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LOCALLY HOMOGENEOUS FINITELY NONDEGENERATE CR-MANIFOLDS

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1. Introduction

Manifolds provided with structures which remain invariant under transitive group actions turned out to be of fundamental importance as they often serve as model spaces for more general objects. Numerous examples from differential geometry (Riemannian symmetric spaces, principal bundles with Cartan structures), topology (Thurston's 8 geometric models in the theory of 3-manifolds) as well as algebraic and complex geometry (bounded symmetric domains, flag manifolds) etc. indicate a lively interest in homogeneous spaces. In this paper, we study homogeneous manifolds from a local point of view in the context of Cauchy-Riemann structures. We further present (up to our knowledge) the first example of a uniformly 3-nondegenerate CR-manifold: Roughly speaking, k-nondegenerate CR-manifolds with $k \geq 2$ have *degenerate* Levi form, but are nondegenerate in a higher-order sense as specified in [5].

The most common way of prescribing a CR-manifold is to describe it locally as a subset in some $\mathbb{C}^n$, given as the zero set of certain defining functions. The characterization of the geometric properties of such a manifold, such as the signature of the Levi form, finite or holomorphic (non)degeneracy, minimality, etc. involves a manipulation of the defining equations, which, in concrete cases, can be quite hard, in particular because the amount of computations grows rapidly with the dimension and the codimension of M. Alternatively, a locally *homogeneous* CR-manifold M, or more precisely its CR-germ $[M, o]$ at a point $o \in M$, can also be described in terms of certain purely Lie algebraic objects. Here, 'locally homogeneous' simply means that there exists a finite dimensional subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{hol}(M, o)$ of infinitesimal CR-transformations of a CR-manifold M, representing the given CR-germ $[M, o]$ such that the natural evaluation map $\epsilon : \mathfrak{g} \to T_oM$ is surjective (see Sections 2 and 3 for definitions and further details). In this context, the algebraic category of *CR-algebras* provides a useful local description of CR-manifolds: There is a natural one-to-one correspondence between the objects in this category (i.e., the CR-algebras $(\mathfrak{g}, \mathfrak{q})$, where roughly speaking $\mathfrak{g}$ is the Lie algebra which infinitesimally and transitively acts on M and $\mathfrak{q}$ is a complex Lie algebra which encodes the CR-structure of M) and the germs of abstract CR-manifolds which are furnished with a locally transitive action of some Lie algebra $\mathfrak{g}$ ($\mathfrak{g}$–homogeneous CR-germs, for short). Of course, this fact is what one would expect; however since we did not find such a statement in the existing literature, in Section 4 we carry out in detail the aforementioned correspondence. The advantage of this coordinate-free point of view is that

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the manipulation of CR-algebras seems to be much easier than the manipulation of the defining equations, provided that there is a simple 'dictionary' which precisely describes complex-geometric properties of a germ of locally homogeneous CR-manifold M in terms of certain purely algebraic conditions of the corresponding CR-algebras.

Taking the correspondence between the CR-algebras and the underlying homogeneous CR-germs for granted, one of the main contributions of this paper is to establish such a 'dictionary', i.e., to formulate purely Lie theoretic and easy to check conditions on CR-algebras which are equivalent to various geometric properties of the underlying homogeneous CR-germs. More precisely, a characterization of holomorphic and finite (non)degeneracy, the nondegeneracy order, minimality, etc. of $[M, o]$ is given in such a way. Furthermore, the Levi form, its signature (in the case of CR-codimension 1) and its higher order analogues, which naturally arise in context of 'uniformly' finitely nondegenerate CR-manifolds, can be expressed by formulas only involving information encoded in a corresponding CR-algebra.

It should be noted that in general a given homogeneous CR-germ $[M, o]$ admits several inequivalent locally transitive actions, hence, in turn there are several CR-algebras (inequivalent in the algebraic category) which characterize one given CR-germ. An example hereunto is a germ of the tube $\mathcal{M} = i\mathbb{R}^m + \mathcal{F}$ over the light cone $\mathcal{F} := \{(x_1, \ldots, x_n) \in \mathbb{R}^m : x_1^2 - \sum_2^m x_j^2 = 0,\ x_1 > 0\}$, see [9]. We underline that the above mentioned geometric properties of M can be extracted from each of the CR-algebras, which can be associated with (M, o). In particular, the knowledge of the full Lie algebra $\mathfrak{hol}(M, a)$ of all infinitesimal CR-transformations is not necessary.

The first part of our paper is devoted to an algebraic characterization of various geometric properties of locally homogeneous CR-manifolds. A key ingredient is a formula which relates the Levi form of M and its higher order analogues to certain Lie bracket expressions in a Lie algebra $\mathfrak{g}$, acting transitively on (M, o) (Main Lemma 1). This enables us in Theorem 1 to characterize the order of nondegeneracy of a locally homogeneous CR-manifold M, as well as to decide whether or not M is holomorphically degenerate. In Theorem 2, the minimality of M is described in terms of the CR-algebra.

In the second part of this paper we provide an example of a homogeneous (hence, uniformly) 3-nondegenerate hypersurface $\mathscr{M}$ in the 7-dimensional Grassmannian of isotropic 2-planes in $\mathbb{C}^7$ with respect to a nondegenerate symmetric 2-form. In this example the first order Levi kernel is 3-dimensional and contains the second order kernel which is 1-dimensional. Note that it is quite easy to produce real-analytic CR-manifolds which are, at some distinguished point p, finitely nondegenerate of an arbitrary high order, but which are Levi nondegenerate at generic points. Besides the tube over the light cone, the only further examples of homogenous 2-nondegenerate CR-manifolds has been presented in [16]. Our hypersurface $\mathscr{M}$ seems to be the first known example of a CR-manifold with a *uniform* order of degeneracy strictly bigger than 2. Note that in [16] orbits of real forms have been studied in a certain subclass of irreducible Hermitian symmetric spaces (of so-called tube type), and the authors prove that all such orbits with nontrivial CR-structure, (i.e., neither open nor totally real) are 2-nondegenerate. Our example is an orbit in a more general flag manifold and we use methods developed in the first part to determine its kind of nondegeneracy. At this point one might expect to find orbits M in complex flag manifolds Z, with uniformly finitely nondegenerate CR-structure of arbitrary high order, provided that the ambient manifold Z is general enough. Surprisingly, at least for hypersurface orbits, this is not the case: In Theorem 3 we give a general upper bound for the order of degeneracy that is valid for all finitely nondegenerate hypersurface orbits in arbitrary flag manifolds. We also give a bound for arbitrary orbits in flag manifolds L/Q with maximal Q. For instance, for all *classical* cases, i.e., where the (connected component of the identity of the) complex group of biholomorphic transformations, $\mathrm{Aut}(Z)^\circ$, is a product of classical simple groups, this upper bound is 3. The methods used to determinate the complex-geometric properties of $\mathscr{M}$ can be generalized to deal with arbitrary orbits of real forms in arbitrary flag manifolds.

As a further application of the above methods we generalize in Theorem 4 a result of Kaup and Zaitsev stated in [16] for certain subclass of irreducible Hermitian symmetric spaces (see the paragraph before 4 for the precise statements) to the more general case of arbitrary flag manifolds Z with second Betti number $b_2(Z)$ equal to 1 (equivalently, the complex parabolic isotropy subgroup Q in $Z = L/Q$ is maximal). Our proof of this theorem does not use Jordan-theoretical methods.

We also like to mention that the methods developed in this paper will be used in the forthcoming article [10] where all 5-dimensional germs of locally homogeneous CR-manifolds with *degenerate* Levi form are classified up to CR-equivalence.

Our paper is organized as follows. In Section 2 we study tensors induced by Lie brackets of vector fields. We also recall basic facts concerning local actions, mainly following the fundamental work of Palais ([18]). The key result of that section is the Main Lemma 1 which clarifies the relation between the just mentioned tensors and certain Lie algebras. Section 3 recalls basic geometric notions concerning CR-manifolds, focusing on the condition of being finitely nondegenerate. In Section 4 we recall the definition of the algebraic category of CR-algebras (this notion is essentially taken from [17]) and show that there is an equivalence between this category and the category of germs of locally homogeneous CR-manifolds. The first part of our paper culminates in Section 5, where we provide a “dictionary”, extracting from a given CR-algebra the information necessary to characterize the complex-geometric properties of the underlying CR-germ as explained above.

In the remaining part of the paper we apply the developed methods to CR-manifolds arising as orbits of real forms in flag manifolds. In Section 6 we collect some basic facts concerning general flag manifolds and orbits of real forms. In Section 7 we present an example of a homogeneous 3-nondegenerate CR-manifold and indicate a root-theoretical method how in general orbits of real forms in arbitrary flag manifolds can be handled. Finally in Section 8 we give an upper bound of degeneracy for certain classes of orbits of real forms in flag manifolds, investigate the existence of nonresonant global vector fields on flag manifolds and give a generalization of the aforementioned theorem in [16].

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2. Tensors and homogeneous manifolds

General notation. Let X be a manifold. Given a vector bundle $\pi : \mathbb{E} \to X$, we write $\Gamma(X, \mathbb{E})$ for the vector space of *smooth* sections over X. If a further specification is necessary, we write $\Gamma^{\omega}(\cdot,\cdot)$ or $\Gamma^{\mathcal{O}}(\cdot,\cdot)$ etc. for the spaces of real-analytic or holomorphic sections, respectively. By $\mathbb{E}_x$ we denote the fibre $\pi^{-1}(x)$ at $x \in X$. As usual, TX stands for the tangent bundle of X and T_xX for the tangent space at x. Given a vector field $\xi \in \Gamma(X, TX)$ we write $\xi_x \in T_xX$ for its value at x. Lie groups are denoted by capital letters $G, H, L, ..$ and the associated Lie algebras by the corresponding fraktur letters $\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$, etc. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is a linear map, we write $\mathfrak{g}^{\pm\theta} := \{v \in \mathfrak{g} : \theta(v) = \pm v\}$ for the subspace(s) of the θ–(anti)fixed points. G° stands for the connected component of the identity of a Lie group G and $e \in G$ for the neutral element. By definition, the Lie bracket in $\mathfrak{g}$ is induced by the Lie bracket of left-invariant vector fields on G. By Ad we denote the adjoint representation of G on $\mathfrak{g}$ and by ad its differential, i.e., $\mathrm{ad}_v(w) = [v, w]$. If the contrary is not explicitely stated all Lie groups and the corresponding Lie algebras are assumed to be of finite dimension. In particular, (locally) "homogeneous" means always (locally) homogeneous under a *finite* dimensional Lie group (algebra). Given a real vector space V, we denote by $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ the formal complexification of V. If the real vector space V is furnished with an endomorphism $J : V \to V$ satisfying $J^2 = -\mathrm{Id}$, we write $V^{1,0}, V^{0,1}$ for the $(\pm i)$–eigenspaces of $J^{\mathbb{C}}$ in $V^{\mathbb{C}}$.

Tensors induced by Lie brackets. Given an arbitrary subbundle $\mathbb{E} \subset TX$, it is well-known that the following $\mathbb{R}$–bilinear map

$$(1) \qquad \Gamma(X, \mathbb{E}) \times \Gamma(X, \mathbb{E}) \longrightarrow \Gamma(X, TX) \big/ \Gamma(X, \mathbb{E}), \quad (\xi, \eta) \mapsto [\xi, \eta] \mod \Gamma(X, \mathbb{E})$$

is, in fact, $C^{\infty}(X)$–bilinear. Hence, it induces a well-defined fibre-wise bilinear map (tensor) $\mathbb{E}_x \times \mathbb{E}_x \to T_xX/\mathbb{E}_x$, i.e., $[\xi, \eta]_x \mod \mathbb{E}_x$ depends only on the values ξ_x, η_x and not on the choice of the local sections ξ, η in $\mathbb{E}$. This is a particular instance of a tensor, induced by Lie brackets; in 1 we study a more general case.

Let X be a manifold which is *homogeneous* with respect to a Lie group G. In this section we clarify the relation between the tensors induced by Lie brackets of vector fields on X as in the above example and the Lie structure of the Lie algebra $\mathfrak{g}$. The main application we have in mind is the determination of the Levi form of a homogeneous CR-manifold M and certain "higher-order" analogues, suitable for the characterization of the k–nondegeneracy of M in the sense of [5]. Since a major part of our paper is concerned with the local structure of a CR-manifold, we recall in the next subsection the basic facts regarding *local* actions, locally homogeneous spaces and some canonical constructions which naturally arise in this context.

Locally homogeneous manifolds and bundles. Due to the fundamental work of Palais, see [18] ([11] is a more up-to-date reference) the topics of this subsection are known. The reader familiar with the global concepts homogeneous spaces or a homogeneous bundles will have no difficulties to define the corresponding objects in the local setting. In the following paragraphs we briefly recall the facts relevant for our purposes.

In the global setting, the fundamental objects are G–manifolds, i.e., manifolds provided with a (left) G–action $\cdot : G\times X \to X$, where G is a Lie group. If a manifold X carries an additional structure, for instance a pseudo-riemannian, symplectic, Cauchy-Riemann, affine etc. structure, then we always require that a given G–action preserves this additional structure and call such an X a G–space. We write $G_x = \{g \in G : g(x) = x\}$ for the isotropy subgroup at $x \in X$ and $\mathfrak{g}_x$ for the corresponding isotropy Lie subalgebra. A *G–homogeneous bundle* $\mathbb{E} \to X$ over such a manifold X is a vector bundle together with a fibre-wise linear action on $\mathbb{E}$ which is a lift of the given G-action on X. (This is the commonly used name although the total space $\mathbb{E}$, if nontrivial, never can be G–homogeneous.) For a G–homogeneous bundle $\mathbb{E}$ over a G–homogeneous manifold the isotropy representation $\imath_x : G_x \times \mathbb{E}_x \to \mathbb{E}_x$ at some fixed but arbitrary $x \in X$ completely determines the global structure of $\mathbb{E}$: The total space $\mathbb{E}$ of this bundle is the twisted product $G \times_{G_x} \mathbb{E}_x := G \times \mathbb{E}_x/\sim$ where the quotient on the right-hand side is taken with respect to the following equivalence relation: $(g_1, v_1) \sim (g_2, v_2)$ iff there exists $h \in G_x$ with $g_2 = g_1 h$ and $v_2 = \imath_x(h^{-1})(v_1)$. Conversely, a representation $H \to \mathrm{GL}(V)$ of a (closed) subgroup of G on some vector space V gives rise to the homogeneous vector bundle $\mathbb{V} := G \times_H V$ over G/H.

All the above objects can be appropriately "localized". Let $e \in G$ be the neutral element. A *local* action of G on a manifold X is a map $\cdot : \mathcal{U} \to X$ where $\mathcal{U} \subset G\times M$ is an appropriate open neighbourhood of $\{e\} \times M$, such that the identity $e{\cdot}x = x$ holds for all $x \in X$ as well as $h \cdot (g \cdot x) = (hg) \cdot x$ if both sides are defined. Without loss of generality we may assume (and do for all what follows) that G is connected and simply connected. A local G–action induces a Lie algebra homomorphism $\Xi : \mathfrak{g} \to \Gamma(X, TX)$, see 3. As shown in [18], each homomorphism $\Xi : \mathfrak{g} \to \Gamma(X, TX)$ induces a local G–action on X. Since the above discussion implies that a local action $\cdot : \mathcal{U} \to X$ and the corresponding homomorphism Ξ are equivalent objects, we call each of them a *local action* (of G or $\mathfrak{g}$, respectively) on X and say that X is a $\mathfrak{g}$–space. Clearly, each global action gives rise to a local one. On the other hand it is known that local actions exist, which cannot be globalized, see [11], p.105 for further details. All above notions can also be applied to space germs. To fix the notation, we write $[X, x]$ for a *space germ* with base point x and representative X. By definition, $[X, x]$ stands for the equivalence class with respect to the following equivalence relation between pairs: $(X, x) \sim (Y, y)$ iff there is a local isomorphism between the pointed spaces (X, x) and (Y, y) i.e., there exist neighborhoods $\mathcal{X} \subset X$ of x, $\mathcal{Y} \subset Y$ of y and a structure preserving isomorphism $\psi : \mathcal{X} \to \mathcal{Y}$ with $\psi(x) = y$. We sometimes also write $[X, x, \mathscr{S}]$ if we wish to stress the additional structure $\mathscr{S}$ under consideration. If a local action is also involved, we write $[\Xi, X, x]$ for the germ of the $\mathfrak{g}$–space X at x (here, $\Xi : \mathfrak{g} \to \Gamma(X, TX)$ is the given local action on a representative X). Note that in the category of real or complex manifolds without additional structures, there is in every dimension only one germ, namely $[\mathbb{R}^n, 0]$ (or $[\mathbb{C}^n, 0]$). This is different in the category of CR-manifolds.

By a *morphism* between the $\mathfrak{g}$–space X and the $\mathfrak{g}'$–space X' we mean a pair (Ψ, ψ), consisting of a map $\Psi : X \to X'$ in the given category and a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\Psi_*(\Xi(v)_x) = \Xi'(\psi(v))_{\Psi(x)}$ for all $v \in \mathfrak{g}$. Analogously, a morphism between the germs $[\Xi, X, x]$ and $[\Xi', X', x']$ is the equivalence class $[\Psi, \psi]$ of a base point preserving morphism $(\Psi, \psi) : X \to X'$ between some representatives

X, X'. Every $\mathfrak{g}$–*equivariant* map (i.e., $\psi = \mathrm{Id}$) is an example of a morphism between two $\mathfrak{g}$–spaces.

We call a local action of $\mathfrak{g}$ on X *effective* if the map Ξ is injective. A global action $G \times X \to X$ is effective in this sense if and only if the subgroup, formed by all elements $g \in G$ which act as the identity on X, is discrete. Dividing $\mathfrak{g}$ (or G) by the ineffectivity ideal $\mathfrak{i} = \ker \Xi$ (resp. by the connected component of $\bigcap_{x \in X} G_x$), every non-effective $\mathfrak{g}$–action can be modified into an effective $\mathfrak{g}/\mathfrak{i}$–action which has set-theoretically the same orbits (resp. local integral manifolds of the integrable distribution $\mathbb{G} \subset TX$, generated by $\Xi(\mathfrak{g})$). A local $\mathfrak{g}$–action on X is called *transitive* at x if the evaluation map $\epsilon_x : \mathfrak{g} \to T_x X$, $v \mapsto \Xi(v)_x$, is surjective. We then say that a $\mathfrak{g}$–space X (or space germ $[X, x]$) is $\mathfrak{g}$–*homogeneous* if $\mathfrak{g}$ acts transitively at all its points, (resp. at x).

