

A CRITICAL-EXPONENT BALIAN–LOW THEOREM

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ABSTRACT. Using a variant of the Sobolev Embedding Theorem, we prove an uncertainty principle related to Gabor systems that generalizes the Balian–Low Theorem. Namely, if $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{p'/2}(\mathbb{R})$ with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then the Gabor system $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$. In the $p = 1$ case, we obtain a generalization of the result in [BCPS].

1. Introduction

Given a function $f \in L^2(\mathbb{R})$ and positive constants α, β , the associated *Gabor system* is

$$\mathcal{G}(f, \alpha, \beta) := \{e^{2\pi i m \beta \cdot} f(\cdot - n\alpha)\}_{m, n \in \mathbb{Z}} \subset L^2(\mathbb{R}),$$

the collection of translates and modulates of f by the lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$. Gabor systems have proven useful in time-frequency analysis as means for generating orthonormal bases or “frames” for $L^2(\mathbb{R})$. A *frame* for a Hilbert space \mathcal{H} is a collection $\{e_n\} \subset \mathcal{H}$ for which one has the modified Parseval relation

$$(1.1) \quad A\|x\|_{\mathcal{H}}^2 \leq \sum_n |\langle x, e_n \rangle|^2 \leq B\|x\|_{\mathcal{H}}^2$$

for all $x \in \mathcal{H}$ and some *frame constants* $A, B > 0$; frames may be viewed as natural generalizations of orthonormal bases. We adopt the terminology “ (A, B) -frame” for a frame with frame constants A and B .

It is natural to consider under what conditions $\mathcal{G}(f, \alpha, \beta)$ generates a frame for $L^2(\mathbb{R})$; the classical Balian–Low Theorem is an instance of the uncertainty principle in this setting (see *e.g.* [Dau]):

Theorem 1.1 (Balian–Low–Coifman–Semmes). *Let $f \in L^2(\mathbb{R})$. If $f \in H^1(\mathbb{R})$ and $\hat{f} \in H^1(\mathbb{R})$, then $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$.¹*

Here $H^1(\mathbb{R})$ denotes the usual L^2 -Sobolev space; thus we see that if f is suitably well-localized in phase space, then it cannot generate a Gabor frame. In light of this result, it is reasonable to ask whether one can alter the regularity assumptions on f and \hat{f} to obtain a similar uncertainty principle.

To date, two significant results in this direction have suggested critical Sobolev regularity assumptions. The first, essentially due to Gröchenig [Grö], is:

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¹Known results show that $\alpha = \beta = 1$ are the “interesting” lattice constants in this setting; see *e.g.* [Dau].

Theorem 1.2. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. If $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{q/2}(\mathbb{R})$, then $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$.*

From the other direction, Benedetto *et al.* prove the following in [BCGP]:

Theorem 1.3. *Let $\frac{1}{p} + \frac{1}{q} > 1$. Then there exists a function $f \in L^2(\mathbb{R})$ such that $\mathcal{G}(f, 1, 1)$ is a frame (in fact an orthonormal basis) and such that $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{q/2}(\mathbb{R})$.*

(In fact, their result is stronger; it allows for stricter regularity conditions than inclusion in the appropriate Sobolev spaces.)

Given these results, it is natural to study the critical exponent case $\frac{1}{p} + \frac{1}{q} = 1$. In [BCPS], Benedetto *et al.* conjectured that in fact Theorem 1.2 can be extended to this range of exponents, and they proved the following “ $(1, \infty)$ endpoint” result:

Theorem 1.4. *If $f \in H^{1/2}(\mathbb{R})$ is supported in the interval $[-1, 1]$, then $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$.*

The main result of this paper is the following theorem, which answers the aforementioned conjecture in the affirmative.

Theorem 1.5.

- (1) *Let $1 < p < \infty$. If $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{p'/2}(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$.*
- (2) *If $f \in H^{1/2}(\mathbb{R})$ has compact support, then $\mathcal{G}(f, 1, 1)$ is not a frame for $L^2(\mathbb{R})$.*

Note that the $p = 2$ case of this theorem is the classical Balian–Low Theorem (Theorem 1.1); part 2 is a slight generalization of Theorem 1.4 (see also Remark (2) below for a further strengthening of this result).

