(\lambda) = \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p} \int_{\mathbb{R}^n} |x|^{\beta} \left(\lambda + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-r)p}{r-p}} dx,$$

$$B(\lambda) = \int_{\mathbb{R}^n} |x|^{\alpha-1+(1-\frac{1}{p})\beta} \left(\lambda + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-p)r}{r-p}} dx.$$

We *claim* that the functions $A$ and $B$ are well defined. In fact, we have

$$\begin{aligned} A(1) &= \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p} \int_{\mathbb{R}^n} |x|^{\beta} \left(1 + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-r)p}{r-p}} dx \\ &= \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p} \int_0^{+\infty} c_{n-1} r^{\beta+n-1} \left(1 + r^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-r)p}{r-p}} dr, \end{aligned}$$

where $c_{n-1}$ denotes the area of the unit sphere in $R^n$. Since

$$\frac{p-1}{p(r-1)} + \frac{\beta}{n} > 0,$$

one has

$$\beta + n - 1 > -1. \tag{2.4}$$

It follows from (1.5) and (2.4) that $A(1)$ converges. On the other hand, one has

$$\begin{aligned} B(1) &= \int_{\mathbb{R}^n} |x|^{\alpha-1+(1-\frac{1}{p})\beta} \left(1 + |x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-p)r}{r-p}} dx \\ &= \int_0^{\infty} c_{n-1} r^{n-1+\alpha-1+\left(1-\frac{1}{p}\right)\beta} \left(1 + r^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-p)r}{r-p}} dr. \end{aligned} \tag{2.5}$$

Since $n + \gamma r > 0$, we have

$$n - 1 + \alpha - 1 + \left(1 - \frac{1}{p}\right)\beta > -1.$$

Also, it is easy to see that (1.5) is equivalent to

$$n - 1 + \alpha - 1 + \left(1 - \frac{1}{p}\right)\beta + \left(1 - \alpha + \frac{\beta}{p}\right)\frac{(1-p)r}{r-p} < -1.$$

Thus $B(1)$ converges. It is easy to see that

$$A(\lambda) = \lambda^{\frac{n+\gamma r}{1-\alpha+\beta/p} - \frac{r(p-1)}{r-p}} \cdot A(1), \quad B(\lambda) = \lambda^{\frac{n+\gamma r}{1-\alpha+\beta/p} - \frac{r(p-1)}{r-p}} \cdot B(1). \tag{2.6}$$

So our *claim* is true.

Since

$$\nabla g = \frac{(1-p)(1-\alpha+\frac{\beta}{p})}{r-p}|x|^{-\alpha+\frac{\beta}{p}}\left(\lambda+|x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{1-r}{r-p}}\nabla|x|,$$

one can easily obtain that

$$\left(\int_{\mathbb{R}^n}|x|^{\alpha p}|\nabla g|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|x|^{\beta}|g|^{\frac{p(r-1)}{p-1}}dx\right)^{\frac{p-1}{p}} = A(\lambda), \tag{2.7}$$

and that

$$\int_{\mathbb{R}^n}|x|^{\gamma r}|g|^r dx = B(\lambda). \tag{2.8}$$

In order to see that (2.3) holds, we need only to show that

$$A(1) = \frac{n+\gamma r}{r}B(1). \tag{2.9}$$

This can be seen as follows.

$$\begin{aligned} & A(1) \\ = {} & \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p}\int_{\mathbb{R}^n}|x|^{\beta}\left(1+|x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{(1-r)p}{r-p}}dx \\ = {} & \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p}\left(\int_{\mathbb{R}^n}|x|^{\alpha-1+(1-\frac{1}{p})\beta}\left(1+|x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{r(1-p)}{r-p}}dx\right. \\ & \left. -\int_{\mathbb{R}^n}|x|^{\alpha-1+(1-\frac{1}{p})\beta}\left(1+|x|^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{p(1-r)}{r-p}}dx\right) \\ = {} & \frac{(p-1)(1-\alpha+\frac{\beta}{p})}{r-p}\left(B(1)+\frac{r-p}{r(p-1)}B'(1)\right). \end{aligned} \tag{2.10}$$

By (2.6),

$$B'(1) = \left(\frac{n+\gamma r}{1-\alpha+\beta/p}-\frac{r(p-1)}{r-p}\right)B(1). \tag{2.11}$$

Combining (2.10) and (2.11), one gets (2.9). This completes the proof of Theorem 1.2.