Analogous to the corresponding notion in the global setting we call a vector bundle $\pi : \mathbb{E} \to X$ over a local G–space X *locally G–homogeneous* or $\mathfrak{g}$–*homogeneous* if the local action of G on X lifts to a local action on the total space $\mathbb{E}$ in such a way that the corresponding local transformations $\mathbb{E} \to \mathbb{E}$ are fibre-wise linear. The germ of a $\mathfrak{g}$–homogeneous bundle (we use the notation $[\mathbb{E}, X, x]$ for it) over a $\mathfrak{g}$–homogeneous space (at x) is determined by the linear representation $\imath_x : \mathfrak{g}_x \to \mathfrak{gl}(\mathbb{E}_x)$ of the isotropy Lie algebra $\mathfrak{g}_x$.

Homogeneous germs associated with a pair of Lie algebras. A $\mathfrak{g}$–homogeneous germ $[\Xi, X, x]$ determines the pair $\mathfrak{g}_x \subset \mathfrak{g}$ of Lie algebras. On the other hand it is known ([18]) that for every given pair $\mathfrak{h} \subset \mathfrak{g}$ of finite-dimensional Lie algebras, there is a germ $[X, x]$ with a transitive local action $\Xi : \mathfrak{g} \to \Gamma(X, TX)$ such that $\mathfrak{h} = \{v \in \mathfrak{g} : \Xi(v)_x = 0\}$, and this $\mathfrak{g}$-homogeneous germ $[\Xi, X, x]$ is unique up to equivariant isomorphisms. A representative $\mathcal{X}$ of $[X, x]$, associated with the pair $\mathfrak{h} \subset \mathfrak{g}$ can be constructed as follows: Select a vector subspace $\mathfrak{W} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{W} \oplus \mathfrak{h}$ and select open neighbourhoods $\mathcal{W} \subset \mathfrak{W}$ and $\mathcal{V} \subset \mathfrak{h}$ of 0 such that the map $\Psi : \mathcal{W} \times \mathcal{V} \to G$, $(w, v) \mapsto (\exp w)(\exp v)$ is a diffeomorphism onto the open subset $\mathcal{G} := \Psi(\mathcal{W} \times \mathcal{V}) \subset G$. Here, G is the simply connected Lie group with Lie algebra $\mathfrak{g}$. Define the following locally closed submanifolds of G,

$$\mathcal{X} := \exp \mathcal{W} \quad \text{and} \quad \mathcal{H} := \exp \mathcal{V}.$$

The local G–action on $\mathcal{X}$ is given as follows: Every element $g \in \mathcal{G}$ has the unique decomposition $\mathbf{x}(g)\mathbf{h}(g)$ with $\mathbf{x}(g) \in \mathcal{X}$ and $\mathbf{h}(g) \in \mathcal{H}$; we consider here $\mathbf{x}, \mathbf{h}$ as maps $\mathbf{x} : \mathcal{G} \to \mathcal{X}$, $\mathbf{h} : \mathcal{G} \to \mathcal{H}$. For every $x \in \mathcal{X}$ and $g \in G$ such that $gx \in \mathcal{G}$ define

$$g \cdot x := \mathbf{x}(gx)\ . \tag{2}$$

This yields a local G–action on $\mathcal{X}$. Let $\Xi_{\mathcal{X}} : \mathfrak{g} \to \Gamma(\mathcal{X}, T\mathcal{X})$ be the corresponding homomorphism. We refer to the pair $\mathfrak{h} \subset \mathfrak{g}$ as to the *infinitesimal model* for $[\Xi, X, x]$ $(= [\Xi_{\mathcal{X}}, \mathcal{X}, e])$. We say that a $\mathfrak{g}$–action on a germ, or that an infinitesimal model $\mathfrak{h} \subset \mathfrak{g}$ is effective if the action of $\mathfrak{g}$ on some representative X has this property.

For *infinite* dimensional Lie algebras $\mathfrak{g}$ we do not know (even if $\dim \mathfrak{g}/\mathfrak{h} < \infty$) whether it is always possible to construct in a meaningful way a germ of a (finite dimensional) manifold with a local transitive action of $\mathfrak{g}$ or even G, if there is an infinite dimensional group G 'associated' with $\mathfrak{g}$.

Bracket map in terms of Lie algebra structure. We are now about to state the main result of this section. It is formulated in greater generality, but we keep in mind that its application will be within the category of CR-manifolds. We start with some preliminary remarks. As already mentioned, every (local) G-action on X induces the map $\Xi : \mathfrak{g} \to \Gamma(X, TX),\ v \mapsto \xi^v$, given by

$$\xi^v_y\, f := \frac{d}{dt}\bigg|_{t=0} f(\exp(-tv)\cdot y)\ , \tag{3}$$

where the f's run through smooth functions defined in a neighborhood of y. This map is a Lie algebra homomorphism. We call each vector field $\xi^v = \Xi(v)$ *fundamental* (with respect to the given local action of G). Unfortunately, the fundamental vector fields and locally homogeneous subbundles of TX over a $\mathfrak{g}$–homogeneous space X seem to be unrelated as in general the fundamental vector fields are *not* invariant under the given local group action. Consequently, given a locally G-homogeneous subbundle $\mathbb{E} \subsetneq TX$ and a fundamental vector field ξ^v such that $\xi^v_y \in \mathbb{E}_y$ for some $y \in X$, the values ξ^v_x may not belong to $\mathbb{E}_x$ for x close to y. Summarizing, fundamental vector fields cannot be in general used for the calculation of the Lie brackets in situations similar to 1.

The set-up in the lemma below is as follows: X stands for an arbitrary but fixed manifold which is locally G–homogeneous at the base point $y \in X$, (G being a Lie group). Let $\Xi : \mathfrak{g} \to \Gamma(X, TX)$ be the corresponding Lie algebra homomorphism and let $\mathfrak{g}_y \subset \mathfrak{g}$ be the isotropy Lie subalgebra at $y \in X$. In particular, we have the canonical identification $T_yX = \mathfrak{g}/\mathfrak{g}_y$. Retaining this notation, we have

Main Lemma 1. *Let $\mathbb{E}^1, \mathbb{E}^2, \mathbb{D}$ be locally G-homogeneous subbundles of TX and let $\mathfrak{e}^1, \mathfrak{e}^2, \mathfrak{d} \subset \mathfrak{g}$ be the corresponding* $\mathrm{ad}(\mathfrak{g}_y)$*–stable linear subspaces such that $\mathbb{E}^j_y = \mathfrak{e}^j/\mathfrak{g}_y$ and $\mathbb{D}_y = \mathfrak{d}/\mathfrak{g}_y$. Assume that the bracket map*

$$[\ ,\ ] : \Gamma(X, \mathbb{E}^1) \times \Gamma(X, \mathbb{E}^2) \longrightarrow \Gamma(X, TX)/\Gamma(X, \mathbb{D})$$

is $C^\infty(X)$–bilinear, i.e., it defines a tensor $b : \mathbb{E}^1 \otimes \mathbb{E}^2 \to TX/\mathbb{D}$. Identify $T_yX/\mathbb{D}_y$ with $\mathfrak{g}/\mathfrak{d}$. Then:

(i) *For arbitrarily chosen representatives $u^1 \in \mathfrak{e}^1$ and $u^2 \in \mathfrak{e}^2$ of any given tangent vectors $\nu^1 \in \mathbb{E}^1_y$, $\nu^2 \in \mathbb{E}^2_y$, we have*

$$b_y(\nu^1, \nu^2) \equiv [u^1, u^2]_{\mathfrak{g}} \bmod \mathfrak{d}\ .$$

Here, the bracket is taken in the Lie algebra $\mathfrak{g}$, and the right-hand side does not depend on the choice of the representatives u^j.

(ii) *For $j = 1, 2$ let $A^j : \mathbb{E}^j \to \mathbb{E}^j$ be locally G–equivariant bundle homomorphisms and $\mathbf{a}^j : \mathfrak{e}^j \to \mathfrak{e}^j$ arbitrary linear maps with $\mathbf{a}^j(\mathfrak{g}_y) \subset \mathfrak{g}_y$, commuting with* ad_{h} *for every $\mathsf{h} \in \mathfrak{g}_y$ and such that the induced maps $\mathbf{a}^j : \mathfrak{e}^j/\mathfrak{g}_y \to \mathfrak{e}^j/\mathfrak{g}_y$ coincide with $A^j : \mathbb{E}^j_y \to \mathbb{E}^j_y$. Then, for u^j and ν^j as above we have*

$$b(A^1(\nu^1), A^2(\nu^2)) \equiv [\mathbf{a}^1(u^1), \mathbf{a}^2(u^2)]_{\mathfrak{g}} \bmod \mathfrak{d}.$$

Proof. Given a space X which is locally G–homogeneous at y, let $\mathfrak{h} := \mathfrak{g}_y \subset \mathfrak{g}$ be the isotropy Lie algebra. We retain here the notation from the preceeding subsection. In particular, we select once and for all a linear subspace $\mathfrak{W} \subset \mathfrak{g}$, complementary to $\mathfrak{h}$. Let then $\mathcal{W} \subset \mathfrak{W}$, $\mathcal{V} \subset \mathfrak{h}$, $\mathcal{X} := \exp \mathcal{W} \subset \mathcal{G} = \mathcal{X}\cdot\mathcal{H} \subset G$ and $\mathbf{x} : \mathcal{G} \to \mathcal{X}$ be as explained in that subsection. We write π for the map $\mathbf{x} : \mathcal{G} \to \mathcal{X}$. By construction of the local G-action on $\mathcal{X}$, see 2, π is a locally G–equivariant (taking the action of G on itself by left translations) and $\pi(e) = e$ (the neutral element of G and a base point in $\mathcal{X} \subset G$). In the following we employ $\mathcal{X}$ at e as a manifold representing the locally G–homogeneous germ $[X, y]$. In order to construct vector fields on $\mathcal{X}$ being also sections of the subbundles $\mathbb{E}^1$ and $\mathbb{E}^2$ we proceed as follows: Let $TG = G \times \mathfrak{g}$ be the trivialization, given by left-invariant vector fields. Given $w \in \mathfrak{g}$, write w_L for the corresponding left invariant vector field on G. For any given element $w \in \mathfrak{g}$ define the following vector field ζ^w on $\mathcal{G}$ by

(4)
$$\zeta^w_{xh} := (xh, \mathrm{Ad}_{h^{-1}}(w)) \in T_{xh}\mathcal{G} \subset \mathcal{G} \times \mathfrak{g} \quad \text{for all } \ g = xh \in \mathcal{G} \ \text{ with } \ x \in \mathcal{X}, \ \ h \in \exp \mathcal{V}\,,$$

and observe that the twist by $\mathrm{Ad}_{h^{-1}}$ makes the locally defined vector fields ζ^w projectable with respect to the projection map $\pi : \mathcal{G} \to \mathcal{X}$. Differentiating the exponential $\exp : \mathfrak{W} \to \mathcal{X}$ we may identify $\mathfrak{W}$ with $T_e\mathcal{X}$. Select arbitrary tangent vectors $\nu^j \in \mathbb{E}^j_e \subset T_e\mathcal{X}$, $j = 1, 2$, and let $w_1, w_2 \in \mathfrak{W}$ be the corresponding elements with respect to the identification $T_e\mathcal{X} \cong \mathfrak{W}$. Extend w^1, w^2 to a basis $w^1, \dots, w^m$ of $\mathfrak{W}$ and let $w^{m+1}, \dots, w^n$ be a basis of $\mathfrak{h}$. We will employ such a basis $w^1, \dots, w^n$ of $\mathfrak{g}$ for the rest of the proof. Since for every $w \in \mathfrak{g}$, ζ^w is π–projectable, define

$$\eta^w := \pi_*(\zeta^w) \ \in \Gamma(\mathcal{X}, T\mathcal{X})\,. \tag{5}$$

Write for short $\zeta^1 := \zeta^{w^1}$, $\zeta^2 := \zeta^{w^2}$ and $\eta^j := \pi_*(\zeta^j)$, $j = 1, 2$. The crucial fact here is that

Claim: For $j = 1, 2$, $\eta^j \in \Gamma(\mathcal{X}, \mathbb{E}^j)$.

Indeed, write $L^g : G \to G$ for the left multiplication by g in G as well as for the induced local map $x \mapsto g \cdot x$ on $\mathcal{X}$. Consider $x \in \mathcal{X}$ as an element in G and observe that $\zeta^j_x = L^x_*(\zeta^j_e)$. Since π is locally G–equivariant, it follows for every $x \in \mathcal{X}$

$$\eta^j_x = \pi_*(L^x_*\zeta^j_e) = L^x_*(\pi_*(\zeta^j_e)) = L^x_*(\eta^j_e) = L^x_*(\nu^j) \in L^x_*(\mathbb{E}^j_e) = \mathbb{E}^j_x. \tag{6}$$

and the claim is proved.

Hence, we have $b(\nu^1, \nu^2) \equiv [\eta^1, \eta^2]_e \bmod \mathbb{D}_e$. On the other hand, the π–projectable vector fields ζ^j satisfy

$$\pi_*[\zeta^1, \zeta^2] = [\pi_*\zeta^1, \pi_*\zeta^2] = [\eta^1, \eta^2]\,. \tag{7}$$

We claim that $[\zeta^1, \zeta^2]_e$ has a simple expression in terms of the Lie brackets in $\mathfrak{g}$ (by definition $[v, w]_{\mathfrak{g}} = [v_L, w_L]$). Since w^j, $1 \le j \le n$, form a basis of $\mathfrak{g}$, the vector fields ζ^j can be written as linear combinations of left-invariant vector fields, i.e.,

$$\zeta^1 = \textstyle\sum_{j=1}^n a_j w^j_L\,, \qquad\qquad \zeta^2 = \sum_{j=1}^n b_j w^j_L$$

with $a_j, b_j \in C^\omega(\mathcal{G})$. By construction, all these functions are constant on $\mathcal{X}$ and we have in particular $a_k|_{\mathcal{X}} = 0$ for $k \ne 1$, and $b_k|_{\mathcal{X}} = 0$ for $k \ne 2$. The following identity is valid at an arbitrary point $x \in \mathcal{X}$:

$$\begin{aligned}
[\zeta^1, \zeta^2]_x = \big[\textstyle\sum a_j w^j_L\,,\ \sum b_k w^k_L\big] &= \\
&= \textstyle\sum_{j,k} a_j(x) b_k(x) [w^j_L, w^k_L] + \sum_{j,k} a_j(x)(w^j_L b_k) w^k_L - \sum_{j,k} b_k(x)(w^k_L a_j) w^j_L = \\
&= [w^1_L, w^2_L] + \textstyle\sum_k (w^1_L b_k)(x) \cdot w^k_L - \sum_j (w^2_L a_j)(x) \cdot w^j_L\,.
\end{aligned}$$

Since $t \mapsto \exp tw^j \in Y$ are the local integral curves at e for w^1_L and w^2_L, it follows $w^1_L b_k\ (e) = w^2_L a_k\ (e) = 0$ for all k, and the above formula, evaluated at e, implies

$[\zeta^1,\zeta^2]_e = [w^1_L, w^2_L]_e = [w^1,w^2]_{\mathfrak g}$. Summarizing, we have proved

(8) $\quad b_e(\nu^1,\nu^2) \;\equiv\; [\eta^1,\eta^2]_e \bmod \mathbb{D}_e \;\equiv\; [\zeta^1,\zeta^2]_e \bmod \mathfrak{d} \;\equiv\; [w^1,w^2]_{\mathfrak g} \bmod \mathfrak{d}.$

which is the formula in (i) for $u^j := w^j \in \mathfrak{W}$. Next, we claim that the elements w^j on the right-hand side of 8 can be replaced by $w^j + h^j$ with arbitrary $h^j \in \mathfrak{h}$: Indeed, from the $C^\infty(\mathcal{X})$–bilinearity of the bracket map follows that $\mathbb{E}^1 + \mathbb{E}^2 \subset \mathbb{D}$, and in turn $\mathfrak{e}^1 + \mathfrak{e}^2 \subset \mathfrak{d}$. Hence, since the $\mathfrak{e}^j$'s are ad($\mathfrak{h}$)–stable, $[w^1,w^2]_{\mathfrak g} \bmod \mathfrak{d} \equiv [w^1+h^1, w^2+h^2]_{\mathfrak g} \bmod \mathfrak{d}$, and the proof of part (i) is now complete.
To show (ii), let $A^j : \mathbb{E}^j \to \mathbb{E}^j$ and $\mathbf{a}^j : \mathfrak{e}^j \to \mathfrak{e}^j$ be as in the assumption part of (ii). By 6 we have for every $x \in \mathcal{X}$ and $w \in \mathfrak{e}^j \cap \mathfrak{W}$

$$A^j(\eta^w_x) = A^j(L^x_*\eta^w_e) = L^x_*(A^j\eta^w_e) = L^x_*(\eta^{\mathbf{a}^j w}_e) = \eta^{\mathbf{a}^j w}_x \qquad \text{i.e.,}\quad A^j(\eta^w) = \eta^{\mathbf{a}^j w}\;.$$

This identity together with 8 implies (ii). □

In Section 5 we apply the Main Lemma to locally homogeneous CR-manifolds and compute their Levi forms and certain higher order tensors. This will enable us to give a simple characterization of the (non)degeneracy type for locally homogeneous CR-manifolds.

3. CR-manifolds and nondegeneracy conditions

In this section we briefly recall some basic facts concerning CR-manifolds and certain geometric properties of them. In particular, we closely examine the condition of being finitely nondegenerate, which plays a major role in the next sections. As a general reference for CR-manifolds, see [4] and [7].