Before proceeding to the proof of this theorem, we provide some remarks on its general philosophy in relation to the history of the problem. In particular, some discussion of the proof of the Balian–Low Theorem 1.1 is in order. The key tool in the original (incomplete) proof given independently by Balian [Bal] and Low [Low] is the *Zak transform*, also known as the *Weil–Brezin map*. For compactly supported $f \in L^2(\mathbb{R})$, the Zak transform $Zf \in L^2_{\text{loc}}(\mathbb{R}^2)$ is given by

$$Zf(x, y) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell y} f(x - \ell).$$

One can view Zf as a function on the unit square $Q_0 := [0, 1) \times [0, 1)$, and in fact Z extends to an isomorphism from $L^2(\mathbb{R})$ to $L^2(Q_0)$. We will develop some background on the Zak transform in Section 2 below. For the present, we note that to prove either Theorem 1.1 or Theorem 1.5, it suffices to show that

$$\text{ess inf } |Zf| = 0$$

under the given regularity assumptions.² Surprisingly, this is the case for *any* function f for which Zf is continuous (see Proposition 2.2 below). In particular, it is worth noting that the proof of this fact is based on a winding number argument and is hence “degree-theoretic” in the topological sense (albeit very simply).

²Here $\text{ess inf } g := \inf \{ \lambda \mid |\{g \leq \lambda\}| > 0 \}$ is the essential infimum of g , where $|E|$ is the Lebesgue measure of a set E .

In their original proofs of Theorem 1.1, Balian and Low claimed that the regularity conditions $f \in H^1$ and $\hat{f} \in H^1$ would force Zf to be continuous; by the remarks above, this would imply the theorem. However, the regularity conditions only imply that $Zf \in H^1_{\text{loc}}(\mathbb{R}^2)$, which is not contained in $C(\mathbb{R}^2)$. This gap in the proof was filled by Coifman and Semmes and presented in [Dau].

In fact, the Coifman–Semmes argument may be viewed as a simple prototype of the VMO-degree construction in the Brezis–Nirenberg theory of [BN95] and [BN96], which heavily influences our approach in the current paper. Broadly speaking, the results of [BN95] and [BN96] show that in many cases VMO maps are as good as continuous maps for the purposes of degree theory. (Here $\text{VMO}(\mathbb{R}^n)$ is Sarason’s space of functions of *vanishing mean oscillation* on \mathbb{R}^n ; see Section 2 below.) In accordance with this principle, Coifman and Semmes first prove that under the given regularity assumptions $Zf \in \text{VMO}(\mathbb{R}^2)$; their argument gives the $n = p = 2$ case of the following endpoint Sobolev embedding theorem (see *e.g.* [BN95]):

Theorem 1.6. *Let $1 \leq p < \infty$, and let $s = p/n$. Then $W^{s,p}(\mathbb{R}^n) \subset \text{VMO}(\mathbb{R}^n)$ with continuous embedding, where $W^{s,p}$ is the usual L^p -Sobolev space.*

This fact is then used to run a modified winding number argument and prove that $\text{ess inf } |Zf| = 0$, from which the theorem follows. (The Coifman–Semmes proof as presented in [Dau] does not explicitly mention VMO, BMO or the above Sobolev embedding, but the methods are present without the terminology.)

For the proof of our main result, Theorem 1.5, we take a parallel approach. As noted, prior to the results of [BCPS], the best known result was Theorem 1.2. This latter follows from the results of [Grö], in which it is shown that under the given regularity assumptions f belongs to the Wiener algebra

$$W(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \sum_{k \in \mathbb{Z}} \sup_{x \in [0,1]} |f(x+k)| < \infty\}.$$

This in turn immediately implies that the Zak transform of f is continuous. However, the proof is invalid for the critical regularity case where $\frac{1}{p} + \frac{1}{q} = 1$, so we expect that Zf “barely” fails to be continuous under our regularity assumptions. Thus it seems reasonable to expect that $Zf \in \text{VMO}(\mathbb{R}^2)$; that this is in fact true is the key step of our proof, established by a variant of the above Sobolev embedding theorem (Theorem 3.1 below). We combine this with a simplified version of the Coifman–Semmes winding number argument (essentially drawn from [BN96]) to yield the final result.

In the sequel, we will write “ $A \lesssim B$ ” if $A \leq cB$ for some universal constant c ; “ $A \sim B$ ” means $A \lesssim B \lesssim A$. Subscripts on the symbols “ \lesssim ” and “ \sim ” will denote dependence of the implied constants.