## 3. Proofs of Theorems 1.3 and 1.4

*Proof of Theorem 1.3.* Let $\theta > 1$ be fixed and set $\rho = \theta^{-1}\mu$. For any $\lambda > 0$, let

$$F(\lambda) = \int_M \frac{\rho^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}dv}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{r(p-1)}{r-p}}}.$$

Let us first verify that $F$ is a well defined $C^1$ function on $(0,+\infty)$. In fact, by Fubini theorem (cf. [SY])

$$F(\lambda) = \int_0^{+\infty}\mathrm{vol}\left\{x : \frac{\rho^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{r(p-1)}{r-p}}}(x) > s\right\}ds. \tag{3.1}$$

Since $M$ has non-negative Ricci curvature, the Bishop-Gromov comparison theorem implies that $\mathrm{vol}[B(x_0,t)] \le |B_n| t^n$ (cf. [BC], [G]). Set

$$\omega = 1-\alpha+\left(\frac{1}{p}-1\right)\beta,\ z=\frac{r(p-1)}{r-p},$$

then we have from (1.5) that

$$n-\omega-1-z(\omega+\beta) < -1. \tag{3.2}$$

Making the variable change

$$s=\frac{t^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}}{\left(\lambda+t^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{r(p-1)}{r-p}}}$$

in (3.1), we get

$$\begin{aligned} F(\lambda) &= \int_0^{+\infty} \mathrm{vol}[B(x_0,\theta t)]\cdot \frac{\omega\lambda+(\omega+z(\omega+\beta))t^{\omega+\beta}}{t^{\omega+1}(\lambda+t^{\omega+\beta})^{z+1}}dt \\ &\le \int_0^{+\infty} \frac{|B_n|\theta^n(\omega\lambda+(\omega+z(\omega+\beta))t^{\omega+\beta})t^{n-\omega-1}}{(\lambda+t^{\omega+\beta})^{z+1}}dt. \end{aligned} \tag{3.3}$$

Since $n-\omega-1=n+\gamma r-1>-1$ and (3.2) holds, we know that $0\le F(\lambda)<+\infty$, $\forall\lambda>0$, that $F$ is differentiable and that

$$F'(\lambda)=-\frac{r(p-1)}{r-p}\int_M \frac{\rho^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}dv}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{p(r-1)}{r-p}}} \tag{3.4}$$

Observe that $|\nabla\rho|=\theta^{-1}$ almost everywhere. By an approximation procedure, we can apply (1.7) to $\left(\lambda+\rho^{1-\alpha+\beta/p}\right)^{-\frac{p-1}{r-p}}$ for every $\lambda>0$ to get

$$\begin{aligned} &F(\lambda) \\ = {}& \int_M \frac{\rho^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}dv}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{r(p-1)}{r-p}}} \\ \le {}& \frac{r(p-1)(1-\alpha+\beta/p)}{(n+\gamma r)(r-p)}\int_M \frac{\rho^{\beta}dv}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{p(r-1)}{r-p}}} \\ = {}& \frac{r(p-1)(1-\alpha+\beta/p)}{(n+\gamma r)(r-p)}\int_M \frac{\rho^{\alpha-1+\left(1-\frac{1}{p}\right)\beta}\left(\rho^{1-\alpha+\frac{\beta}{p}}+\lambda-\lambda\right)dv}{\left(\lambda+\rho^{1-\alpha+\frac{\beta}{p}}\right)^{\frac{p(r-1)}{r-p}}} \\ = {}& \frac{r(p-1)(1-\alpha+\beta/p)}{(n+\gamma r)(r-p)}\left(F(\lambda)+\frac{r-p}{r(p-1)}\lambda F'(\lambda)\right) \end{aligned} \tag{3.5}$$

Thus we have

$$-\frac{1-\alpha+\beta/p}{n+\gamma r}\lambda F'(\lambda)\le\left(\frac{r(p-1)(1-\alpha+\beta/p)}{(n+\gamma r)(r-p)}-1\right)F(\lambda), \tag{3.6}$$

or

$$-\lambda F'(\lambda) \le lF(\lambda), \tag{3.7}$$

where

$$\begin{aligned} l &= \frac{r(p-1)}{r-p} - \frac{n+\gamma r}{(1-\alpha+\beta/p)} \\ &= \frac{r(p-1)}{r-p} - \frac{n+\beta}{(1-\alpha+\beta/p)} - 1 \\ &= \frac{p(r-1)(1-\alpha+\beta/p)-(r-p)(n+\beta)}{(1-\alpha+\beta/p)(r-p)} \\ &> 0, \end{aligned}$$

where in the last inequality, we used (1.5).

Consider the function $B : (0, +\infty) \to I\!R$ introduced in the last section. By (2.6)

$$B(\lambda) = \lambda^{-l} \cdot B(1) \tag{3.8}$$

and so

$$-\lambda B'(\lambda) = lB(\lambda). \tag{3.9}$$

We claim that if for some $\lambda_0 > 0$, $F(\lambda_0) < B(\lambda_0)$, then $F(\lambda) < B(\lambda),\ \forall \lambda \in (0, \lambda_0]$. In order to see this, suppose that there exists some $\tilde{\lambda} \in (0, \lambda_0)$ such that $F(\tilde{\lambda}) \ge B(\tilde{\lambda})$. Set

$$\lambda_1 = \sup\{\lambda < \lambda_0; F(\lambda) = B(\lambda)\}.$$

Then for any $\lambda \in [\lambda_1, \lambda_0],\ 0 < F(\lambda) \le B(\lambda)$ and so we have from (3.7) and (3.9) that

$$\lambda(F'(\lambda) - B'(\lambda)) \ge l(B(\lambda) - F(\lambda)) \ge 0, \quad \forall \lambda \in [\lambda_1, \lambda_0].$$

It follows that the function $F(\lambda) - B(\lambda)$ is increasing on $[\lambda_1, \lambda_0]$. Consequently, we have

$$0 = (F - B)(\lambda_1) \le (F - B)(\lambda_0) < 0.$$

This is a contradiction. Thus the above *claim* is true.