Definition 1. *An abstract CR-manifold is a smooth manifold M together with a subbundle $\mathbb{H} \subset TM$ (we call it the complex subbundle) and a vector bundle endomorphism $J : \mathbb{H} \to \mathbb{H}$ with $J^2 = -\mathrm{Id}$ (the so-called* partial almost complex structure*) such that for all $\xi,\eta \in \Gamma(X,\mathbb{H})$ the condition $[\xi,\eta] - [J\xi, J\eta] \in \Gamma(M,\mathbb{H})$ is fulfilled and, in addition*[1]*, the Nijenhuis tensor*

$$N(J)(\xi,\eta) = [J\xi,J\eta] - [\xi,\eta] - J([\xi,J\eta] + [J\xi,\eta]),\ \xi,\eta \in \Gamma(M,\mathbb{H}),$$

of J vanishes.

In this paper we almost exclusively investigate manifolds which are locally homogeneous under some Lie group. Every smooth manifold furnished with a smooth locally transitive action of a finite dimensional Lie group automatically carries a real-analytic structure, compatible with the group action. We assume from now on (if the contrary is not explicitly stated) that

all manifolds, actions and subbundles are real-analytic.

However, the sections in such subbundles may be only smooth. Due to the well-known embedding theorem of Andreotti-Fredricks ([2]), every (formally integrable) real-analytic CR-manifold $(M,\mathbb{H},J)$ admits a CR-embedding $\imath : M \to Z$ into a complex manifold $(Z,\hat{J})$: This means $\mathbb{H}_x = T_xM \cap \hat{J}T_xM$, $J = \hat{J}|_{\mathbb{H}}$ (the embedding is CR) and $T_xM + \hat{J}T_xM = T_xZ$ (the embedding is *generic*) for all $x \in M$. We write

[1]Some authors does not require the condition '$N(J) = 0$' of formal integrability in the definition of an abstract CR-manifold.

here $\hat{J} : TZ \to TZ$ for the bundle isomorphism (= complex structure), induced by the multiplication with $i = \sqrt{-1}$ in local coordinate charts. A CR-manifold M is called *minimal* at $o \in M$ if for each locally closed submanifold $Y \subset M$ such that $o \in Y$ and $\mathbb{H}_y \subset T_y Y$ for all $y \in Y$ automatically $[M, o] = [Y, o]$.

Infinitesimal CR-transformations. Let $M = (M, \mathbb{H}, J)$ be a real-analytic CR-manifold. Call a vector field $\xi \in \Gamma^\omega(M, TM)$ an *infinitesimal CR-transformation* if the corresponding local 1-parameter subgroup Ψ_t^ξ given by ξ acts by local CR-transformations of M, i.e., $(\Psi_t^\xi)_* \circ J = J \circ (\Psi_t^\xi)_*$. Write $[M, o]$ for the germ of M at the base point o. We also call $[M, o]$ a *CR-germ*. Define $\mathfrak{hol}(M) \subset \Gamma^\omega(M, TM)$ (resp. $\mathfrak{hol}(M, o)$, if dealing with germs) as the subspace consisting of (germs of) infinitesimal CR-transformations of M; (the elements in $\mathfrak{hol}(M, o)$ not necessarily vanish at o). The spaces $\mathfrak{hol}(M)$ and $\mathfrak{hol}(M, o)$ are Lie algebras, with Lie structure induced by the usual Lie brackets of vector fields. Let now $M \hookrightarrow Z$ be a generic embedding in a complex manifold. In the above definition we do *not require* that an infinitesimal CR-transformation on an embedded CR-manifold $M \hookrightarrow Z$ is a restriction of a holomorphic vector field on an open neighborhood of M in Z. However, due to Proposition 12.4.22 in [4], this follows automatically. Finally, by a *holomorphic vector field* on a complex manifold Z we mean a holomorphic section in the real tangent bundle TZ. A (real-analytic, connected) CR-manifold M is called *holomorphically degenerate* at $o \in M$ if there exists an infinitesimal CR-transformation ξ, defined locally in a neighborhood of o in $M \subset (Z, \hat{J})$ such that the local vector field $\hat{J}\xi$ (along M) is also tangent to M.

The notion of *k*-nondegeneracy. A basic invariant of a CR-manifold M is its vector-valued Levi form $\mathscr{L}^M$, or the canonical J–invariant alternating 2-form ω^M : $\mathbb{H} \oplus \mathbb{H} \to TM/_{\mathbb{H}}$. This 2-form is simply the tensor induced by Lie brackets (as in 1 with $\mathbb{E} = \mathbb{H}$; in terms of encoded information $\mathscr{L}^M$ and ω^M are equivalent). The Levi form[2] $\mathscr{L}^M$, is a J–invariant sesquilinear tensor, given by

$$\mathscr{L}^M : \mathbb{H} \otimes \mathbb{H} \to T^{\mathbb{C}}M/_{\mathbb{H}^{\mathbb{C}}}, \quad \mathscr{L}^M(u, v) = \omega^M(u, v) + i\,\omega^M(Ju, v).$$

A complexified version of the Levi form is the tensor $\mathscr{L}^1 : \mathbb{H}^{0,1} \otimes \mathbb{H}^{1,0} \to T^{\mathbb{C}}M/_{\mathbb{H}^{\mathbb{C}}}$ induced by Lie brackets of local sections in $\mathbb{H}^{0,1}$ and $\mathbb{H}^{1,0}$.

In order to study the degeneracy of a CR-manifold, set $\mathbb{F}^{0,1}_{(0)} := \mathbb{H}^{0,1}$, $\mathbb{F}^{1,0}_{(0)} := \overline{\mathbb{F}^{0,1}_{(0)}}$, and define

$$\mathbb{F}^{0,1}_{(1)} := \{\xi \in \mathbb{F}^{0,1}_{(0)} : \mathscr{L}^1(\xi\ ,\ \mathbb{H}^{1,0}) = 0\}.$$

In general, the fibre dimension of $\mathbb{F}^{0,1}_{(1)}$ may vary. A CR-manifold is called *Levi-nondegenerate* or 1–*nondegenerate at* $x \in M$ if the fibre of $\mathbb{F}^{0,1}_{(1)}$ at x is zero. The notion of k-nondegeneracy of M at a point x has been originally defined in [5] (see also Sec. 11.1 in [4]) for arbitrary CR-manifolds. For a general CR-manifold, the order k of nondegeneracy at $x \in M$ may vary from point to point and can be arbitrarily high at certain distinguished points, but often is 1 at generic points. It seems to be much harder to construct CR-manifolds which are everywhere k–nondegenerate and

[2] We took the definition from [15]. It differs from the Levi form used by some other authors (e.g. [4]) by the factor $i/2$

$k >> 1$. In this paper we study CR-manifolds of "uniform degeneracy", i.e., when the dimensions of all fibre-wise and recursively defined subspaces $(\mathbb{F}^{0,1}_{(k)})_x \subset T^{\mathbb{C}}_x M$ (as in 1), do not depend on $x \in M$ and form well-defined sub*bundles* of $T^{\mathbb{C}}M$. For example, (locally) homogeneous CR-manifolds belong to this class. In this situation it is possible to detect the order of degeneracy by certain tensors $\mathscr{L}^{k+1}$ which can be defined in a coordinate-free way. In the uniform situation, such tensors (the construction of which we will shortly recall) seem to be easier to handle than the original criterion, which involves computation of higher order derivatives of defining functions (see Prop. 11.2.4 in [4]).

We recall now the construction of the tensors $\mathscr{L}^k$ in a form suitable for our purposes (essentially following the approach in Appendix of [16]). The Levi form $\mathscr{L}^1 : \mathbb{F}^{0,1}_{(0)} \otimes \mathbb{F}^{1,0}_{(0)} \to T^{\mathbb{C}}M/H^{\mathbb{C}}M$ has been already defined. Define recursively the following subsets (which by the uniform degeneracy assumption on M are all *subbundles*):

$$\mathbb{F}^{0,1}_{(k)} := \{\xi \in \mathbb{F}^{0,1}_{(k-1)} : \mathscr{L}^k(\xi\ ,\ \mathbb{H}^{1,0}) = 0\}\ , \tag{1}$$

and the following maps, induced by Lie brackets:

$$\mathscr{L}^{k+1} : \mathbb{F}^{0,1}_{(k)} \otimes \mathbb{H}^{1,0} \longrightarrow \mathbb{F}^{0,1}_{(k-1)} \oplus \mathbb{H}^{1,0} \Big/ \mathbb{F}^{0,1}_{(k)} \oplus \mathbb{H}^{1,0} \quad \subset \quad \mathbb{H}^{\mathbb{C}} \Big/ \mathbb{F}^{0,1}_{(k)} \oplus \mathbb{H}^{1,0}\ . \tag{2}$$

The fact that all $\mathscr{L}^{k+1}$'s are well-defined tensors follows from the formula $-d\theta(\varphi \otimes \eta) = \theta([\varphi, \eta])$, where φ and η are local sections in $\mathbb{F}^{0,1}_{(k)}$ and $\mathbb{H}^{1,0}$, respectively, and the θ's run through all 1-forms $\theta : T^{\mathbb{C}}M \to \mathbb{C}$ which vanish on $\mathbb{F}^{0,1}_{(k)} \oplus \mathbb{H}^{1,0}$. By construction, for each CR-manifold of uniform degeneracy there is the following filtration of $\mathbb{H}^{0,1}$ by complex subbundles: $\mathbb{H}^{0,1} = \mathbb{F}^{0,1}_{(0)} \supset \mathbb{F}^{0,1}_{(1)} \supset \mathbb{F}^{0,1}_{(2)} \supset \cdots$. The property of being k–nondegenerate is characterized in the following

Proposition 1. *Let $\mathbb{F}^{0,1}_{(j)}$, $j = 0, 1, 2, ...$ be the subbundles as defined in* 1*. A CR-manifold M of uniform degeneracy is k–nondegenerate if and only if $\mathbb{F}^{0,1}_{(k-1)} \neq \mathbb{F}^{0,1}_{(k)} = 0$.*

For locally homogeneous CR-manifolds the subbundles and tensors, as defined in 1 and 2, respectively, can be characterized in Lie algebraic terms. In particular, the geometric notion of k–nondegeneracy can be completely described in terms of a filtration of certain subalgebras, as will be shown in Section 5.

4. Homogeneous CR-germs and CR-algebras

In this section we show that every germ $[M, o]$ of a locally homogeneous real-analytic CR-manifold M (homogeneous CR-germ, for short) can be described by an algebraic datum, for instance by a CR-algebra. Vice versa, every CR-algebra gives rise to a homogeneous CR-germ and all these assignments are functorial. We start by recalling the definition of the category of CR-algebras.

The category of CR-algebras. To fix notation, let $\mathfrak{g}$ stand for a real Lie algebra, let $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ be its complexification and $\psi^{\mathbb{C}}$ the complexification of a real homomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$. As before, we write $\mathfrak{l}$ for the complexification $\mathfrak{g}^{\mathbb{C}}$ and σ for the unique complex conjugation $\mathfrak{l} \to \mathfrak{l}$, fixing the real form $\mathfrak{g} \subset \mathfrak{l}$. The following definition is essentially taken from [17]. A pair, consisting of a finite-dimensional real Lie algebra $\mathfrak{g}$ and a complex subalgebra $\mathfrak{q}$ of $\mathfrak{l} := \mathfrak{g}^{\mathbb{C}}$ is called a *CR-algebra*. In contrast to [17], here we require the finite dimensionality of $\mathfrak{g}$. A *morphism* $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}', \mathfrak{q}')$ is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ with $\psi^{\mathbb{C}}(\mathfrak{q}) \subset \mathfrak{q}'$. We refer to the category in which the objects are CR-algebras and the morphisms are as just described as to the category of *CR-algebras*, or, for short, $\boldsymbol{A_{CR}}$.

On the geometric side there is the **category of homogeneous CR-germs.** The objects in this category are $\mathfrak{g}$-homogeneous CR-germs $[\Xi, M, o]$ for some $\mathfrak{g}$ and the morphisms $[\Psi, \psi]$ are equivalence classes of $(\Psi, \psi) : (\Xi, M, o) \to (\Xi', M', p)$ where $\Psi : (M, \mathbb{H}, J) \to (M', \mathbb{H}', J')$ is a CR-map (i.e., $\Psi_*(\mathbb{H}) \subset \mathbb{H}'$ and $\Psi_* \circ J(v) = J' \circ \Psi_*(v)$ for all $v \in \mathbb{H}$ and $\psi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism such that $\Psi_*(\Xi(v)_x) = \Xi'(\psi(v))_{\Psi(x)}$ for all $x \in M$ in a neighbourhood of $o \in M$). This coincide with the general definition of a morphism between an $\mathfrak{g}$– and $\mathfrak{g}'$–space as given in the subsection "*locally homogeneous manifolds and bundles*" of section 2. We refer to this category as to the category of *homogeneous CR-germs* (and write $\boldsymbol{CR_{ho}}$, for short).

Remark. In our definition of local homogeneity of a germ $[X, x, \mathscr{S}]$ we have assumed that the Lie algebra of infinitesimal transformations, $\mathfrak{g}$, which is transitive on $[X, x, \mathscr{S}]$ is finite dimensional. This is not a substantial restriction in the category of CR-germs: For each CR-germs $[X, x]$ for which there exists a possibly infinite dimensional Lie algebra $\widehat{\mathfrak{g}}$ of infinitesimal real-analytic CR-transformations, transitive on X at x, there also exists a *finite* dimensional Lie algebra $\mathfrak{g}$ of infinitesimal CR-transformations which is transitive on X at x. This follows from Prop. 3.1 in [6], which implies that such an X (generically embedded in some complex manifold Z) is locally CR-equivalent at x to a product $M_1 \times \mathbb{C}^p$, where M_1 is a finitely nondegenerate CR-manifold.

Various notions of equivalence. Disregarding for a moment local actions, there is also the category $\boldsymbol{CR_o}$, consisting of germs of real-analytic CR-manifolds as objects and real-analytic (germs of) base point preserving CR-maps $(M, o) \to (M', o')$ as morphisms. We have then the obvious forgetful functor $\boldsymbol{CR_{ho}} \rightsquigarrow \boldsymbol{CR_o}$. Note, however, that the notion of an isomorphism is different in these two categories: Two homogeneous CR-germs $[\Xi, M, o]$ and $[\Xi', M', o']$ may be non-isomorphic in $\boldsymbol{CR_{ho}}$, though the underlying CR-germs are CR-equivalent, i.e., isomorphic in $\boldsymbol{CR_o}$. To distinguish these two notions of an isomorphism, we refer to $[\Xi, M, o]$ and $[\Xi', M', o']$ as *isomorphic* if there is an isomorphism between them in $\boldsymbol{CR_{ho}}$, and as *CR-equivalent* if $[M, o]$ and $[M', o']$ are isomorphic in $\boldsymbol{CR_o}$

Functors. There is a functor $\mathscr{G}$ from the category of CR-algebras to the category of homogeneous CR-germs (this has also been remarked in [17]). Given a CR-algebra $(\mathfrak{g}, \mathfrak{q})$ set $\mathfrak{l} := \mathfrak{g}^{\mathbb{C}}$. Let $[Z, o]$ be the germ of a complex $\mathfrak{l}$–homogeneous manifold with the infinitesimal model $\mathfrak{l} \supset \mathfrak{q}$ and let Z be a locally L–homogeneous representative. The CR-germ $[\Xi, M, o]$ is then given by a real-analytic submanifold $M \subset Z$ with $o \in M$, which is the local integral submanifold of the Frobenius distribution $\mathbb{M} \subset TZ$, generated by the vector fields $\Xi(v)$, $v \in \mathfrak{g}$. As a consequence of $\mathfrak{g} + i\mathfrak{g} = \mathfrak{l}$, this CR-manifold is generically embedded in Z.

Let $\psi : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}', \mathfrak{q}')$ be a morphism between two CR-algebras. Let (Z, o) (resp. (Z', o')) be a representative of the $\mathfrak{l}$–homogeneous ($\mathfrak{l}'$–homogeneous) germ, determined by the complex infinitesimal model $\mathfrak{l} \supset \mathfrak{q}$ (or $\mathfrak{l}' \supset \mathfrak{q}'$, respectively). Then $\psi^{\mathbb{C}}$ induces (possibly after shrinking Z) an $(\mathfrak{l}, \mathfrak{l}')$–equivariant, holomorphic and base point preserving map $\Psi : Z \to Z'$, which maps $M \subset Z$ to $M' \subset Z'$. Hence, the restriction of Ψ to M is a real-analytic CR-map and yields a morphism between the homogeneous CR-germs $[\Xi, M, o]$ and $[\Xi', M', o']$.

There exists also a functor $\mathscr{A}$ in the opposite direction. Let a $\mathfrak{g}$–homogeneous CR-germ $[\Xi, M, o]$ be given. Due to [2], there exists a complex manifold Z such that a representative M is generically CR-embedded in Z. The only point here is that this embedding is automatically locally equivariant with respect to $\mathfrak{g}$: This is a consequence of the extension results in [4] (Corollaries 12.4.17 and 1.7.13) and our assumption that $\mathfrak{g}$ is finite dimensional. Hence, possibly after shrinking Z, we may assume that for each $v \in \mathfrak{g}$ the vector field $\Xi(v)$ is the restriction of a holomorphic vector field on Z. Therefore, we can consider Ξ as a Lie algebra homomorphism $\mathfrak{g} \to \Gamma^{\mathcal{O}}(Z, TZ)$. Since the Lie algebra $\Gamma^{\mathcal{O}}(Z, TZ)$ is complex, Ξ extends to a complex homomorphism $\Xi^{\mathbb{C}} : \mathfrak{l} \to \Gamma^{\mathcal{O}}(Z, TZ)$. Define the complex isotropy subalgebra $\mathfrak{q} := \{w \in \mathfrak{l} : \Xi(w)_o = 0\}$. The pair $\mathscr{A}[\Xi, M, o] := (\mathfrak{g}, \mathfrak{q})$ is a CR-algebra and we call it the CR-algebra *associated with* $[\Xi, M, o]$. Define $\mathfrak{g}_o := \mathfrak{g} \cap \mathfrak{q}$. Observe that $\mathfrak{g} \supset \mathfrak{g}_o$ is the infinitesimal real model for $[\Xi, M, o]$ and $\mathfrak{l} \supset \mathfrak{q}$ the infinitesimal model for $[\Xi^{\mathbb{C}}, Z, o]$. A word of caution: Even if $\Xi : \mathfrak{g} \to \Gamma(Z, TZ)$ is injective, i.e., the original $\mathfrak{g}$–action is effective, the complexification $\Xi^{\mathbb{C}}$ may not be injective, i.e., the sum $\Xi(\mathfrak{g}) + J\Xi(\mathfrak{g})$ may not be direct in $\Gamma(Z, TZ)$.