2. Background and preliminaries: The Zak transform and VMO

We begin by recalling some basic facts about the Zak transform. As stated above, for compactly supported $f \in L^2(\mathbb{R})$, the Zak transform of f is defined (almost everywhere) by

$$Zf(x, y) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell y} f(x - \ell).$$

It is easily seen that Zf verifies the “quasi-periodicity” relations

$$(2.1) \quad \begin{aligned} Zf(x+1, y) &= e^{2\pi i y} Zf(x, y) \\ Zf(x, y+1) &= Zf(x, y), \end{aligned}$$

so that Zf is completely determined by its values on the unit cube $Q_0 \subset \mathbb{R}^2$. As mentioned above, Z actually extends to a unitary isomorphism from $L^2(\mathbb{R})$ to $L^2(Q_0) \cong L^2(\mathbb{T}^2)$. This can easily be seen by examining its action on the orthonormal basis $\{e_{m,n}\}$ of $L^2(\mathbb{R})$, where

$$e_{m,n}(x) = e^{2\pi i n x} \mathbf{1}_{[0,1)}(x-m), \quad m, n \in \mathbb{Z};$$

this basis is mapped to the usual Fourier basis of $L^2(\mathbb{T}^2)$ by the Zak transform. Thus we may view Z as a map from $L^2(\mathbb{R})$ to either $L^2(\mathbb{T}^2)$ or $L^2_{\text{loc}}(\mathbb{R}^2)$.

Zf provides a time-frequency representation of f ; in fact, viewed as an element of $L^2(\mathbb{T}^2)$, Zf is the Fourier transform of the *Gabor coefficients* $(f_{m,n}) \in \ell^2(\mathbb{Z} \times \mathbb{Z})$, defined by

$$f_{m,n} = \langle f, e_{m,n} \rangle.$$

Similarly, the Zak transform is intimately connected with the frame properties of Gabor systems.

Proposition 2.1. *Let $f \in L^2(\mathbb{R})$. Then $\mathcal{G}(f, 1, 1)$ is an (A, B) -frame for $L^2(\mathbb{R})$ if and only if $A^{1/2} \leq |Zf| \leq B^{1/2}$ almost everywhere.*

This is complemented by the following somewhat curious fact, as mentioned above.

Proposition 2.2. *If $f \in L^2(\mathbb{R})$ has continuous Zak transform, then Zf must have a zero.*

For the proofs of these results and more on the Zak transform, see *e.g.* [Dau] and [Fol]. In light of Proposition 2.1, we see that in order to prove an obstruction result such as the Balian–Low Theorem 1.1 or Theorem 1.5, it suffices to show that $\text{ess inf } |Zf| = 0$. We will accomplish this in part by proving an analogue of Proposition 2.2 (Proposition 4.1 below).

We now discuss the regularity properties of the Zak transform of a function f satisfying some given time-frequency localization (or regularity) conditions. For convenience, we introduce the notation $S_{p,q}$ with $0 < p, q < \infty$ for the Hilbert space

$$S_{p,q} := \left\{ g \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |\hat{g}(\xi_1, \xi_2)|^2 (1 + |\xi_1|^p + |\xi_2|^q) d\xi_1 d\xi_2 < \infty \right\},$$

equipped with the norm

$$\|g\|_{S_{p,q}} = \left(\int_{\mathbb{R}^2} |\hat{g}(\xi_1, \xi_2)|^2 (1 + |\xi_1|^p + |\xi_2|^q) d\xi_1 d\xi_2 \right)^{1/2}.$$

$S_{p,q}$ should be thought of as a modified Sobolev space; when $p = q$, $S_{p,p}$ coincides with the usual inhomogeneous Sobolev space $H^{p/2}(\mathbb{R}^2)$, with equivalent norms.

The Zak transform of a function f , being a time-frequency representation of f , naturally inherits the smoothness properties of f and \hat{f} in the following sense.

Lemma 2.3. *Let $f \in H^{s_1}(\mathbb{R})$ and $\hat{f} \in H^{s_2}(\mathbb{R})$ with $s_1, s_2 > 0$. Then for any smooth, compactly supported function $\varphi \in C_c^\infty(\mathbb{R}^2)$, we have $\varphi Zf \in S_{2s_1, 2s_2}$.*

Proof. For $j = 1, 2$, we write ∇_j for the j -th distributional partial derivative operator on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$; ∇ denotes the distributional derivative on $\mathcal{S}'(\mathbb{R})$. Similarly, for $s \geq 0$, we define the inhomogeneous fractional derivatives $\langle \nabla_j \rangle^s$ as Fourier multipliers on $\mathcal{S}'(\mathbb{R}^2)$ with symbols $\langle \xi_j \rangle^s$.³

Recalling the definition

$$Zf(x_1, x_2) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell x_2} f(x_1 - \ell)$$

for compactly supported f , it is easy to check that for all $f \in H^s(\mathbb{R})$

$$Z(\nabla^n f) = \nabla_1^n(Zf)$$

in the sense of distributions for any nonnegative integer $n \leq s$.