Before we can finish the proof of Theorem 1.3, we will need the following lemma.

**Lemma 3.1.** We have

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{B(\lambda)} \ge \theta^n > 1.$$

*Proof.* Fix a small $\epsilon > 0$. Since

$$\lim_{u \to 0} \frac{\mathrm{vol}[B(x_0, u)]}{V_0(u)} = 1,$$

there exists a $\delta > 0$ such that

$$\mathrm{vol}[B(x_0, h)] \ge (1 - \epsilon)V_0(h),\ \forall\ h \le \frac{\delta}{\theta},$$

where $V_0(h)$ denotes the volume of an $h$-ball in $\mathbb{R}^n$. It then follows from (3.3) that

$$\begin{aligned} F(\lambda) &\geq \int_0^{\delta/\theta} \mathrm{vol}[B(x_0,\theta t)] \frac{\omega\lambda + (\omega + z(\omega+\beta))t^{\omega+\beta}}{t^{\omega+1}(\lambda + t^{\omega+\beta})^{z+1}} dt \\ &\geq (1-\epsilon)\theta^n \int_0^{\delta/\theta} V_0(t) \frac{\omega\lambda + (\omega + z(\omega+\beta))t^{\omega+\beta}}{t^{\omega+1}(\lambda + t^{\omega+\beta})^{z+1}} dt \\ &= (1-\epsilon)\theta^n \lambda^{-l} \int_0^{\delta/(\theta\lambda^{\frac{1}{\omega+\beta}})} V_0(s) \frac{\omega + (\omega + z(\omega+\beta))s^{\omega+\beta}}{s^{\omega+1}(1 + s^{\omega+\beta})^{z+1}} ds. \end{aligned}$$

On the other hand, it is easy to see that

$$B(\lambda) = \lambda^{-l} \int_0^{+\infty} V_0(s) \frac{\omega + (\omega + z(\omega+\beta))s^{\omega+\beta}}{s^{\omega+1}(1 + s^{\omega+\beta})^{z+1}} ds.$$

Thus

$$\liminf_{\lambda\to 0} \frac{F(\lambda)}{G(\lambda)} \geq (1-\epsilon)\theta^n.$$

Letting $\epsilon \to 0$, one finishes the proof of Lemma 3.1.

Let us go on the proof of Theorem 1.3. It follows from the above *Claim* and Lemma 3.1 that $F(\lambda) \geq B(\lambda)$ for every $\lambda > 0$, that is

$$\int_0^{+\infty} (\mathrm{vol}[B(x_0,\theta t)] - V_0(t)) \frac{\omega\lambda + (\omega + z(\omega+\beta))t^{\omega+\beta}}{t^{\omega+1}(\lambda + t^{\omega+\beta})^{z+1}} dt \geq 0.$$

Letting $\theta \to 1$, we have

$$\int_0^{+\infty} (\mathrm{vol}[B(x_0,t)] - V_0(t)) \frac{\omega\lambda + (\omega + z(\omega+\beta))t^{\omega+\beta}}{t^{\omega+1}(\lambda + t^{\omega+\beta})^{z+1}} dt \geq 0, \quad \forall \lambda > 0.$$

Since $M$ has non-negative Ricci curvature, we have $\mathrm{vol}[B(x_0,t)] \leq V_0(t),\ \forall t > 0$. It then follows from (3.11) that $\mathrm{vol}[B(x_0,t)] = V_0(t)$ for almost every $t \geq 0$, and thus every $t \geq 0$ by continuity. Consequently, $M$ is isometric to $R^n$ by the equality case in Bishop-Gromov's theorem. This completes the proof of Theorem 1.3.

*Proof of Theorem 1.4.* Let $p_t(x,y)$ be the heat kernel on $M$. By [BCL], there exists a positive constant $C_0 > 0$ such that

$$\sup_{x,y\in M} p_t(x,y) \leq \frac{C_0}{t^{n/2}}, \quad \forall t > 0.$$

In [LY], Li and Yau showed that there is a positive constant $C_1(n)$ depending on the dimension of $M$, such that

$$C_1(n) \leq \liminf_{t\to\infty} \mathrm{vol}[B(x,\sqrt{t})] p_t(x,y), \quad \forall x,y \in M.$$

Combining the above two inequalities, we know that

$$\liminf_{r\to+\infty} \frac{\mathrm{vol}[B(x,r)]}{r^n} \geq C_1 C_0^{-1} > 0.$$

Thus $M$ has large volume growth (cf. [S2]). On the other hand, the work of Druet, Hebey and Vaugon [DHV] implies that the sectional curvature of $M$ is identically zero. A theorem of Marenich and Toponogov [MT] states that a complete Riemannian manifold with non-negative sectional curvature and large volume growth is diffeomorphic to a Euclidean space. Combining all these facts, we know that $M$ is isometric to $R^n$. This completes the proof of Theorem 1.4.

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