It follows that a base point preserving morphism $(\Psi, \psi) : (\Xi, M, o) \to (\Xi', M', o')$ induces a morphism of the associated CR-algebras: The only point which has to be checked is that the complexification of $\psi : \mathfrak{g} \to \mathfrak{g}'$ maps $\mathfrak{q}$ to $\mathfrak{q}'$: Again by the extension result from [4], a representative $\Psi : M \to M'$ extends to a holomorphic map $\widehat{\Psi} : Z \to Z'$. By the identity principle, $\widehat{\Psi}$ is equivariant with respect to $\mathfrak{l}$ and $\mathfrak{l}'$. Since $\widehat{\Psi}$ preserves the base points, the inclusion $\psi^{\mathbb{C}}(\mathfrak{q}) \subset \mathfrak{q}'$ follows from $\widehat{\Psi}_*(\Xi(w)_o) = \Xi'(\psi^{\mathbb{C}}(w))_{\hat{\Psi}(o)} = \Xi'(\psi^{\mathbb{C}}(w))_{o'}$. Summarizing, we have

Proposition 2. *The above defined covariant functors*

$$\boldsymbol{A_{CR}} \overset{\mathscr{G}}{\rightsquigarrow} \boldsymbol{CR_{ho}} \quad \text{and} \quad \boldsymbol{CR_{ho}} \overset{\mathscr{A}}{\rightsquigarrow} \boldsymbol{A_{CR}}$$

are mutually quasi-inverse and yield an equivalence of the two categories.

5. Geometric properties of a germ, determined by a CR-algebra

As seen in the previous section, the germ at o of a locally homogeneous CR-manifold M is completely determined by the corresponding CR-algebra. In this section we show explicitly how the geometric information encoded in a CR-algebra $(\mathfrak{g}, \mathfrak{q})$ associated with a CR-germ $[M, o]$ can be extracted. In particular, we give a description of the subbundles $\mathbb{H}, \mathbb{H}^{0,1}, \mathbb{H}^{1,0}, \mathbb{F}^{0,1}_{(k)}$ of $T^{\mathbb{C}}M$ in terms of certain quotients of subspaces of $\mathfrak{g}$ and $\mathfrak{l}$. The main results of this section are 1) a characterization of the k–nondegeneracy and the holomorphic nondegeneracy of a CR-germ $[M, o]$ in terms of Lie algebraic properties of an associated CR-algebra (Theorem 1; see also the following remarks), and 2) Theorem 2, in which the minimality of M is characterized in a similar fashion.

Let $(\mathfrak{g},\mathfrak{q})$ be a given CR-algebra and M a CR-manifold, representing the underlying CR-germ $[M,o]$, generically embedded into a complex manifold Z such that $[Z,o]$ is the $\mathfrak{l}$-homogeneous germ of complex manifold with the infinitesimal model $\mathfrak{l}\supset\mathfrak{q}$. Recall our notation: $\mathfrak{l}=\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{g}_o:=\mathfrak{g}\cap\mathfrak{q}$ and $\sigma:\mathfrak{l}\to\mathfrak{l}$ is the involutive automorphism with $\mathfrak{l}^\sigma=\{v\in\mathfrak{l}:\sigma(v)=v\}=\mathfrak{g}$ (compare Section 4). Since the vector bundles $TM,\mathbb{H},\mathbb{H}^{1,0},\mathbb{H}^{0,1},T^{1,0}Z$, etc. are locally homogeneous with respect to the given transitive local actions on M and Z, they are determined by its single fibre at $o\in M$. Clearly, $T_oM=\mathfrak{g}/\mathfrak{g}_o$, $T_oZ=\mathfrak{l}/\mathfrak{q}$, and $T_o^{\mathbb{C}}M=\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_o^{\mathbb{C}}$. We further need to specify $\mathbb{H}_o$ $\mathbb{H}_0^{0,1}$ and $\mathbb{H}_o^{0,1}$ as subspaces of the preceeding quotients of Lie algebras. We proceed with preparatory observations.

• The real isotropy Lie algebra $\mathfrak{g}_o$ is a real form of $\mathfrak{q}\cap\sigma\mathfrak{q}$ (this has already been observed in [21]). Hence, the complexified tangent space $T_o^{\mathbb{C}}M$ can be identified with the quotient $\mathfrak{l}/\mathfrak{q}\cap\sigma\mathfrak{q}$.

• Define the subspace $\mathfrak{H}:=(\mathfrak{q}+\sigma\mathfrak{q})^\sigma=(\mathfrak{q}+\sigma\mathfrak{q})\cap\mathfrak{g}$ of $\mathfrak{g}$. Note that $[\mathfrak{g}_o,\mathfrak{H}]\subset\mathfrak{H}$ and observe that the map $\mathfrak{q}\to\mathfrak{H}$, $w\mapsto w+\sigma w$ is surjective. By elementary linear algebra, $\mathfrak{H}/\mathfrak{g}_o=\mathfrak{g}/\mathfrak{g}_o\cap i(\mathfrak{g}/\mathfrak{g}_o)=\mathbb{H}_o$. Here, $\mathfrak{g}/\mathfrak{g}_o$ is considered as a subset of the complex vector space $\mathfrak{l}/\mathfrak{q}$.

• The invariant partial complex structure $J:\mathbb{H}\to\mathbb{H}$ induced by the CR-embedding $M\hookrightarrow Z$, is determined by $J_o:\mathbb{H}_o\to\mathbb{H}_o$, i.e., by the corresponding endomorphism $J_o:\mathfrak{H}/\mathfrak{g}_o\to\mathfrak{H}/\mathfrak{g}_o$. Given an arbitrary $x\in\mathfrak{H}\subset\mathfrak{g}$, it follows

$$J_o:\mathfrak{H}/\mathfrak{g}_o\to\mathfrak{H}/\mathfrak{g}_o,\quad (x+\mathfrak{g}_o)\mapsto y+\mathfrak{g}_o, \tag{1}$$

where $y\in\mathfrak{g}$ is an arbitrary element with $y+ix\in\mathfrak{q}$. Complexifying J_o it is readily seen that $\mathbb{H}_o^{1,0}=\mathrm{Eig}(J_o^{\mathbb{C}},i)=\sigma\mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q}$ which is a subspace of $\mathbb{H}_o^{\mathbb{C}}=\mathfrak{H}^{\mathbb{C}}/\mathfrak{g}_o^{\mathbb{C}}=\mathfrak{q}+\sigma\mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q}$. We summarize the above results, i.e., the identifications of the various fibres at o with the corresponding quotients of Lie algebras in the diagram below:

$$\begin{array}{ccccccccc}
\mathfrak{H}/\mathfrak{g}_o & = & \mathbb{H}_o & & & & & & \\
\cap & & \cap & & & & & & \\
\mathfrak{g}/\mathfrak{g}_o & = & T_oM & \hookrightarrow & T_oZ & = & \mathfrak{l}/\mathfrak{q} & & \\
\cap & & \cap & & \cap & & \cap & & \\
\mathfrak{l}/\mathfrak{q}\cap\sigma\mathfrak{q} & = & T_o^{\mathbb{C}}M & \hookrightarrow & T_o^{\mathbb{C}}Z & = & \mathfrak{l}^{\mathbb{C}}/\mathfrak{q}^{\mathbb{C}} & & \\
\cup & & \cup & & \| & & \| & & \\
\sigma\mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q} & = & \mathbb{H}_o^{1,0} & \hookrightarrow & T_o^{1,0}Z & = & \mathfrak{l}^{1,0}/\mathfrak{q}^{1,0} & = & \mathfrak{l}/\mathfrak{q} \\
\oplus & & \oplus & & \oplus & & \oplus & & \\
\mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q} & = & \mathbb{H}_o^{0,1} & \hookrightarrow & T_o^{0,1}Z & = & \mathfrak{l}^{0,1}/\mathfrak{q}^{0,1} & \overset{\sigma}{\cong} & \mathfrak{l}/\sigma\mathfrak{q}\ .
\end{array} \tag{2}$$

Finite nondegeneracy in terms of CR-algebras. In the next paragraphs we repeatedly apply the Main Lemma 1 to the various tensors associated with a locally homogeneous CR-manifold M as described in Section 3. We obtain in that way expressions for all $\mathscr{L}^k$'s in terms of Lie brackets in the Lie algebra $\mathfrak{l}$. Here, $\mathfrak{l}=\mathfrak{g}^{\mathbb{C}}$ comes from the CR-algebra, associated to a given $\mathfrak{g}$–homogeneous CR-germ $[M,o]$. Keeping in mind the identifications 2, let $\mathbf{j}:\mathfrak{H}\to\mathfrak{H}$ be the $\mathrm{ad}(\mathfrak{g}_o)$–equivariant linear map with $\mathbf{j}(\mathfrak{g}_o)\subset\mathfrak{g}_o$ such that the induced map $\mathbf{j}:\mathfrak{H}/\mathfrak{g}_o\to\mathfrak{H}/\mathfrak{g}_o$ coincides with J_o.

The Main Lemma 1 applied to $\mathcal{L}^M$ and ω^M (tensors defined in Section 3) yields

$$(3)\qquad \begin{array}{ll} \omega_o^M : \mathfrak{H}/\mathfrak{g}_o \times \mathfrak{H}/\mathfrak{g}_o \to \mathfrak{g}/\mathfrak{H}, & (u,v) \mapsto [u,v]_{\mathfrak{g}} \bmod \mathfrak{H} \\ \mathscr{L}_o^M : \mathfrak{H}/\mathfrak{g}_o \times \mathfrak{H}/\mathfrak{g}_o \to \mathfrak{l}/\mathfrak{q}+\sigma\mathfrak{q}, & (u,v) \mapsto [u,v]_{\mathfrak{g}} + i\,[\mathbf{j}(u),v]_{\mathfrak{g}} \bmod \mathfrak{q}+\sigma\mathfrak{q} \end{array},$$

where $u,v \in \mathfrak{H}$ are arbitrary representatives of the cosets in $\mathfrak{H}/\mathfrak{g}_o$. The complexification of ω^M, restricted to $\mathbb{H}^{0,1} \times \mathbb{H}^{1,0}$, i.e., the tensor $\mathscr{L}^1$, invariant under the local action of G, coincides after identifying $\mathbb{H}^{0,1} \cong \mathbb{H}^{1,0} \cong \mathbb{H}$ with the Levi form $\mathscr{L}^M$ up to the factor 2. As also a complex version of the Main Lemma remains valid (i.e., replacing TX and its subbundles by complex subbundles of $T^{\mathbb{C}}X$ and the subalgebras of $\mathfrak{g}$ by complex subalgebras of $\mathfrak{l} = \mathfrak{g}^{\mathbb{C}}$) the tensor $\mathscr{L}^1$ at o is expressed by the following formula:

$$(4)\qquad \mathscr{L}_o^1 : \mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q} \times \sigma\mathfrak{q}/\mathfrak{q}\cap\sigma\mathfrak{q} \to \mathfrak{l}/(\mathfrak{q}+\sigma\mathfrak{q})\ , (u,v) \mapsto [u,v]_{\mathfrak{l}} \bmod (\mathfrak{q}+\sigma\mathfrak{q})\ .$$

For short, write $\mathfrak{q}^{(0)} := \mathfrak{q}$, $\mathfrak{q}^{(\infty)} := \mathfrak{q}\cap\sigma\mathfrak{q}$. Hence, the (left-) kernel of $\mathscr{L}^1$ is $\mathfrak{q}^{(1)}/\mathfrak{q}\cap\sigma\mathfrak{q}$, with

$$(5)\qquad \mathfrak{q}^{(1)} := \{w \in \mathfrak{q} : [w,\sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}\}$$

This subspace coincides with the normalizer $N_{\mathfrak{q}}(\mathfrak{q}+\sigma\mathfrak{q}) = \{u \in \mathfrak{q} : [u,\mathfrak{q}+\sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}\}$ and consequently is a complex subalgebra. For bigger k, it is also readily seen that the recursively defined locally homogeneous subbundles $\mathbb{F}^{0,1}_{(k)}$ (1), i.e., their fibres at o, and the 'higher order' tensors $\mathscr{L}^{k+1}$ (2) can be expressed by the formulae:

$$(6)\qquad (\mathbb{F}^{0,1}_{(k)})_o = \mathfrak{q}^{(k)}/\mathfrak{q}^{(\infty)} \quad \text{with} \quad \mathfrak{q}^{(k)} := \{w \in \mathfrak{q}^{(k-1)} : [w,\sigma\mathfrak{q}^{(0)}] \subset \mathfrak{q}^{(k-1)} + \sigma\mathfrak{q}^{(0)}\}\ .$$

$$(7)\qquad \begin{array}{cccccc} \mathscr{L}_o^{k+1} : \mathfrak{q}^{(k)}/\mathfrak{q}^{(\infty)} & \times & \sigma\mathfrak{q}^{(0)}/\mathfrak{q}^{(\infty)} & \longrightarrow & \mathfrak{q}^{(k-1)}+\sigma\mathfrak{q}^{(0)} \big/ \mathfrak{q}^{(k)}+\sigma\mathfrak{q}^{(0)} \\ (u & , & v) & \longmapsto & [u,v] \bmod \mathfrak{q}^{(k)}+\sigma\mathfrak{q}^{(0)}. \end{array}$$

Here and above, the right-hand sides do not depend on the choice of the representatives u and v. Recall that $\mathfrak{H} := (\mathfrak{q}+\sigma\mathfrak{q})^{\sigma}$. We have the auxiliary

Lemma 2. *Let $(\mathfrak{g},\mathfrak{q})$ be a CR-algebra and let $\mathfrak{q}^{(k)}$ be the subspaces of $\mathfrak{q}$, defined in 6. Then*

(i) *All subspaces occurring in the filtration $\mathfrak{q} = \mathfrak{q}^{(0)} \supset \mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots \supset \mathfrak{q}^{(\infty)}$ are complex* subalgebras *of $\mathfrak{q}$.*

(ii) *Define $\mathfrak{F}^{(k)} := (\mathfrak{q}^{(k)}+\sigma\mathfrak{q}^{(k)})^{\sigma}$ for $k \geq 0$. Then $\mathfrak{F}^{(k)}$ (and $\mathfrak{q}^{(k)}+\sigma\mathfrak{q}^{(k)}$) are subalgebras for all $k \geq 1$. $\mathfrak{F}^{(0)} = \mathfrak{H}$ is a subalgebra if and only if the CR-germ associated with $(\mathfrak{g},\mathfrak{q})$ is Levi flat.*

Proof. ad (i): Clearly, $\mathfrak{q} = \mathfrak{q}^{(0)}$ is a subalgebra. Assume inductively that for all j with $j < k$ the subspaces $\mathfrak{q}^{(j)}$ are subalgebras. To conclude that $\mathfrak{q}^{(k)} \subset \mathfrak{q}^{(k-1)}$ is also a subalgebra, observe that for $u,v \in \mathfrak{q}^{(k)}$

$$\begin{array}{rcl} [[u,v],\sigma\mathfrak{q}] & \subset & [u,[v,\sigma\mathfrak{q}]] + [v,[u,\sigma\mathfrak{q}]] \subset [u,\mathfrak{q}^{(k-1)}] + [v,\mathfrak{q}^{(k-1)}] + \mathfrak{q}^{(k-1)} + \sigma\mathfrak{q} \subset \\ & \subset & \mathfrak{q}^{(k-1)} + \sigma\mathfrak{q}\ , \end{array}$$

i.e., $[u,v] \in \mathfrak{q}^{(k)}$.

ad (ii): First we show that $\mathfrak{F} := \mathfrak{F}^{(1)}$ is a subalgebra. We claim that the subspace

$N_{\mathfrak{l}}(\mathfrak{q}+\sigma\mathfrak{q}) \cap (\mathfrak{q}+\sigma\mathfrak{q})$ is a subalgebra: Indeed for every u, v from the above space $[u,v] \in N_{\mathfrak{l}}(\mathfrak{q}+\sigma\mathfrak{q})$ (as the normalizer $N_{\mathfrak{l}}(\star)$ is a subalgebra) and $[u,v] \in [u, \mathfrak{q}+\sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}$. Our next observation is that

$$\mathfrak{q}^{(1)} + \sigma\mathfrak{q}^{(1)} \;=\; N_{\mathfrak{l}}(\mathfrak{q}+\sigma\mathfrak{q}) \cap (\mathfrak{q}+\sigma\mathfrak{q}).$$

The inclusion "$\subset$" follows directly from the definition of $\mathfrak{q}^{(1)}$. The opposite inclusion can be seen as follows: $u, \sigma v \in N_{\mathfrak{l}}(\mathfrak{q}+\sigma\mathfrak{q})$ with $u, v \in \mathfrak{q}$ implies $[u, \sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}$ and $\sigma[v, \sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}$, i.e., $u, v \in \mathfrak{q}^{(1)}$.

Assume now inductively that for all j with $j \leq k$ the subspaces $\mathfrak{q}^{(j)} + \sigma\mathfrak{q}^{(j)}$ are subalgebras. We claim that

$$(\dagger) \qquad \mathfrak{q}^{(k+1)} + \sigma\mathfrak{q}^{(k+1)} = N_{\mathfrak{l}}(\mathfrak{q}^{(k)} + \sigma\mathfrak{q}) \cap (\mathfrak{q}^{(k)} + \sigma\mathfrak{q}^{(k)}) \cap N_{\mathfrak{l}}(\mathfrak{q} + \sigma\mathfrak{q}^{(k)})$$

The subspace on the right-hand side is a Lie subalgebra as a consequence of the induction hypothesis. It remains to prove the identity (†). Both inclusions, "$\subset$" and "$\supset$" can be checked by a straightforward computation. The last statement in (ii), concerning Levi-flatness, follows immediately from the formula (3). □

We are now in the position to characterize holomorphic (non)degeneracy in terms of a purely algebraic condition on CR-algebras. Note that a homogeneous CR-germ $[M,o]$ is holomorphically nondegenerate if and only it is k-nondegenerate for some finite k. This follows from Theorem 11.5.1 in [4], applied to the homogeneous case. In the following theorem we use the notation and constructions from Section 4.