Now for $j = 1, 2$ and $s \geq 0$, let $H_j^s(\mathbb{R}^2)$ denote the modified Sobolev space

$$H_j^s(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) \mid \|f\|_{H_j^s} = \|\langle \nabla_j \rangle^s f\|_2 < \infty\}.$$

Fix a compactly supported bump function $\varphi \in C_c^\infty(\mathbb{R}^2)$. When k is an integer, the Leibniz rule for weak derivatives yields

$$\begin{aligned} \|\varphi Zf\|_{H_1^k} &\lesssim_k \|\varphi Zf\|_{L^2(\mathbb{R}^2)} + \sum_{m=0}^k \|(\nabla_1^m \varphi)(\nabla_1^{k-m} Zf)\|_2 \\ &= \|\varphi Zf\|_{L^2(\mathbb{R}^2)} + \sum_{m=0}^k \|(\nabla_1^m \varphi) Z(\nabla^{k-m} f)\|_2 \\ &\lesssim_{k, \varphi} \sum_{m=0}^k \|Z(\nabla^m f)\|_{L^2(\text{supp}(\varphi))} \\ &\lesssim_\varphi \|f\|_{H^k(\mathbb{R})}. \end{aligned}$$

The last two inequalities follow from the quasi-periodicity relations (2.1) and the unitarity of Z viewed as a map into $L^2(\mathbb{T}^2)$, which imply that

$$\|Zg\|_{L^2(K)} \lesssim_K \|g\|_{L^2(\mathbb{R})}$$

for all compact $K \subset \mathbb{R}^2$. Thus φZ is a bounded linear operator from $H^k(\mathbb{R})$ to $H_1^k(\mathbb{R}^2)$ for integer values of k . The spaces $H_1^s(\mathbb{R}^2)$ can be interpolated via the complex method as with the traditional Sobolev spaces (see *e.g.* [AF]), so we obtain in fact that φZ is bounded from $H^s(\mathbb{R})$ to $H_1^s(\mathbb{R}^2)$ for all $s \geq 0$. (Equivalently, one can work on the Fourier transform side and appeal to Stein's weighted interpolation theorem; see [Ste56].)

A similar argument also shows that

$$\langle \nabla_2 \rangle^s (\varphi Zf) \in L^2(\mathbb{R}^2)$$

whenever $\hat{f} \in H^s(\mathbb{R})$, once one applies the well-known fact that

$$Z\hat{f}(x, y) = e^{2\pi i xy} Zf(-y, x),$$

³Here $\langle a \rangle = (1 + |a|^2)^{1/2}$.

which follows from the Poisson Summation Formula. So when $f \in H^{s_1}(\mathbb{R})$ and $\hat{f} \in H^{s_2}(\mathbb{R})$, we have

$$(\langle \nabla_1 \rangle^{s_1} + \langle \nabla_2 \rangle^{s_2})(\varphi Z f) \in L^2(\mathbb{R}^2)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$. The lemma then follows immediately from Plancherel's Theorem. \square

Finally, we recall some basic facts about the space $\text{VMO}(\mathbb{R}^n)$. Recall that $\text{BMO}(\mathbb{R}^n)$ is the space of functions (modulo constants) of bounded mean oscillation on \mathbb{R}^n ,

$$\text{BMO}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) \mid \|f\|_{\text{BMO}} := \sup_Q \left(\int_Q |f(x) - \int_Q f| dx \right) < \infty \right\},$$

where the supremum is taken over cubes Q in \mathbb{R}^n , and

$$\int_E g := \frac{1}{|E|} \int_E g$$

denotes the average of a function g over a Lebesgue measurable set E . We define $\text{VMO}(\mathbb{R}^n)$ to be the closure of the uniformly continuous functions in the BMO -norm. We will also use a more concrete characterization of VMO : $f \in \text{BMO}(\mathbb{R}^n)$ is in VMO if and only if

$$\lim_{a \rightarrow 0} \sup_{|Q| \leq a} \left(\int_Q |f(x) - \int_Q f| dx \right) = 0.$$

For the proof of this and other equivalent characterizations of VMO , see [Sar].

3. The embedding into VMO

As a first step, we establish that whenever $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{p'/2}(\mathbb{R})$ for $1 < p, p' < \infty$ with p and p' conjugate, we have $Zf \in \text{VMO}(\mathbb{R}^2)$. The key step of our proof is the following analogue of the endpoint Sobolev embedding in Theorem 1.6 for the spaces $S_{p,q}$.

Theorem 3.1. *Let $1 < p < \infty$, and let p' be the conjugate exponent to p . Then $S_{p,p'} \subset \text{VMO}(\mathbb{R}^2)$ with*

$$\|f\|_{\text{BMO}} \lesssim \|f\|_{S_{p,p'}}.$$

Proof. Without loss of generality, we assume $1 < p < 2$, so that $p' > 2 > p$. (The case $p = 2$ is a special case of Theorem 1.6, as mentioned above.) We will use the Littlewood–Paley characterization of BMO .

Let $\{\psi_k\}_{k \in \mathbb{Z}}$ be a Littlewood–Paley partition of unity on the frequency space \mathbb{R}^2 , so that each ψ_k is a nonnegative smooth bump function supported on the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, with

$$\sum_k \psi_k(\xi) = 1, \text{ a.e. } \xi \in \mathbb{R}^2.$$

Let P_k denote the corresponding Littlewood–Paley projection operators, so that each P_k is a Fourier multiplier with symbol ψ_k . Then for all $f \in L^2(\mathbb{R}^2)$, we have

$$\|f\|_{\text{BMO}(\mathbb{R}^2)} \sim_c \sup_Q \left(\int_Q \sum_{k \geq -\log_2 \ell(Q) + c} |P_k f|^2 \right)^{1/2},$$

where the supremum is taken over cubes $Q \subset \mathbb{R}^2$ of dyadic side lengths $\ell(Q)$, and $c \geq 0$. For our purposes, it suffices to take $c = 3$. (This is essentially a discrete version of Theorem 3 in Chapter IV, §4.3 of [Ste93]; see also §4.5 of the same.)

Let $f \in L^2(\mathbb{R}^2)$, and fix a cube $Q \subset \mathbb{R}^2$ with $\ell(Q) = 2^{-k_0}$. As a first step, we prove the estimate

$$(3.1) \quad \oint_Q |P_k f|^2 \lesssim \int_{\mathbb{R}^2} |\widehat{P_k f}(\xi)|^2 (1 + |\xi_1|^p + |\xi_2|^{p'}) d\xi_1 d\xi_2$$

for each Littlewood–Paley piece $P_k f$ with $k \geq k_0 + 3$. Let $\psi \in \mathcal{S}(\mathbb{R}^2)$ be a nonnegative Schwartz function adapted to Q , such that $\psi \geq 1$ on Q and $\hat{\psi}$ is supported on the cube of length $2^{k_0} = \ell(Q)^{-1}$ centered at 0. Then ψ will satisfy the estimate

$$\|\psi\|_2 \lesssim 2^{-k_0},$$

with implied constant independent of Q . Then we have

$$(3.2) \quad \oint_Q |P_k f|^2 \leq 2^{2k_0} \|\psi P_k f\|_{L^2(\mathbb{R}^2)}^2 = 2^{2k_0} \sum_{\substack{J \text{ dyadic} \\ \ell(J)=2^{k_0}}} \|\hat{\psi} * \widehat{P_k f}\|_{L^2(J)}^2,$$

where the last sum is taken over the disjoint cubes J in the dyadic mesh at scale 2^{k_0} . Let w_p be the weight function

$$w_p(\xi) = (1 + |\xi_1|^p + |\xi_2|^{p'})^{1/2}.$$

Then for each J , Young’s inequality and Cauchy–Schwarz yield

$$\|\hat{\psi} * \widehat{P_k f}\|_{L^2(J)} \leq \|\hat{\psi}\|_2 \|\widehat{P_k f}\|_{L^1(3J)} \lesssim 2^{-k_0} \|\widehat{P_k f} \cdot w_p\|_{L^2(3J)} \|w_p^{-1}\|_{L^2(3J)}$$

since $\hat{\psi}$ is supported on a cube of side length $2^{k_0} = \ell(J)$ (here $3J$ denotes the cube with the same center as J and $\ell(3J) = 3\ell(J)$). Summing over J as in (3.2) yields