Theorem 1. *Let $(\mathfrak{g}, \mathfrak{q})$ be a given CR-algebra and $[M,o]$ the corresponding homogeneous CR-germ, generically embedded into the germ $[Z,o]$. Let $\mathfrak{q}^{(\bullet)}$ be the filtration by subalgebras as in 2.ii. Then , for every integer $k \geq 1$*

- (i) *(M,o) is k–nondegenerate if and only if $\mathfrak{q}^{(k-1)} \neq \mathfrak{q}^{(k)} = \mathfrak{q}^{(\infty)}$, and then $k \leq \dim \mathfrak{q}^{(0)} - \dim \mathfrak{q}^{(\infty)}$.*
- (ii) *(M,o) is holomorphically degenerate if and only if there exists a complex subalgebra $\mathfrak{r} \subset \mathfrak{l} = \mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{q} \subsetneq \mathfrak{r} \subset \mathfrak{q}+\sigma\mathfrak{q}$. The latter condition implies the existence of a locally equivariant CR-morphism $\Psi : M \to M'$, whose fibres are positive-dimensional complex submanifolds of Z and M' is a representative of the $\mathfrak{g}$–homogeneous CR-germ $[M', o']$ associated with $(\mathfrak{g}, \mathfrak{r})$.*

Proof. The first part is an immediate consequence of Proposition 1 and formula 6. For the proof of part (ii) let $\mathfrak{q} = \mathfrak{q}^{(0)} \supset \mathfrak{q}^{(1)} \supset \cdots \supset \mathfrak{q}^{(\infty)}$ the filtration of $\mathfrak{q}$ with $\mathfrak{q}^{(k)}$ as in 6. As already mentioned, in the locally homogeneous case the holomorphic degeneracy of $[M,o]$ is equivalent to the fact that $[M,o]$ is not finitely nondegenerate. Thus, according to (i), there exists $n \in \mathbb{N}$ such that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} \neq \mathfrak{q}^{(\infty)}$. By the very construction of $\mathfrak{q}^{(n)}$, this implies $[\mathfrak{q}^{(n)}, \sigma\mathfrak{q}] \subset \mathfrak{q}^{(n)} + \sigma\mathfrak{q}$. Since $\mathfrak{q}^{(n)}$ is a subalgebra by Lemma 2, $\mathfrak{q}^{(n)} + \sigma\mathfrak{q}$ is a subalgebra, as well. Define

$$\mathfrak{r} := \sigma\mathfrak{q}^{(n)} + \mathfrak{q} = \sigma(\mathfrak{q}^{(n)} + \sigma\mathfrak{q})$$

and note that $\mathfrak{r} \supsetneq \mathfrak{q}$ and $\mathfrak{r} + \sigma\mathfrak{r} = \mathfrak{q} + \sigma\mathfrak{q}$. This proves the existence of $\mathfrak{r}$ as claimed.
For the "if" part, let $[M', o']$ be the CR-germ, associated with the CR algebra $(\mathfrak{g}, \mathfrak{r})$. As before, let $M \subset Z$ be a R-manifold, representing the CR-germ associated with $(\mathfrak{g}, \mathfrak{q})$ and generically embedded into the complex manifold Z, as explained in the subsection of Section 4 concerning functors. Since $\mathfrak{q} \subset \mathfrak{r}$, the identity map on $\mathfrak{l} = \mathfrak{g}^{\mathbb{C}}$ induces a morphism $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{r})$, and by Proposition 2 a CR-morphism $\Psi : (M,o) \to (M', o')$

which is the restriction of a holomorphic surjective morphism $\widehat{\Psi} : (Z, o) \to (Z', o')$ (shrinking Z and Z', if necessary). We claim that the germ of the fibre $\Psi^{-1}(o')$ coincides with the germ of the fibre $\widehat{\Psi}^{-1}(o')$: This can be seen by comparing the dimensions: we will prove that the injective map

$$T_o(\Psi^{-1}(o')) = \mathfrak{g} \cap \mathfrak{r}/\mathfrak{g} \cap \mathfrak{q} \longrightarrow \mathfrak{r}/\mathfrak{q} = T_o(\widehat{\Psi}^{-1}(o'))$$

is also surjective: Indeed, select a linear subspace $\mathfrak{q}^\wedge \subset \mathfrak{q}$ such that $\mathfrak{q} = (\mathfrak{q} \cap \sigma\mathfrak{q}) \oplus \mathfrak{q}^\wedge$. Then $\sigma\mathfrak{q} = (\mathfrak{q} \cap \sigma\mathfrak{q}) \oplus \sigma\mathfrak{q}^\wedge$ and $\mathfrak{r} = \mathfrak{q} \oplus \mathfrak{r}^\wedge$ with $\mathfrak{r}^\wedge = \mathfrak{r} \cap \sigma\mathfrak{q}^\wedge$. It follows $\mathfrak{g} \cap \mathfrak{r}/\mathfrak{g} \cap \mathfrak{q} \cong (\mathfrak{r}^\wedge \oplus \sigma\mathfrak{r}^\wedge)^\sigma$, hence $\dim(\mathfrak{r}^\wedge \oplus \sigma\mathfrak{r}^\wedge)^\sigma = \dim \mathfrak{r}/\mathfrak{q}$ as claimed. □

Remark.
In the remark following Prop. 13.3 in [17], and also in Theorem 3.2 in [1], it is claimed that "the CR-germ $[M, o]$, associated with $(\mathfrak{g}, \mathfrak{q})$ is holomorphically nondegenerate if and only if $(\mathfrak{g}, \mathfrak{q})$ is 'ideal nondegenerate' (as defined in Sec.2 in [17])". The following examples show that this cannot be true.

Let $Z := L/Q$ be a complex flag manifold with L *simple* and Q a parabolic subgroup. Let $G \subset L$ be a real form and σ the corresponding complex conjugation. Clearly, every orbit $M := G{\cdot}z$ in Z, gives rise to the CR-germ $[M, z]$ and the corresponding CR-algebra is $(\mathfrak{g}, \mathfrak{l}_z)$. Here, $\mathfrak{l}_z$ stands for the complex isotropy Lie algebra at z. In each such Z there is precisely one closed G–orbit ([21]). On the other hand, every CR-algebra $(\mathfrak{g}, \mathfrak{q})$, describing a non-complex CR-germ (for instance a non-open G–orbit) such that $\mathfrak{g}$ simple, trivially is 'ideal nondegenerate'. However, we give examples of closed G–orbits in flag manifolds which are holomorphic *degenerate*.

Lemma 3. *Assume that $M := G{\cdot}z_c \subset Z = L/Q$ is a closed orbit such that there exist complex subalgebras $\mathfrak{p} \subset \mathfrak{l}$ with $\mathfrak{l}_{z_c} \subsetneq \mathfrak{p} \subset \mathfrak{l}_{z_c} + \sigma\mathfrak{l}_{z_c}$. Then there exists a closed G-orbit M' in a flag manifold Z' and a locally trivial holomorphic fibration $\pi : Z \to Z'$ with positive dimensional complex fibres, biholomorphic to a complex flag manifold F, such that $\pi(M) = M'$ and $M = \pi^{-1}(M')$. In particular, $\pi : M \to M'$ is a locally trivial CR-fibration with compact complex fibres and is* locally *CR-equivalent to the product $M' \times \mathbb{C}^{\dim F}$.*

Proof. We give here an elementary argument which does not use our Theorem 1. As $\mathfrak{p}$ contains $\mathfrak{q} := \mathfrak{l}_{z_c}$, it is parabolic. Let $P = N_L(\mathfrak{p}) = N_L(\mathfrak{p})^\circ$ be the corresponding parabolic subgroup. The inclusion $Q \subset P$ induces the *locally trivial holomorphic fibration* $\pi : L/Q \to L/P$. Its typical fibre is the flag manifold $F := P/Q$. Let σ denote the involution which determines the real form G (or $\mathfrak{g}$). The orbit $M' = G{\cdot}\pi(z_c)$ is the closed orbit in L/P. A key point is that $\pi^{-1}(M') = M$: Since $\mathfrak{q} + \sigma\mathfrak{q} = \mathfrak{p} + \sigma\mathfrak{p}$, a simple linear algebraic argument shows that

$$\dim_{\mathbb{R}} M \cap \pi^{-1}(\pi(z_c)) = \dim {}^{(\mathfrak{p}+\sigma\mathfrak{p})^\sigma}/{}_{(\mathfrak{q}+\sigma\mathfrak{q})^\sigma} = \dim_{\mathbb{R}} {}^{\mathfrak{p}}/{}_{\mathfrak{q}} = \dim_{\mathbb{R}} \pi^{-1}(\pi(z_c));$$

hence, $M \cap \pi^{-1}(\pi(z_c))$ is open in the fibre $\pi^{-1}(\pi(z_c))$; since M is closed by assumption, $M \cap \pi^{-1}(\pi(z_c))$ is closed; since the fibres of $\pi : L/Q \to L/P$ are connected, it follows $\pi^{-1}(M') = M$. Consequently, for every open neighbourhood $U' \subset L/P$ of $\pi(z_c)$ such that $\pi^{-1}(U') \cong U' \times F$, we have $\pi^{-1}(U' \cap M') \cong (U' \cap M') \times F$, i.e., M is locally CR-equivalent to a product as claimed above. In particular, M is then holomorphically degenerate. □

For instance, let $L := \mathrm{SL}_5(\mathbb{C})$, let P and Q be given in terms of matrices as follows

$$Q := \left\{ \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \right\}, \quad P := \left\{ \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \right\}. \text{ Define } R_5 := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

and let $\sigma : \mathfrak{sl}_5(\mathbb{C}) \to \mathfrak{sl}_5(\mathbb{C})$, $X \mapsto R_5(-\overline{X}^t)R_5$ be the antilinear involution, i.e., $\mathfrak{g} = \mathfrak{sl}_5(\mathbb{C})^\sigma \cong \mathfrak{su}(2,3)$ and $G = SU(2,3)$. It is then clear that $\mathfrak{p} \subset \mathfrak{q} + \sigma\mathfrak{q}$. In this particular example $Z := L/Q \cong \mathbb{F}_{3,4}(\mathbb{C}^5)$, $Z' := L/P \cong \mathbb{F}_3(\mathbb{C})$, where $\mathbb{F}_{p_1,..,p_k}(\mathbb{C}^n)$ stands for the manifold of $(p_1,\dots,p_k)$-flags $E_{p_1} \subset \dots \subset E_{p_k}$ in $\mathbb{C}^n$. Let z_c be the point $1{\cdot}Q \in L/Q$. Using the characterization of closed orbits in flag manifolds given in [21], a simple check shows that $M := G{\cdot}z_c$ is closed. The associated CR-algebra is $(\mathfrak{g}, \mathfrak{q})$. The corresponding holomorphic fibre bundle map $\pi : L/Q \to L/P$ is then simply $\pi : \mathbb{F}_{3,4}(\mathbb{C}^5) \to \mathbb{F}_3(\mathbb{C}^5)$, $(E_3, E_4) \mapsto E_3$. The fibres are isomorphic to $\mathbb{P}_1$. As $\mathfrak{su}(2,3)$ is simple, the CR-algebra associated with any $\mathrm{SU}(2,3)$–homogeneous CR-manifold, in particular $(\mathfrak{su}(2,3), \mathfrak{q})$ is 'ideal nondegenerate'.

Minimality in terms of CR-algebras. Recall that a CR-manifold $(M, \mathbb{H}, J)$ is called *minimal* at $o \in M$ if for each locally closed submanifold $Y \subset M$ such that $o \in Y$ and $\mathbb{H}_y \subset T_yY$ for all $y \in Y$ the identity $T_oY = T_oM$ holds, i.e., $[M, o] = [Y, o]$. In the locally homogeneous situation the property of being minimal at one particular point is equivalent to the minimality at all points of M. As before, $\mathfrak{H} = (\mathfrak{q} + \sigma\mathfrak{q})^\sigma \subset \mathfrak{g}$ and $\mathbb{H}_o \cong \mathfrak{H}/\mathfrak{g}_o$.

Theorem 2. *Given a CR-algebra* $(\mathfrak{g}, \mathfrak{q})$, *let* (M, o) *be the underlying CR-germ. Then* M *is minimal at* o *if and only if the smallest subalgebra of* $\mathfrak{g}$, *which contains* $\mathfrak{H}$, *is* $\mathfrak{g}$ *itself.*

Proof. The minimality condition can be reformulated as follows: Define inductively the following ascending chain of subbundles (associated with the locally homogeneous CR-manifold M):

$$\mathbb{H}^{(0)} := \mathbb{H}, \quad \mathbb{H}^{(j+1)} := \mathbb{H}^{(j)} + [\mathbb{H}^{(j)}, \mathbb{H}^{(j)}] \quad \text{for } j \geq 0 .$$

Here, $[\mathbb{H}^{(\ell)}, \mathbb{H}^{(\ell)}]$ stands for the subbundle generated by all brackets $[\xi, \eta]$, where ξ, η run through local sections in $\mathbb{H}^{(\ell)}$. The minimality of M is equivalent to the condition $\bigcup_{k\geq 0} \mathbb{H}^{(k)} = TM$. In our situation all subbundles $\mathbb{H}^{(k)}$ are homogeneous; hence, they are completely determined by the fibres at $o \in M$. Let $\mathfrak{H}^{(k)} \subset \mathfrak{g}$ denote the subspaces containing $\mathfrak{g}_o$ such that $\mathbb{H}_o^{(k)} = \mathfrak{H}^{(k)}/\mathfrak{g}_o$ for all k. Observe that the map

$$\Gamma(M, \mathbb{H}^{(k)}) \times \Gamma(M, \mathbb{H}^{(k)}) \longrightarrow \Gamma(M, \mathbb{H}^{(k+1)})/\Gamma(M, \mathbb{H}^{(k+1)}),$$

given by the Lie brackets is $C^\infty(M)$–bilinear. Consequently, we can employ the Main Lemma 1: The corresponding tensor $\mathbb{H}_o^{(k)} \times \mathbb{H}_o^{(k)} \to \mathbb{H}_o^{(k+1)}/\mathbb{H}_o^{(k)}$ is simply given by the Lie bracket in $\mathfrak{g}$. This yields an inductive definition of all $\mathfrak{H}^{(k)}$: The subspace $\mathfrak{H}^{(k+1)}$ is generated by elements $u \in \mathfrak{H}^{(k)}$ and all Lie brackets $[u, v]_\mathfrak{g}$, $u, v \in \mathfrak{H}^{(k)}$. If the smallest Lie algebra in $\mathfrak{g}$ which contains $\mathfrak{H}^{(0)}$, coincides with $\mathfrak{g}$ then $\bigcup_{k\geq 0} \mathfrak{H}^{(k)} = \mathfrak{g}$

and consequently $\bigcup_{k\geq 0}\mathbb{H}^{(k)} = TM$, i.e., M is minimal. The opposite direction, i.e., "M minimal implies $\mathfrak{g}$ is the smallest subalgebra containing $\mathfrak{H}$" is easier to see: The existence of a proper subalgebra of $\mathfrak{g}$ which contains $\mathfrak{H}$, would imply the existence of an integral manifold (Nagano leaf) through o, strictly lower-dimensional than M. But this contradicts the minimality of M. □

6. Orbits in complex flag manifolds: General facts

For the remainder of this paper, we concentrate our study on homogeneous CR-manifolds which arise as orbits in complex flag manifolds. In order to formulate our main results, we first need to fix our notation. We also collect results, concerning flag manifolds Z and the combinatorial description of the various orbits $M \subset Z$ of certain real semisimple groups. This is the content of the present section. As a general reference for flag manifolds, see [14]; for the combinatorics of orbits in flag manifolds we refer to the seminal paper [21].

We start with a geometric definition. A complex *flag manifold* is a compact Kähler manifold Z which is homogeneous under some Lie group and subject to the topological condition $b_1(Z) = 0$. Particular examples of flag manifolds are projective spaces $\mathbb{P}_n$, Grassmannians $\mathbb{F}_k(\mathbb{C}^n)$ and quadrics $\boldsymbol{Q}_n$. The group acting transitively on Z can be further specified: It is known ([13]) that $\mathrm{Aut}(Z)$ is complex semisimple. As for any compact complex space $\mathrm{Aut}(Z)$ is a (finite dimensional) Lie group in the CO-topology, and we write $\mathfrak{aut}(Z)$ for its Lie algebra. Further properties, following from the above definition are: Z is projective (i.e., an algebraic subset of some $\mathbb{P}_N$) and rational, (i.e., Z contains a Zariski open subset $E \cong \mathbb{C}^n$). The isotropy subgroups of any complex semisimple Lie group L acting transitively on a flag manifold are parabolic. (There are instances with $L \subsetneq \mathrm{Aut}(Z)$.) This is a key point which makes it possible to describe flag manifolds by root-theoretical methods. Later on we will give more details concerning this fact.

We come now to the main objects of our interest: CR-submanifolds in flag manifolds, arising as orbits of certain group actions. Let L be an arbitrary complex semisimple Lie group, acting transitively on a flag manifold Z. Select a real form G of L and let $\sigma : \mathfrak{l} \to \mathfrak{l}$ be the conjugation with $\mathfrak{g} = \mathfrak{l}^\sigma$. The basic facts are: There are finitely many G–orbits in a given Z. In particular, open G–orbits exist. The closure of each orbit is a finite union of orbits of lower dimensions. For every flag manifold Z, L as above and a real form $G \subset L$ there is precisely one closed G–orbit $Y \subset Z$ (see [21]). As $G \times L/Q \to L/Q$ is an action by biholomorphic transformations, every orbit $M := G \cdot z \subset Z = L/Q$, $z \in Z$, is a CR-submanifold. Each such CR-manifold is generically embedded in Z but may be not minimal.