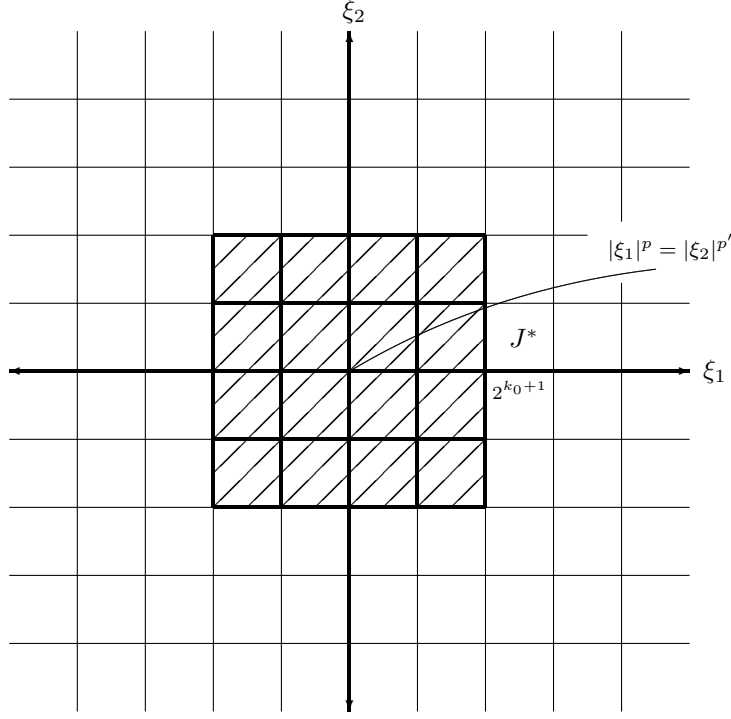
$$\oint_Q |P_k f|^2 \lesssim \sum_{J \in \mathcal{J}_{k_0}} \|\widehat{P_k f} \cdot w_p\|_{L^2(3J)}^2 \|w_p^{-1}\|_{L^2(3J)}^2.$$

Here we write \mathcal{J}_{k_0} for the collection of “admissible” cubes on which $\widehat{P_k f}$ does not vanish identically for every $k \geq k_0 + 3$. In particular, since $\widehat{P_k f}$ is supported on the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, \mathcal{J}_{k_0} omits cubes sufficiently close to the origin (*viz.*, the shaded cubes in Figure 1 below).

It so happens that this is enough to ensure that for all $J \in \mathcal{J}_{k_0}$

$$(3.3) \quad \|w_p^{-1}\|_{L^2(3J)} \lesssim 1,$$

where the implied constant is *independent* of the scale parameter k_0 . By symmetry and monotonicity considerations on w_p , and since $p < p'$, it suffices to bound the term corresponding to $J^* = [2 \cdot 2^{k_0}, 3 \cdot 2^{k_0}) \times [0, 2^{k_0})$ (see Figure 1). Now we have the

FIGURE 1. Inadmissible cubes for \mathcal{J}_{k_0} , cube J^* (at scale $k_0 \gg 1$).

estimate

$$\begin{aligned}
\|w_p^{-1}\|_{L^2(3J^*)}^2 &= \int_{2^{k_0}}^{2^{k_0+2}} \int_{-2^{k_0}}^{2^{k_0+1}} \frac{d\xi_2 d\xi_1}{1 + |\xi_1|^p + |\xi_2|^{p'}} \\
&\lesssim \int_{2^{k_0}}^{2^{k_0+2}} \int_0^{2^{k_0+1}} \frac{d\xi_2 d\xi_1}{1 + |\xi_1|^p + |\xi_2|^{p'}} \\
&\leq \int_{2^{k_0}}^{2^{k_0+2}} \int_0^{\xi_1^{p/p'}} \xi_1^{-p} d\xi_2 d\xi_1 + \int_{2^{k_0}}^{2^{k_0+2}} \int_{\xi_1^{p/p'}}^{2^{k_0+1}} \xi_2^{-p'} d\xi_2 d\xi_1 \\
&\lesssim 1 + 2^{k_0(2-p')},
\end{aligned}$$

which is bounded for $k_0 \geq 0$ as $p' > 2$. For $k_0 < 0$, we simply note that $|3J^*| \lesssim 1$ and $\|w_p^{-1}\|_\infty = 1$, and we obtain (3.3). This implies

$$\int_Q |P_k f|^2 \lesssim \sum_{\substack{J \text{ dyadic} \\ \ell(J)=2^{k_0}}} \|\widehat{P_k f} \cdot w_p\|_{L^2(3J)}^2 \lesssim \|\widehat{P_k f} \cdot w_p\|_{L^2(\mathbb{R}^2)}^2,$$

which is the desired estimate (3.1) on $P_k f$. Summing this estimate in $k \geq k_0 + 3$ and taking the supremum over all cubes Q , we obtain the BMO estimate

$$\|f\|_{\text{BMO}} \lesssim \|f\|_{S_{p,p'}}.$$

But since Schwartz functions are dense in $S_{p,p'}$, we actually have $S_{p,p'} \subset \text{VMO}(\mathbb{R}^2)$, as VMO is the BMO-closure of the uniformly continuous functions. This concludes the proof of the theorem. \square

From this and Lemma 2.3 (combined with the quasi-periodicity property (2.1)) we obtain:

Corollary 3.2. *For $1 < p < \infty$, if $f \in H^{p/2}(\mathbb{R})$ and $\hat{f} \in H^{p'/2}(\mathbb{R})$, then $Zf \in \text{VMO}(\mathbb{R}^2)$.*

We now turn to the “endpoint” regularity case where $p = 1$; of course, the dual localization condition “ $\hat{f} \in H^{\infty/2}$ ” requires suitable interpretation. For our purposes, as in Theorem 1.5.2, we will take this to mean f has compact support; this is less restrictive than the condition $\text{supp}(f) \subset [-1, 1]$ in Theorem 1.4. In this setting, we will show directly that $Zf \in \text{VMO}(\mathbb{R}^2)$, provided that $Zf \in L^\infty(\mathbb{R}^2)$. (The additional boundedness assumption on Zf will be acceptable for our purposes, in light of Proposition 2.1.)

Lemma 3.3. *Suppose $f \in H^{1/2}(\mathbb{R})$ has compact support, with $Zf \in L^\infty(\mathbb{R}^2)$. Then $Zf \in \text{VMO}(\mathbb{R}^2)$.*

Proof. Fix a large cube $Q^* \subset \mathbb{R}^2$ such that the unit cube Q_0 is contained in the interior of Q^* . By the quasi-periodicity relations (2.1) for the Zak transform, it suffices to prove $Zf \in \text{VMO}(Q^*)$, in the sense that

$$\lim_{a \rightarrow 0} \sup_{\substack{|Q| < a, \\ Q \subset Q^*}} \left(\int_Q |Zf(x) - \int_Q Zf| dx \right) = 0.$$

But since f has compact support, its Zak transform is a *finite* sum

$$Zf(x, y) = \sum_{|\ell| \leq N} e^{2\pi i \ell y} f(x - \ell)$$

for $(x, y) \in Q^*$. We have $\|f\|_\infty \leq \|Zf\|_\infty < \infty$; this can be seen for instance by fixing x and viewing $Zf(x, y)$ as a Fourier series in y . Since $f \in H^{1/2}(\mathbb{R})$, we also have $f \in \text{VMO}(\mathbb{R})$ by Theorem 1.6.

A simple calculation shows that if $g, h \in \text{VMO}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then their tensor product $g \otimes h$ lies in $\text{VMO}(\mathbb{R}^2)$. The restriction of Zf to Q^* agrees with a finite sum of such tensor products, so we have $Zf \in \text{VMO}(Q^*)$, and the lemma follows. \square

4. The winding number argument

Recall that the Zak transform of a function f satisfies the quasi-periodicity relations (2.1):

$$\begin{aligned} Zf(x+1, y) &= e^{2\pi i y} Zf(x, y) \\ Zf(x, y+1) &= Zf(x, y) \end{aligned}$$

for a.e. $(x, y) \in \mathbb{R}^2$; as mentioned before, any continuous function satisfying these relations must have a zero. In fact, the same is essentially true of bounded VMO functions.

Lemma 4.1. *Suppose $F \in \text{VMO}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ satisfies*

$$\begin{aligned} F(x+1, y) &= e^{2\pi i y} F(x, y) \\ F(x, y+1) &= F(x, y) \end{aligned}$$

almost everywhere. Then $\text{ess inf } |F| = 0$.

Proof. ⁴ We proceed by contradiction. By scaling, we may assume that there exists $d > 0$ such that

$$d \leq |F| \leq 1$$

almost everywhere. Let $Q_\varepsilon(x, y)$ denote the cube of side length ε centered at $(x, y) \in \mathbb{R}^2$, and define

$$F_\varepsilon(x, y) := \oint_{Q_\varepsilon(x, y)} F,$$

the average of F over $Q_\varepsilon(x, y)$. F_ε is continuous and satisfies the modified quasi-periodicity relations

$$\begin{aligned} F_\varepsilon(x+1, y) &= e^{2\pi i y} F_\varepsilon(x, y) + \Phi_\varepsilon(x, y) \\ (4.1) \quad F_\varepsilon(x, y+1) &= F_\varepsilon(x, y), \end{aligned}$$

where the error term Φ_ε satisfies

$$|\Phi_\varepsilon(x, y)| \lesssim \varepsilon.$$

Moreover, for ε sufficiently small, F_ε is also bounded from below, as we now show. Since $F \in \text{VMO}(\mathbb{R}^2)$, we may choose ε_0 so that