Flag manifolds and the orbits of real forms can be handled most effectively using the combinatorics of parabolic Lie subgroups and subalgebras as well as root theory. We recall here briefly the relevant facts. Let $Z = L/Q$ be a flag manifold. A *Borel subgroup* of a complex semisimple Lie group L is a maximal solvable connected subgroup $B \subset L$, and a *parabolic subgroup* Q is any proper subgroup of L which contain a Borel subgroup. All Borel subgroups are conjugate in L, and there are finitely many conjugacy classes of parabolic subgroups. Since the normalizer $N_L(\mathfrak{q}) = \{g \in L : \mathrm{Ad}_g(\mathfrak{q}) = \mathfrak{q}\}$ of the Lie algebra $\mathfrak{q}$ of a parabolic Q coincides with Q, elements of the quotient L/Q are in 1-to-1 correspondence with elements of the conjugacy class of Q. This means that a parabolic subgroup Q_z, conjugate to Q, determines uniquely a point $z \in L/Q$. Following a common convention (which takes into account the aforementioned correspondence) we write Q_z instead of L_z for the (parabolic) isotropy subgroup at a point z in $Z = L/Q$. All these isotropy subgroups are connected; consequently there is a bijection between the parabolic subgroups $Q_z \subset L$ and the parabolic Lie subalgebras $\mathfrak{q}_z \subset \mathfrak{l}$.

Let $G \subset L$ be a real form. Given a CR-manifold $M = G{\cdot}z \subset Z$, the CR-algebra, associated with the $\mathfrak{g}$–homogeneous CR-germ (M, z) is the pair $(\mathfrak{g}, \mathfrak{q}_z)$. In section 5 we have explained that the spaces $\mathfrak{q}_z \cap \sigma\mathfrak{q}_z$ $\mathfrak{q}_z + \sigma\mathfrak{q}_z$ as well as certain intermediate subalgebras $\mathfrak{q}_z^{(k)}$ between $\mathfrak{q}_z \cap \sigma\mathfrak{q}_z$ and $\mathfrak{q}_z$ play an important rôle in the description of the geometric properties of M. The technical key point here is that every complex subalgebra $\mathfrak{q}_z \cap \sigma\mathfrak{q}_z$ contains a σ–stable Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$ i.e., a maximal Abelian subalgebra, consisting of ad-semisimple elements. This leads to a root-theoretical description of all relevant subspaces and subalgebras. We briefly recall the basic facts and notions (see [8] as a general reference for root systems and related topics).

Every of the aforementioned complex subspaces, containing $\mathfrak{t}$, is the direct sum of $\mathfrak{t}$ and certain 1-dimensional root spaces $\mathfrak{l}_\alpha = \{v \in \mathfrak{l} : [x, v] = \alpha(x){\cdot}v \text{ for all } x \in \mathfrak{t}\}$, where $\alpha : \mathfrak{t} \to \mathbb{C}$ is a non-zero linear functional. In particular, there is the decomposition into root spaces: $\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{t}^* \setminus \{0\}} \mathfrak{l}_\alpha$. Every nonzero linear functional $\alpha \in \mathfrak{t}^*$ which nontrivially occurs in this decomposition is called a *root*. The set of all roots of $\mathfrak{l}$ (with respect to a Cartan subalgebra $\mathfrak{t}$) is denoted by $\Phi = \Phi(\mathfrak{t})$ and called the *root system*. There exists a real form $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{t}$ such that all roots $\alpha \in \Phi$ are $\mathbb{R}$–valued on $\mathfrak{t}_{\mathbb{R}}$, i.e., Φ can be considered as a finite subset of the real vector space $(\mathfrak{t}_{\mathbb{R}})^*$. By selecting a separating hyperplane in $\mathfrak{t}^*_{\mathbb{R}}$, for instance an element $x^{\mathrm{reg}} \in \mathfrak{t}_{\mathbb{R}}$ such that $\alpha(x^{\mathrm{reg}}) \neq 0$ for all $\alpha \in \Phi$, one obtains a decomposition $\Phi = \Phi^+ \cup \Phi^-$ with $\Phi^+ = \{\alpha \in \Phi : \alpha(x^{\mathrm{reg}}) > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \alpha(x^{\mathrm{reg}}) < 0\} = -\Phi^+$ in positive and negative roots. Of course, the selection of a positive root system Φ^+ depends on the choice of a separating hyperplane. A Lie-theoretical interpretation of the subset $\Phi^+ \subset \Phi(\mathfrak{t})$ is the following: $\mathfrak{b}(\Phi^+) := \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi^+} \mathfrak{l}_\beta$ is a Borel subalgebra. Further, the set of all positive subsystems in $\Phi(\mathfrak{t})$ is in 1-to-1 correspondence with Borel subalgebras $\mathfrak{b} \subset \mathfrak{l}$ which contain $\mathfrak{t} : \Phi^+ \leftrightsquigarrow \mathfrak{b}(\Phi^+)$. To underline the connection between $\mathrm{ad}(\mathfrak{t})$–stable subspaces $\mathfrak{p} \subset \mathfrak{l}$ and subsets of Φ we simply write $\Phi(\mathfrak{p}, \mathfrak{t}) \subset \Phi$ for the subset consisting of those roots which occur in the decomposition $\mathfrak{p} = \mathfrak{t} \cap \mathfrak{p} \ \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{p},\mathfrak{t})} \mathfrak{l}_\alpha$. Every positive system Φ^+ contains a unique root basis $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$, i.e., a vector basis of $\mathfrak{t}^*$ such that each root $\beta \in \Phi$ can be uniquely written as $\beta = \sum n_j \alpha_j$ with $n_j \in \mathbb{Z}$ and either all n_j are non-negative or all n_j are non-positive. The particular geometry of Π, considered as a subset of the Euclidean space $\mathfrak{t}^*_{\mathbb{R}}$, (with Killing form as the scalar product) is depicted by the Dynkin diagram, see [8], chap. VI, §4 for further details. Parabolic subalgebras can be characterized as follows: By definition, $\mathfrak{q}$ contains a Borel subalgebra $\mathfrak{b}$. Select a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q}$. As commonly agreed, one declares the set of roots occurring in the decomposition of $\mathfrak{b}$ as a *negative* root subsystem: $\Phi^- := \Phi(\mathfrak{b}, \mathfrak{t})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ be the corresponding root basis.

Then the parabolic subalgebra $\mathfrak{q}$ determines the set $Q^r := \Pi \cap \Phi(\mathfrak{q}, \mathfrak{t})$. A fundamental fact in this context is that there is a bijection between proper subsets of Π and parabolic subalgebras $\mathfrak{q}$ containing the fixed $\mathfrak{b} \supset \mathfrak{t}$. Each conjugacy class of parabolic subalgebras in $\mathfrak{l}$ contains precisely one of the above parabolics $\mathfrak{q} \supset \mathfrak{b} \supset \mathfrak{t}$. Let $\mathfrak{q}^{\mathrm{nil}}$ be the nilpotent radical of $\mathfrak{q}$. Then $\mathfrak{q}^{\mathrm{nil}}$ is a direct sum of root spaces: $\mathfrak{q}^{\mathrm{nil}} = \bigoplus_{\Phi^{-n}} \mathfrak{l}_\alpha$ where $\Phi^{-n} := \Phi(\mathfrak{q}^{\mathrm{nil}}, \mathfrak{t}) \subset \Phi^-$. Set $\Phi^r := \Phi(\mathfrak{q}, \mathfrak{t}) \setminus \Phi^{-n}$, $\Phi^n := \Phi \setminus \Phi(\mathfrak{q}, \mathfrak{t}) = -\Phi^{-n}$ and $\mathfrak{q}^{\pm n} := \bigoplus_{\Phi^{\pm n}} \mathfrak{l}_\alpha$. Then there are the following decompositions:

$$(1) \qquad \mathfrak{q} = \mathfrak{q}^r \oplus \mathfrak{q}^{-n} \quad \text{and} \quad \mathfrak{l} = \mathfrak{q}^n \oplus \mathfrak{q}^r \oplus \mathfrak{q}^{-n} \quad \text{with} \quad \mathfrak{q}^r := \mathfrak{t} \oplus \bigoplus_{\Phi^r} \mathfrak{l}_\beta.$$

Let $Q^{\pm n}, Q^r$ be the subgroups in L, corresponding to the Lie subalgebras $\mathfrak{q}^{\pm n}$ and $\mathfrak{q}^r$, respectively. We like to underline that given $\mathfrak{q}$ the above decompositions depend on the choice of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}$. Let $z \in Z = L/Q$ be an arbitrary base point and $\mathfrak{q}_z$ the corresponding isotropy Lie algebra. Select a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}_z$. Then $E := Q_z^n \cdot z \subset Z$ is a Zariski open subset, algebraically isomorphic to $\mathbb{C}^k$, to which we refer as to the *big cell* in Z, centered around z. Each such big cell provides a coordinate chart in Z.

Let a real form $G \subset L$ and the corresponding involution $\sigma : \mathfrak{l} \to \mathfrak{l}$ be given. Select an arbitrary base point $z \in Z$ and let $\mathfrak{q}_z$ be the corresponding parabolic isotropy subalgebra. A further key result is that $\mathfrak{q}_z$ contains a σ–stable Cartan subalgebra $\mathfrak{t}$. Hence, σ induces the automorphism $\sigma : \Phi(\mathfrak{t}) \to \Phi(\mathfrak{t})$, $\sigma(\alpha)\ (v) := \overline{\alpha(\sigma v)}$. The root subsets, determined by the subspaces $\mathfrak{q}_z \cap \sigma\mathfrak{q}_z \subset \mathfrak{q}_z^{(\ell)} \subset \mathfrak{q}_z + \sigma\mathfrak{q}_z$ can be completely determined only provided that the involution $\sigma : \Phi \to \Phi$ and the subset $\Phi(\mathfrak{q}_z, \mathfrak{t}) \subset \Phi$ are known explicitly. Below, this procedure will be exemplified on the particular example of a hypersurface orbit $\mathscr{M}$ in a flag manifold $Z = \mathrm{SO}(7, \mathbb{C})/Q$.

7. A 3-nondegenerate homogeneous CR-manifold

In this section we present an example of a homogeneous CR-manifold $\mathscr{M}$ which turns out to be 3-nondegenerate. This CR-manifold occurs as a real hypersurface in a 7-dimensional complex manifold Z. We first give an explicit description of $\mathscr{M}$ in purely geometric terms: Write $V := \mathbb{C}^7$ and select:

• A symmetric 2-form $b : V \times V \to \mathbb{C}$ which is nondegenerate, for example $b(\boldsymbol{z}, \boldsymbol{w}) := \sum_{j=1}^{7} z_j w_{8-j}$ where $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{C}^7$. It determines the special orthogonal group and the corresponding Lie algebra:

(1)

$$\mathrm{SO}(7, \mathbb{C}) = \mathrm{SO}(V, b) := \{A \in \mathrm{SL}(7, \mathbb{C}) : b(Av, Aw) = b(v, w) \text{ for all } v, w \in V\}$$
$$\mathfrak{so}(7, \mathbb{C}) = \mathfrak{so}(V, b) := \{X \in \mathfrak{sl}(7, \mathbb{C}) : b(Xv, w) + b(v, Xw) = 0 \text{ for all } v, w \in V\}$$

Here, $\mathrm{SL}(7, \mathbb{C}) = \{A \in \mathbb{C}^{7\times 7} : \det A = 1\}$, $\mathfrak{sl}(7, \mathbb{C}) = \{X \in \mathbb{C}^{7\times 7} : \operatorname{trace} X = 0\}$ and $\mathbb{C}^{p\times q}$ stands for the space of complex $p \times q$ matrices.

• Select a real form $G \subset L := \mathrm{SO}(7, \mathbb{C})$, isomorphic to $\mathrm{SO}(4, 3)^\circ$ as follows: Choose a totally real vector subspace $V^{\mathbb{R}} \cong \mathbb{R}^7$ in V such that the restriction $b|_{V^{\mathbb{R}}}$ is a real symmetric 2-form of signature (4,3). This is the case for $V^{\mathbb{R}} := \mathbb{R}^7 \subset \mathbb{C}^7 = V$ and b chosen as above. The corresponding involutive automorphism $\sigma : \mathrm{SO}(7, \mathbb{C}) \to \mathrm{SO}(7, \mathbb{C})$

is then simply the map $A \mapsto \overline{A}$. Extend $b|_{V^{\mathbb{R}}}$ to a Hermitian 2-form on V by $h^b(\boldsymbol{z},\boldsymbol{w}) := b(\boldsymbol{z},\overline{\boldsymbol{w}})$ (both have signature (4,3), i.e., 4 positive and 3 negative eigenvalues).

• The ambient complex manifold Z is the Grassmannian $\mathbb{F}_2^b(V)$, consisting of all 2-dimensional subspaces (2-planes) $F \subset V$ which are b–isotropic. $\mathbb{F}_2^b$ is a 7-dimensional flag manifold, homogeneous under $\mathrm{SO}(V,b)$, and can naturally be considered as a submanifold of the Grassmannian $\mathbb{F}_2(V)$ of all 2-planes in V. $\mathbb{F}_2(V)$ is a 10-dimensional flag manifold, homogeneous under $\mathrm{SL}(7,\mathbb{C})$. As already mentioned, flag manifolds are rational, in particular $\mathbb{F}_2^b$ admits a coordinate chart (big cell) $E \subset \mathbb{F}_2^b$ centered around an arbitrarily chosen point $z \in \mathbb{F}_2^b$. For example, let $e_1, \dots, e_7$ be the canonical basis of the vector spaces $\mathbb{C}^7 = V$. Then $\mathbb{C}e_6 \oplus \mathbb{C}e_7$ is a b–isotropic 2-plane, hence, a point in $Z = \mathbb{F}_2^b(\mathbb{C}^7)$ which we denote for the rest of this paper by z_o. The big cell around z_o is given explicitly as follows:

$$(\boldsymbol{z},\boldsymbol{w},u) \overset{\Psi}{\longmapsto} \mathbb{C}\begin{pmatrix} u - c(\boldsymbol{z},\boldsymbol{w}) \\ -c(\boldsymbol{z},\boldsymbol{z}) \\ z_3 \\ z_2 \\ z_1 \\ 1 \\ 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} -c(\boldsymbol{w},\boldsymbol{w}) \\ -u - c(\boldsymbol{z},\boldsymbol{w}) \\ w_3 \\ w_2 \\ w_1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{F}_2^b(\mathbb{C}^7)\ . \tag{2}$$

Here, $\boldsymbol{z},\boldsymbol{w} \in \mathbb{C}^3$, $u \in \mathbb{C}$ and $c(\boldsymbol{z},\boldsymbol{w}) := \frac{1}{2}(z_1w_3 + z_2w_2 + z_3w_1)$.

• Finally, consider the set $\mathscr{H} \subset \mathbb{F}_2^b(V)$ of all those planes $F \in \mathbb{F}_2^b(V)$ such that $h^b|_F$ is degenerate. This is a (singular) 13-dimensional real hypersurface in Z which decomposes into finitely many G–orbits. From the mere definition of $\mathscr{H}$ follows that the intersection $\mathscr{H} \cap \Psi(\mathbb{C}^7)$ is the zero set of the following polynomial function $\rho : \mathbb{C}^7 \to \mathbb{R}$ of degree 4 (the 2×2 matrix below represents $b|_{\Psi(\boldsymbol{z},\boldsymbol{w},u)}$)

$$\rho = \det\begin{pmatrix} c(\boldsymbol{w},\boldsymbol{w}) - 2c(\overline{\boldsymbol{w}},\boldsymbol{w}) + c(\overline{\boldsymbol{w}},\overline{\boldsymbol{w}}) & c(\boldsymbol{z},\boldsymbol{w}) - 2c(\overline{\boldsymbol{z}},\boldsymbol{w}) + c(\overline{\boldsymbol{z}},\overline{\boldsymbol{w}}) + i\Im u \\ c(\boldsymbol{z},\boldsymbol{w}) - 2c(\boldsymbol{z},\overline{\boldsymbol{w}}) + c(\overline{\boldsymbol{z}},\overline{\boldsymbol{w}}) - i\Im u & c(\boldsymbol{z},\boldsymbol{z}) - 2c(\overline{\boldsymbol{z}},\boldsymbol{z}) + c(\overline{\boldsymbol{z}},\overline{\boldsymbol{z}}) \end{pmatrix}\ . \tag{3}$$

For example, $\mathscr{H}$ contains the closed and totally real G–orbit $\mathbb{F}_2^b(\mathbb{R}^7) \cong \mathscr{Y} = G \cdot z_0$ ($\dim_{\mathbb{R}} \mathscr{Y} = 7$), one orbit $\mathscr{M}$ with $\dim_{\mathbb{R}} \mathscr{M} = 13$ and also further orbits of intermediate dimensions. A base point $z_{\mathscr{M}} \in \mathscr{M}$ is, for instance, the 2–plane

$$F_{\mathscr{M}} := \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 1 \\ 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -i\sqrt{2} \\ 0 \\ 0 \\ 1 \end{pmatrix}\ .$$

It corresponds to the point $(-i,0,0\ ,\ 0,-i\sqrt{2},0\ ,\ 0) \in \mathbb{C}^7$ in the coordinate system 2. $h^b|_{F_{\mathscr{M}}}$ has one positive and one 0 eigenvalue. It should be noted, however, that in contrary to the classical Grassmannian $\mathbb{F}_2(V)$, where an orbit $\mathrm{SU}(4,3)\cdot F$ is completely characterized by the signature of $h^b|_F$, the set of all 2-planes $F \in \mathbb{F}_2^b(V)$ such that $h^b|_F$ has a fixed signature is in general a union of *several* G–orbits (this is, for instance, the case for all those F's with one positive and one 0 eigenvalue). A straightforward check shows that ρ is a defining function for $\mathscr{M}$ at $z_{\mathscr{M}}$, i.e., $d\rho(z_{\mathscr{M}}) \neq 0$. Once a defining function is given, one could use the original definition of k–nondegeneracy (see Prop. 11.2.4 in [4]) to verify that $\mathscr{M}$ is 3-nondegenerate (at $z_{\mathscr{M}}$ hence, by homogeneity, by each point of $\mathscr{M}$).

Instead of a direct examination of this equation which is a well-known procedure we rather give a description of the corresponding CR-algebra and use Theorem 1 to compute the order of nondegeneracy of $\mathscr{M}$. This method can actually be generalized to arbitrary orbits of real forms in general flag manifolds.

A root theoretical description. Our first task is to identify the conjugacy class of the parabolic isotropy subalgebra $\mathfrak{q}_{z_{\mathscr{M}}} \subset \mathfrak{l} \cong \mathfrak{so}(7,\mathbb{C})$ for $Z = \mathbb{F}_2^b$ in terms of root subsystems, or equivalently, to identify the corresponding subset of $\mathcal{Q}^r = \Pi \cap \Phi(\mathfrak{q}_{z_{\mathscr{M}}}, \mathfrak{t})$ of a root basis $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ of $\mathfrak{l} = \mathfrak{so}(7,\mathbb{C})$. Since it depends on the ambient flag manifold Z only, and not on a particular choice of a base point and a Cartan subalgebra contained in the corresponding isotropy subalgebra, we carry out our computations at the point $z_o = \mathbb{C}e_6 \oplus \mathbb{C}e_7 \in \mathbb{F}_2^b(\mathbb{C}^7)$ in the closed orbit $\mathscr{Y}$. In terms of matrix subalgebras of $\mathfrak{so}(7,\mathbb{C}) \subset \mathbb{C}^{7\times 7}$, a σ–stable CSA $\mathfrak{a}$ and the complex parabolic isotropy subalgebra $\mathfrak{q}_{z_o}$ have the following shape:

(4)
$$\mathfrak{a} = \mathfrak{a}_{a_1,a_2,a_3} = \left\{\begin{pmatrix} a_1 &&&&&& \\ & a_2 &&&&& \\ && a_3 &&&& \\ &&& 0 &&& \\ &&&& -a_3 && \\ &&&&& -a_2 & \\ &&&&&& -a_1 \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_2 & y_1 & y_2 & 0 & 0 & 0 \\ v_3 & v_4 & y_3 & 0 & -y_2 & 0 & 0 \\ v_5 & v_6 & 0 & -y_3 & -y_1 & 0 & 0 \\ z & 0 & -v_6 & -v_4 & -v_2 & -x_4 & -x_2 \\ 0 & -z & -v_5 & -v_3 & -v_1 & -x_3 & -x_1 \end{pmatrix}\right\} = \mathfrak{q}_{z_o}$$

For the positive system $\Phi^+ \subset \Phi(\mathfrak{a})$ corresponding to the Borel subalgebra consisting of upper-triangular matrices in $\mathfrak{so}(7,\mathbb{C})$, the root basis Π of Φ^+ consists of the linear functionals $\alpha_1 := a_1 - a_2$, $\alpha_2 := a_2 - a_3$ and $\alpha_3 := a_3$ (we consider the a_j's as coordinate functions on $\mathfrak{a}$). A glance at the above matrices shows that $\Phi(\mathfrak{q}_{z_o}, \mathfrak{a}) \cap \Pi = \{\alpha_1, \alpha_3\}$ which are the two endpoints of the Dynkin diagram $\circ\!\!-\!\!\!-\!\!\circ\!\!\Rightarrow\!\!\circ$. The induced involution $\sigma : \Phi \to \Phi$ is the identity. Consequently $\mathfrak{q}_{z_o} + \sigma\mathfrak{q}_{z_o} = \mathfrak{q}_{z_o}$ in accordance with the fact that the orbit $G{\cdot}z_o$ is totally real.

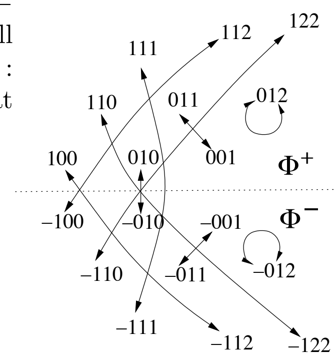
A bit more involved is the determination of the corresponding objects at the point $z_{\mathscr{M}}$. For, select the element

$$u := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{2}i & 0 & 0 & 1 \\ 0 & \sqrt{2} & \sqrt{2}i & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}i & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}i \\ 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2}i & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2}i & \sqrt{2} & 0 \\ 1 & 0 & 0 & -\sqrt{2}i & 0 & 0 & 1 \end{pmatrix} \in L,$$

observe that $u{\cdot}z_o = z_{\mathscr{M}}$, $\sigma(u) = u^{-1}$ and $u^8 = \mathrm{Id}$. Hence, $\mathfrak{t} := \mathrm{Ad}_u(\mathfrak{a})$ is a σ–stable Cartan subalgebra in $\mathfrak{q}_{z_{\mathscr{M}}} = \mathrm{Ad}_u(\mathfrak{q}_{z_o})$. Furthermore, $\mathrm{Ad}^*_{u^{-1}}$ yields a bijection

$\Phi(\mathfrak{a}) \to \Phi(\mathfrak{t})$. With respect to that identification, let $\alpha_1', \alpha_2', \alpha_3' \in \Phi(\mathfrak{t})$ be the images of the simple roots $\alpha_1, \alpha_2, \alpha_3 \in \Pi(\mathfrak{a}) \subset \Phi(\mathfrak{a})$. The knowledge of $\sigma(\alpha_1')$, $\sigma(\alpha_2')$ and $\sigma(\alpha_3')$ completely determines $\sigma : \Phi(\mathfrak{t}) \to \Phi(\mathfrak{t})$. This involutive automorphism is depicted in the figure below. The subspaces $\mathfrak{q}_{z_\mathscr{M}} \cap \sigma\mathfrak{q}_{z_\mathscr{M}}$, $\mathfrak{q}_{z_\mathscr{M}}^{(k)}$ and $\mathfrak{q}_{z_\mathscr{M}} + \sigma\mathfrak{q}_{z_\mathscr{M}}$ are determined by the corresponding subsets $\Phi(\mathfrak{q}_{z_\mathscr{M}} \cap \sigma\mathfrak{q}_{z_\mathscr{M}}, \mathfrak{t})$, $\Phi(\mathfrak{q}_{z_\mathscr{M}}^{(k)}, \mathfrak{t})$, etc. The latter sets of roots can be directly read off the diagram. For short, the digits stand for the coefficients in the expression of a root β with respect to the basis $\alpha_1', \alpha_2', \alpha_3'$. For instance, "$-012$" stands for $-\alpha_2' - 2\alpha_3'$ and $\mathfrak{l}_{-012} := \mathfrak{l}_{-\alpha_2'-2\alpha_3'}$. The arcs connect all pairs $\beta, \sigma(\beta)$. Hence, they completely determine $\sigma : \Phi(\mathfrak{t}) \to \Phi(\mathfrak{t})$. A glance at that diagram shows that

$$\begin{aligned}
\mathfrak{q}^{(\infty)} = \mathfrak{q}^{(3)} &= \mathfrak{t} \oplus \mathfrak{l}_{100} \oplus \mathfrak{l}_{-112} \oplus \mathfrak{l}_{-001} \oplus \mathfrak{l}_{-011} \oplus \mathfrak{l}_{-012} \\
\mathfrak{q}^{(2)} &= \mathfrak{q}^{(3)} \oplus \mathfrak{l}_{-122} \\
\mathfrak{q}^{(1)} &= \mathfrak{q}^{(2)} \oplus \mathfrak{l}_{-010} \oplus \mathfrak{l}_{-111} \\
\mathfrak{q} = \mathfrak{q}^{(0)} &= \mathfrak{q}^{(1)} \oplus \mathfrak{l}_{-100} \oplus \mathfrak{l}_{-110} \oplus \mathfrak{l}_{001} \quad .
\end{aligned}$$



This shows that $\mathscr{M}$ is a 3-nondegenerate homogeneous CR-manifolds. Furthermore, since $\mathfrak{q}_{z_\mathscr{M}}$ is a maximal parabolic subalgebra, Theorem 2 implies immediately that $\mathscr{M}$ is minimal.

Remark. The above choice of u is not accidental. In the context of partial Cayley transforms, u is the element $u_{\alpha_2} u_{\alpha_1+\alpha_2+\alpha_3}$ in L with respect to the strongly orthogonal subset $\{\alpha_2, \alpha_1 + \alpha_2 + \alpha_3\} \subset \Phi(\mathfrak{a})$. (A subset $\{\beta_1, \ldots, \beta_r\} \subset \Phi(\mathfrak{a})$ is called strongly orthogonal if for all $i \neq j$ with $1 \leq i, j \leq r$ neither $\beta_i + \beta_j$ nor $\beta_i - \beta_j$ is an element in $\Phi(\mathfrak{a})$; further $u_{\beta_j} := \exp \frac{\pi i}{4}(e_{\beta_j} + e_{-\beta_j})$ for certain appropriately chosen $e_{\beta_j} \in \mathfrak{g}_{\beta_j}$ and $e_{-\beta_j} \in \mathfrak{g}_{-\beta_j}$.) In such a way, with $z_o \in \mathscr{Y} \subset Z$ as the starting point, base points of all G turn, induced involutions $\sigma : \Phi(\mathfrak{t}) \to \Phi(\mathfrak{t})$ where $\mathfrak{t}$ runs through σ–stable Cartan subalgebras in the various subalgebras $\mathfrak{q}_y$, can also be explicitely characterized involving only information encoded in the corresponding partial Cayley transforms. We will explain this in greater details in a forthcoming paper.

8. Degeneracy of general orbits in flag manifolds

The hypersurface G-orbit $\mathscr{M}$, defined in the previous section, is a particular example of a finitely nondegenerate CR-manifold. One would expect that there exist G-orbits in flag manifolds which are finitely nondegenerate of arbitrary high order, provided the flag manifold is general enough. Surprisingly, at least for hypersurface orbits, this is not true as Theorem 3 shows. If L/Q is a flag manifold such that Q is maximal parabolic, then an upper bound restricting the nondegeneracy order is valid even for all non-open orbits.

We need some preparations. A *maximal* (proper) parabolic subalgebra $\mathfrak{q}$ with $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{l}$ ($\mathfrak{t}$ is a Cartan and $\mathfrak{b}$ a Borel subalgebra) determines a simple root

$\alpha_q \in \Pi = \{\alpha_1, ..., \alpha_r\}$ such that $\Phi(\mathfrak{t}, \mathfrak{q}) \cap \Pi = \Pi \setminus \{\alpha_q\}$. Let then $\hat{c}(\mathfrak{q}) := \max\{c_q(\gamma) : \gamma \in \Phi^+\}$, where $c_q := c_q(\gamma)$ is the q^{th} coefficient in the expression $\gamma = c_1(\gamma)\alpha_1 + \cdots + c_q(\gamma)\alpha_q + \cdots + c_r(\gamma)\alpha_r$. It is well-known ([13]) that $Q \subset L$ is maximal parabolic if and only if $b_2(L/Q) = 1$. The Hermitian symmetric spaces (see [12], Chap. VIII) form a proper subset of such flag manifolds. For example $\hat{c}(\mathfrak{q}) = 1$ if $Z = L/Q$ is a Hermitian symmetric space with $\mathrm{Aut}(Z)^\circ = L$ (see [20]).

If $L = \prod L_j$ is a direct product of simple complex Lie groups, $G_j \subset L_j$ arbitrary real forms and $G = \prod G_j$ then the corresponding G–orbit M in $Z = L/Q = \prod L_j/Q_j$ is also a direct product $M = \prod M_j$ (in the category of CR-manifolds). We then restrict our considerations to the flag manifolds L/Q with L simple:

Theorem 3. *Let $Z := L/Q$ be an arbitrary flag manifold where L is a simple complex group and $G \subset L$ a real form. Let $M := G{\cdot}z$ be an orbit in Z.*

- (i) *Assume that M is a real hypersurface in Z. Then M is holomorphically nondegenerate if and only if Q is a maximal parabolic subgroup of L.*
- (ii) *Assume that $b_2(Z) = 1$, i.e., $\mathfrak{q}$ is maximal parabolic, and M is not open in Z. Let $k(M)$ denote the order of nondegeneracy of M. Then $k(M) \le \hat{c}(\mathfrak{q}) + 1 \le 7$ (with $\hat{c}(\mathfrak{q})$ as defined above). In particular, $k(M) \le 3$ if L is not an exceptional simple group.*

Proof. We may assume that $\mathfrak{q} := \mathfrak{q}_z$ is the complex isotropy Lie subalgebra at $z \in Z$, a base point in the given homogeneous CR-manifold $M = G{\cdot}z \subset L/Q = Z$.
If $\mathfrak{q}$ is maximal, then every subalgebra $\widehat{\mathfrak{q}} \supset \mathfrak{q}$ with $\widehat{\mathfrak{q}} \subset \mathfrak{q} + \sigma\mathfrak{q} \ne \mathfrak{l}$ obviously satisfies $\widehat{\mathfrak{q}} = \mathfrak{q}$. Due to 1.ii, it follows that M is holomorphically nondegenerate[3].
To prove the "only if" part, select $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q} = \mathfrak{q}_z$ with a σ–stable Cartan subalgebra $\mathfrak{t}$. The assumption $\mathrm{codim}_Z(M) = 1$ implies that $\mathfrak{q} + \sigma\mathfrak{q}$ is a hyperplane in $\mathfrak{l}$ (compare 2), i.e., there exists $\gamma \in \Phi(\mathfrak{t})$ with $\sigma(\gamma) = \gamma$ such that $\mathfrak{l} = (\mathfrak{q} + \sigma\mathfrak{q}) \oplus \mathfrak{l}_\gamma$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the simple roots in $\Phi^+ = -\Phi(\mathfrak{b}, \mathfrak{t})$, $\gamma = \sum n_j\alpha_j$ and $\mathcal{Q}^r := \Phi(\mathfrak{q}, \mathfrak{t}) \cap \Pi$. Write $\mathrm{supp}_\Pi(\gamma) := \{\alpha_j \in \Pi : n_j > 0\}$. Select an element $\hat{\beta} \in \mathrm{supp}_\Pi(\gamma) \setminus \mathcal{Q}^r$ (this is possible since $\mathrm{supp}_\Pi(\gamma) \not\subset \mathcal{Q}^r$). Then $\Pi \setminus \{\hat{\beta}\} \supset \mathcal{Q}^r$ and consequently the maximal parabolic subalgebra $\widehat{\mathfrak{q}}$ with $\Phi(\widehat{\mathfrak{q}}, \mathfrak{t}) \cap \Pi = \Pi \setminus \{\hat{\beta}\}$, contains $\mathfrak{q}$ and satisfies the inclusions $\mathfrak{q} \subset \widehat{\mathfrak{q}} \subset \mathfrak{q} + \sigma\mathfrak{q}$. If $\widehat{\mathfrak{q}} \supsetneq \mathfrak{q}$, i.e., $\mathfrak{q}$ is not maximal parabolic, the corresponding orbit $G \cdot z \subset Z$ is holomorphically degenerate, due to Theorem 1.ii. The proof of part (i) is now complete.
To prove part (ii), let as before $\mathfrak{q} = \mathfrak{q}_z$ be the complex isotropy subalgebra at $z \in M \subset Z$. Select $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q}$. Then $\mathfrak{l} = (\mathfrak{q} + \sigma\mathfrak{q}) \oplus \mathfrak{l}_\Gamma$ where $\mathfrak{l}_\Gamma := \bigoplus_{\gamma\in\Gamma} \mathfrak{l}_\gamma$ for a certain subset $\Gamma \subset \Phi$. For instance, $|\Gamma|$ is the CR-codimension of M in Z. Let $\alpha_q \in \Pi$ be the simple root such that $\Phi(\mathfrak{q}, \mathfrak{t}) \cap \Pi = \Pi \setminus \{\alpha_q\}$. Note that then $\alpha_q \in \mathrm{supp}_\Pi(\gamma)$ for every $\gamma \in \Gamma$. Further, the simple root α_q determines the following $\mathbb{Z}$–filtration $\bigoplus_{-\infty}^{\infty} \mathfrak{l}_j$, where the homogeneous parts are given by

$$\mathfrak{l}_j := \begin{cases} \bigoplus_{c_q(\beta)=j} \mathfrak{l}_\beta & \text{for } j \ne 0 \quad (\beta = c_1(\beta)\alpha_1 + \cdots + c_q(\beta)\alpha_q + \cdots + c_r(\beta)\alpha_r) \\ \mathfrak{q}^r & \text{for } j = 0 \quad \text{(see 1 for the definition of } \mathfrak{q}^r) \end{cases}$$

[3]This argument remains valid for an arbitrary non-open orbit M in L/Q, Q being maximal parabolic: Consequently, every non-open orbit in L/Q is holomorphically nondegenerate; compare 4.i.

and $\mathfrak{l}_j = 0$ for $|j| > c := \hat{c}(\mathfrak{q})$. We have then $\mathfrak{q} = \bigoplus_{j=-c}^{0} \mathfrak{l}_j, \quad \mathfrak{q}^{-n} = \bigoplus_{j=-c}^{-1} \mathfrak{l}_j$. For short, write

$$\mathfrak{q}^{n\setminus\Gamma} := \bigoplus_{\substack{c_q(\beta)>0\\ \beta\notin\Gamma}} \mathfrak{l}_\beta \qquad \subset \ \mathfrak{q}^n$$

and note that $\sigma\mathfrak{q} = \mathfrak{q}^{n\setminus\Gamma} \oplus (\mathfrak{q} \cap \sigma\mathfrak{q}) = \mathfrak{q}^{n\setminus\Gamma} \oplus \mathfrak{q}^{(\infty)}$. Let $\mathfrak{q} = \mathfrak{q}^{(0)} \supset \mathfrak{q}^{(1)} \supset \cdots \supset \mathfrak{q}^{(\infty)}$ be the filtration as defined in 6. For a fixed $m = 0, 1, 2...$ let $\mathfrak{q}_{-c}^{(m)} \oplus \cdots \oplus \mathfrak{q}_{-1}^{(m)} \oplus \mathfrak{q}_0^{(m)}$ for the corresponding gradation of the $\mathfrak{q}^{(m)}$'s with $\mathfrak{q}_j^{(m)} = \mathfrak{q}^{(m)} \cap \mathfrak{l}_j$. Since $\mathfrak{t} \subset \mathfrak{q}^{(\infty)}$ and all subalgebras $\mathfrak{q}^{(m)}$ are ad$(\mathfrak{q}^{(\infty)})$–stable, the condition defining $\mathfrak{q}^{(m)}$ (see 6) is equivalent to

$$(\star) \quad \mathfrak{q}^{(m)} = \bigoplus \mathfrak{q}_j^{(m)} \quad \text{with} \quad \mathfrak{q}_j^{(m)} =$$
$$\{v \in \mathfrak{q}_j^{(m-1)} : [v\ , \mathfrak{q}^{n\setminus\Gamma}] \subset \mathfrak{q}_{j+1}^{(m-1)} \oplus \cdots \oplus \mathfrak{q}_0^{(m-1)} \oplus \mathfrak{q}^{n\setminus\Gamma}\}$$
$$\text{for all } m \geq 0,\ j \leq 0\ .$$

The key ingredient in the proof of statement (ii) is the following

Claim: For every $p \geq 0$ we have $\mathfrak{q}_{-p}^{(p+1)} = \mathfrak{q}_{-p}^{(p+2)} = \cdots = \mathfrak{q}_{-p}^{(\infty)}$.

We give an inductive proof. For $p = 0$, the condition $(\star)$ reads $\mathfrak{q}_0^{(j)} = \{v \in \mathfrak{q}_0^{(j-1)} : [v, \mathfrak{q}^{n\setminus\Gamma}] \subset \mathfrak{q}^{n\setminus\Gamma}\}$. Since the condition "$[v, \mathfrak{q}^{n\setminus\Gamma}] \subset \mathfrak{q}^{n\setminus\Gamma}$" does not depend on j, we conclude $\mathfrak{q}_0^{(1)} = \mathfrak{q}_0^{(2)} = \cdots = \mathfrak{q}_0^{(\infty)}$. Assume that the claim is already proved for $p = 0, 1, ..., m$. By induction hypothesis, for every $\ell \geq m+2$ and $v \in \mathfrak{q}_{-m-1}^{(\ell)}$ we have

$$\begin{aligned}[v, \mathfrak{q}^{n\setminus\Gamma}] \quad \subset \quad & \mathfrak{q}_{-m}^{(\ell-1)} \oplus \mathfrak{q}_{-m+1}^{(\ell-1)} \oplus \cdots \oplus \mathfrak{q}_0^{(\ell-1)} \oplus \mathfrak{q}^{n\setminus\Gamma} = \\ & = \mathfrak{q}_{-m}^{(m+1)} \oplus \mathfrak{q}_{-m+1}^{(m+1)} \oplus \cdots \oplus \mathfrak{q}_0^{(m+1)} \oplus \mathfrak{q}^{n\setminus\Gamma}\end{aligned}$$

and consequently the condition imposed on v is the same independently whether $v \in \mathfrak{q}_{-m-1}^{(m+2)}$ or $v \in \mathfrak{q}_{-m-1}^{(\ell)}$. This proves the claim. □

Due to the above claim, after at most $\hat{c}(\mathfrak{q}) + 1$ steps, the filtration $\mathfrak{q}^{(\bullet)}$ becomes stationary. Since $\mathfrak{q}$ is maximal in $\mathfrak{l}$, the non-open orbits $G{\cdot}z$ are finitely nondegenerate by Theorem 1, i.e., $\mathfrak{q}^{(\hat{c}(\mathfrak{q})+1)} = \mathfrak{q}^{(\infty)}$. This shows also that the order of degeneracy is at most $\hat{c}(\mathfrak{q}) + 1$.
The values of $\hat{c}(\mathfrak{q})$ for various $\mathfrak{q}$'s are bounded by the highest coefficient $C_\mathfrak{l}$ of the highest root in $\Phi(\mathfrak{l}, \mathfrak{t})$. A glance at the table of highest roots for the classical and exceptional simple Lie algebras $\mathfrak{l}$ yields $\hat{c}(\mathfrak{q}) \leq C_\mathfrak{l} \leq 2$ in the classical cases and $C_{\mathfrak{e}_6} = C_{\mathfrak{g}_2} = 3$, $C_{\mathfrak{e}_7} = C_{\mathfrak{f}_4} = 4$ and $C_{\mathfrak{e}_8} = 6$ in the exceptional cases (see [8]). This completes the proof of Theorem 3. □

Let G be a real form of L and $M = G{\cdot}z$ stand for a G-orbit in the flag manifold $Z = L/Q$.

Exercise. Modify the methods of the above proof and formulate the corresponding statements for simple real forms G of complex type, i.e., when the complexification $\mathfrak{l}$ of $\mathfrak{g}$ is a product of two simple complex Lie algebras.

Problem. Generalize the methods of the above proof and formulate the corresponding statements in the case of arbitrary holomorphically nondegenerate orbits in flag manifolds Z with non-maximal Q (i.e., $b_2(Z) \geq 2$)

Recall that $\mathscr{M}$ denotes the 13-dimensional CR-manifold, defined in the previous chapter and $\mathfrak{hol}(\mathscr{M}, z_{\mathscr{M}})$ is the Lie algebra of germs of all infinitesimal CR-transformations of $\mathscr{M}$ at $z_{\mathscr{M}}$ (see Section 3). By the finite nondegeneracy of $\mathscr{M}$ we have $\dim \mathfrak{hol}(\mathscr{M}, z_{\mathscr{M}}) < \infty$, and clearly $\mathfrak{so}(4,3) \subset \mathfrak{hol}(\mathscr{M}, z_{\mathscr{M}})$. We do not know, however, if this inclusion is proper.

Under certain circumstances, e.g., in case of the existence of so-called nonresonant vector fields on Z, more precisely, *for holomorphically nondegenerate and minimal CR-submanifolds $M \subset \mathbb{C}^k$ such that $\mathfrak{g} \subset \mathfrak{hol}(M, o)$ is reductive, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{l} \subset \mathfrak{hol}(\mathbb{C}^k, o)$ contains a nonresonant vector field and $\mathfrak{l}$ generates the tangent space $T_o\mathbb{C}^k$*, the following equality hold: $\mathfrak{g} = \mathfrak{hol}(M, o)$. This has been proved in [16], Prop. 3.8. However, these methods cannot be applied to $\mathscr{M}$ as $\mathfrak{l} = \mathfrak{so}(7, \mathbb{C})$ does not contain a vector field which is nonresonant on $\mathrm{SO}(7, \mathbb{C})/Q_{z_{\mathscr{M}}}$.

Recall that a holomorphic vector field η, locally defined around $0 \in \mathbb{C}^k$ with $\eta_0 = 0$ is called nonresonant if the set $\Lambda \subset \mathbb{C}$ of eigenvalues of the linear part η^{lin} of η (η^{lin} considered as linear map $\mathbb{C}^k \to \mathbb{C}^k$) fulfills the condition $\sum_{\lambda \in \Lambda} n_\lambda \lambda \notin \Lambda$ for every family of integers $n_\lambda \geq 0$ with $\sum n_\lambda \geq 2$. Note that the property of being (non)resonant can be defined for vector fields on general manifold as it does not depend on the choice of local coordinates centered at the base point. The aforementioned proposition from [16] has been applied to all orbits $M = G{\cdot}z$ in a Hermitian symmetric space *of tube type* $Z = L/Q$ which are neither open nor totally real. Later on, we will show that it also holds for all G–orbits of the above type in *arbitrary* Hermitian symmetric spaces. First, however, we clarify the existence of nonresonant globally defined vector fields in general flag manifolds:

Lemma 4. *Let $Z = L/Q$ be an arbitrary flag manifold and $o \in Z$ a base point. Then $\mathfrak{l} \subset \mathfrak{hol}(Z, o) \cong \mathfrak{hol}(\mathbb{C}^m, 0)$ contains a nonresonant vector field if and only if Z is Hermitian and $\mathfrak{l}$ coincides with the Lie algebra $\mathfrak{aut}(Z)$ of* $\mathrm{Aut}(Z)$*. In the Hermitian case there always exists a vector field δ in $\mathfrak{aut}(Z)_o \subset \mathfrak{hol}(\mathbb{C}^m, 0)$ with linear part equal to* Id *(hence, δ is a particular example of a nonresonant vector field).*

Proof. Let $Q = Q_o$, Q^n and $\mathfrak{q}^{\pm n}, \mathfrak{q}^r$, $\Phi^{\pm n}, \Phi^r$ be as in the paragraph around 1 (with respect to some well-chosen Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}$ which will be specified later). For $v \in \mathfrak{l}$ let ξ^v be the corresponding fundamental vector field on Z (compare 3). It vanishes at $o \in Z$ if and only if $v \in \mathfrak{q}$. Recall that $\mathbb{C}^m \cong Q^n{\cdot}o \subset Z$ is a big cell in Z, centered at o. A straightforward computation shows that the linear part of $\xi^v|_{Q^n \cdot o}$, considered as a map $T_oZ \to T_oZ$ ($T_oZ = \mathfrak{l}/\mathfrak{q}$) coincides with $-\mathrm{ad}_v : \mathfrak{l}/\mathfrak{q} \to \mathfrak{l}/\mathfrak{q}$. We have to examine the eigenvalues of ad_v with various $v \in \mathfrak{q}$: Without loss of generality we may assume that $-\mathrm{ad}_v$ is semisimple: Indeed the nilpotent part of the Jordan decomposition of ad_v obviously plays no rôle; furthermore, the semisimple part of

$\mathrm{ad}_v : \mathfrak{l} \to \mathfrak{l}$, $v \in \mathfrak{q}$ is equal to ad_{v^s} for some ad-semisimple element v^s in $\mathfrak{q}$. The ad-semisimple element $v = v^s$ is contained in some Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}$. As declared at the beginning of the proof, we consider the decompositions $\mathfrak{q} = \mathfrak{q}^r \oplus \mathfrak{q}^{-n}$ and $\Phi(\mathfrak{t}) = \Phi^n \cup \Phi^r \cup \Phi^{-n}$ with respect to such $\mathfrak{t}$. In particular, $\mathfrak{t} \subset \mathfrak{q}^r$. Select now a Borel subalgebra $\mathfrak{b}$ with $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q}$ and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the simple roots of $\Phi^+ := -\Phi(\mathfrak{b}, \mathfrak{t})$. As a $\mathfrak{t}$–module, $\mathfrak{l}/\mathfrak{q}$ is isomorphic to the direct sum $\bigoplus_{\beta \in \Phi^n(\mathfrak{t})} \mathfrak{l}_\beta$. We **claim** that if L/Q is *not* a Hermitian symmetric space with $\mathfrak{l} = \mathfrak{aut}(Z)$ then there exist $\alpha, \beta \in \Phi^n$ with $\alpha + \beta \in \Phi^n$ and consequently $\mathrm{ad}_v : \mathfrak{l}/\mathfrak{q} \to \mathfrak{l}/\mathfrak{q}$ contains $\alpha(v), \beta(v)$ and $\alpha(v) + \beta(v)$ in the set of its eigenvalues. Once this claim is proved, it follows that *none* of the vector fields ξ^v on Z is nonresonant.
It remains to prove the above claim: As shown in [20], a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{l}$ is exactly the complex isotropy subalgebra of a Hermitian symmetric space $Z = L/Q$ with $\mathfrak{l} = \mathfrak{aut}(Z)$ if the following holds: $\Pi \cap \Phi(\mathfrak{q}, \mathfrak{t}) = \Pi \setminus \{\alpha_q\}$ and the simple root α_q has the property

$$(\star\star) \qquad \gamma = \sum c_j(\gamma)\alpha_j \in \Phi(\mathfrak{t}) \implies c_q(\gamma) \in \{-1, 0, 1\}$$

Note that in the Hermitian-symmetric situation $\Phi^n = \{\gamma \in \Phi : c_q(\gamma) = 1\}$. In all other cases, that is, either $\Pi \cap \Phi(\mathfrak{q}, \mathfrak{t})$ has strictly less than $r - 1$ elements or $\{c_q(\gamma) : \gamma \in \Phi^n\}$ contains at least $\{1, 2\}$, by some standard properties of root systems (see, e.g., [8], Chap.VI, §1.6, Prop. 19 and Cor. 3) there exist $\alpha, \beta \in \Phi^n$ with $\alpha + \beta \in \Phi$ ($c_q(\alpha + \beta) \geq 2$) and the claim is proved.

For Hermitian symmetric spaces, the existence of vector fields ξ with $\xi^{\mathrm{lin}} = \mathrm{Id}$ (in a coordinate system given by a big cell) is well-known. □

Every bounded symmetric domain $D \subset \mathbb{C}^k$ can be realized as a G–orbit in a flag manifold $Z = Z(D) = G^{\mathbb{C}}/Q$ which is uniquely determined by D ($G = \mathrm{Aut}(D)^\circ$ is a semisimple real Lie group in the CO-topology). Furthermore, a big cell $E \subset Z(D)$ can be selected that $D \subset E$ is a bounded symmetric domain. Each Hermitian symmetric flag manifold $Z = L/Q$ arises that way. A bounded symmetric domain D (the corresponding Lie group $\mathrm{Aut}(D)^\circ$ and the corresponding flag manifold $Z(D)$ is called of *tube type* if there exists a big cell $E' \subset Z(D)$ and a real vector subspace $V \subset E'$ such that $E' = V \oplus iV$, $D \subset E'$ and $D = V + (iV \cap D)$, i.e., D is a tube over $iV \cap D$. As simplest example of this situation consider an open $\mathrm{SL}(2, \mathbb{R})$–orbit D in $\mathbb{P}_1 = Z(D)$; D can be realized 1) as the unit disc $\mathbb{B}^1$ (i.e., a bounded symmetric domain) in certain $E \subset \mathbb{P}_1$, or also 2) as the upper half-plane in certain $E' \subset \mathbb{P}_1$; both, E and E' are big cells in $\mathbb{P}_1$, algebraically biholomorphic to $\mathbb{C}$. Higher-dimensional Euclidean balls $\mathbb{B}^n \subset \mathbb{C}^n$ are not of tube type.

Motivated by a theorem of Tanaka on real hyperquadrics ([19]), in [16] pairs (M, Z) with similar properties have been considered. A pair (M, Z) is called of class $\mathfrak{C}$ if Z is a compact Hermitian symmetric space and M is an orbit of a real form G of $\mathrm{Aut}(Z)$ such that $\dim \mathfrak{hol}(M, a) < \infty$. Pairs (M, Z), (M', Z') of class $\mathfrak{C}$ have the remarkable extension property that every CR-isomorphism $\varphi : U \to U'$, where $U \subset M$ and $U' \subset M'$ are some open connected subsets in M and M' respectively, extends to a biholomorphic map $\hat{\varphi} : Z \to Z'$ with $\hat{\varphi}(M) = M'$. A key point here is to characterize those real forms and orbits M in compact Hermitian symmetric spaces such that (M, Z) is of class $\mathfrak{C}$. Using Jordan-theoretical methods it has been proved (Theorem

4.7 in [16]) that if G is a real form of *tube type* and Z the corresponding compact Hermitian symmetric spaces that all G–orbits in Z which are neither closed nor open are minimal 2-nondegenerate CR-manifolds and hence of class $\mathfrak{C}$. The aforementioned theorem from [16] can be generalized as follows:

Theorem 4. *Let L be a complex simple Lie group, $G \subset L$ an arbitrary real form and $Q \subset L$ a parabolic subgroup.*

(*i*) *Assume that Q is maximal parabolic. Then every non-open G–orbit M in $Z := L/Q$ is holomorphically nondegenerate. All such orbits are also minimal, except for the totally real ones.*

(*ii*) *In particular, if $Z = L/Q$ is an arbitrary irreducible Hermitian symmetric space and $G \subset \mathrm{Aut}(Z)^\circ = L$ an* arbitrary *real form then every G–orbit $M \subset Z$ which is not open is k–nondegenerate with $k \leq 2$. For every such orbit M, which in addition is not totally real, (M, Z) belong to the class $\mathfrak{C}$.*

(*iii*) *If Q is not maximal, then there always exist non-open holomorphically degenerate G–orbits in Z.*

Proof. Let $\sigma : \mathfrak{l} \to \mathfrak{l}$ be the involution given by the real form $G \subset L$. Let $\mathfrak{q}_z$ be the isotropy Lie algebra at a point $z \in Z = L/Q$, $M := G{\cdot}z$ the orbit with the inherited CR-structure such that $\mathfrak{q}_z + \sigma\mathfrak{q}_z \neq \mathfrak{l}$ (i.e., M is not open). Since by maximality of $\mathfrak{q}_z$ the only Lie algebra, properly containing $\mathfrak{q}_z$ is $\mathfrak{l}$ itself, there is no subalgebra $\widehat{\mathfrak{q}}$ with $\mathfrak{q}_z \subsetneq \widehat{\mathfrak{q}} \subset \mathfrak{q}_z + \sigma\mathfrak{q}_z$). Theorem 1 implies then the first part of (i). Assume now in addition that $\mathfrak{q}_z + \sigma\mathfrak{q}_z = \mathfrak{q}_z$ (i.e., M is not totally real). It is then clear that the smallest Lie algebra which contains $\mathfrak{q}_z + \sigma\mathfrak{q}_z \neq \mathfrak{q}_z$ coincides with $\mathfrak{l}$. In view of Theorem 2, this fact completes the proof of (i).

To prove (ii) let $\mathfrak{q}$ be a complex isotropy subalgebra of a compact Hermitian symmetric space $Z = L/Q$. The upper bound for the order of nondegeneracy k of an arbitrary non-open orbit M in Z is provided by Theorem 3 together with the well-known technical fact that $\hat{c}(\mathfrak{q}) = 1$ (as already explained in $(\star\star)$ in the proof of Lemma 4) [4].

If Q is not maximal, there exists a maximal parabolic Q', containing Q, such that G is not transitive on L/Q'. Consequently, the corresponding L–equivariant holomorphic map $\pi : {}^{L}\!/_{Q} \to {}^{L}\!/_{Q'} =: Z'$ has positive-dimensional complex fibres. Let $M' \subset Z'$ be an arbitrary G–orbit which is not open. Then $\pi^{-1}(M')$ consists of finitely many G–orbits. In particular there exists an orbit M which is open in $\pi^{-1}(M')$. The fibres of the restriction $\pi : M \to M'$ (which is a CR-map) are then complex manifolds and consequently M is locally equivalent to a product of a CR-manifold and a positive dimensional complex manifold. This proves the existence of a holomorphically degenerate orbit M. ☐

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