$$\oint_{Q_\varepsilon(x, y)} |F - F_\varepsilon(x, y)| \leq \frac{d}{2}$$

for all $(x, y) \in \mathbb{R}^2$ and $\varepsilon < \varepsilon_0$. Then simply applying the triangle inequality, we have

$$\begin{aligned} |F_\varepsilon(x, y)| &= \oint_{Q_\varepsilon(x, y)} |F_\varepsilon(x, y)| \\ &\geq \oint_{Q_\varepsilon(x, y)} |F| - \oint_{Q_\varepsilon(x, y)} |F - F_\varepsilon(x, y)| \geq \frac{d}{2}, \end{aligned}$$

since we assume $|F| \geq d$ almost everywhere. Thus $\frac{d}{2} \leq |F_\varepsilon| \leq 1$ almost everywhere for $\varepsilon < \varepsilon_0$.

However, this is impossible, as the relations (4.1) force the curve $\Gamma_\varepsilon := F_\varepsilon(\partial Q_0)$ to have nonzero winding number about 0 for ε sufficiently small, where $Q_0 = [0, 1] \times [0, 1]$ is the unit cube in \mathbb{R}^2 . To make this contradiction more precise, we give the same argument as Coifman and Semmes. Note that since F_ε is continuous with $\frac{d}{2} \leq |F_\varepsilon| \leq$

⁴This argument is almost identical to that of Coifman and Semmes in [Dau]; the difference is that they use a *quantitative* VMO estimate, whereas we need to make do with only the qualitative assumption that $F \in \text{VMO}$. We present the whole argument for self-containment.

1, we can define a continuous branch γ_ε of $\log F_\varepsilon$. From the modified quasi-periodicity conditions (4.1), we have

$$(4.2) \quad \begin{aligned} \gamma_\varepsilon(x+1, y) &= \gamma_\varepsilon(x, y) + 2\pi i j + 2\pi i y + \Psi_\varepsilon(x, y) \\ \gamma_\varepsilon(x, y+1) &= \gamma_\varepsilon(x, y) + 2\pi i k \end{aligned}$$

for all x, y in some simply connected neighborhood U of Q_0 . Here $j, k \in \mathbb{Z}$ are constant on U by continuity of γ_ε , and

$$|\Psi_\varepsilon| \leq -\log \left(1 - \frac{|\Phi_\varepsilon|}{|F_\varepsilon|} \right) \lesssim \frac{|\Phi_\varepsilon|}{|F_\varepsilon|}$$

provided that $|\Phi_\varepsilon|/|F_\varepsilon|$ is sufficiently small. This can be arranged by taking ε sufficiently small, since $|\Phi_\varepsilon| \lesssim \varepsilon$ and $|F_\varepsilon| \geq d/2$; thus for ε small we have

$$|\Psi_\varepsilon| < 1$$

on U . To obtain the contradiction, we simply compute

$$\begin{aligned} 0 &= \gamma_\varepsilon(1, 0) - \gamma_\varepsilon(0, 0) + \gamma_\varepsilon(1, 1) - \gamma_\varepsilon(1, 0) \\ &\quad + \gamma_\varepsilon(0, 1) - \gamma_\varepsilon(1, 1) + \gamma_\varepsilon(0, 0) - \gamma_\varepsilon(0, 1) \\ &= \Psi_\varepsilon(0, 0) - \Psi_\varepsilon(0, 1) - 2\pi i \neq 0, \end{aligned}$$

since $|\Psi_\varepsilon| < 1$. Thus our original assumption that $|F| \geq d$ almost everywhere must be false, and hence $\text{ess inf } |F| = 0$ as desired. \square

From this, we can deduce our main result, Theorem 1.5. Suppose $f \in L^2(\mathbb{R})$ satisfies either of the prescribed time-frequency regularity conditions, and suppose furthermore that $\mathcal{G}(f, 1, 1)$ is an (A, B) -frame for $L^2(\mathbb{R})$. Then by Proposition 2.1, we have

$$A^{1/2} \leq |Zf| \leq B^{1/2} \quad a.e.$$

Moreover, by either Corollary 3.2 or Lemma 3.3, $Zf \in \text{VMO}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. But, by Lemma 4.1, this is impossible; this contradiction concludes the proof of the theorem.

5. Remarks

- (1) The second part of the proof of Lemma 4.1 essentially shows that the continuous maps F_ε all have nonzero degree at the point 0, for ε sufficiently small. In the context of [BN95] and [BN96], this is a manifestation of the stability of degree under VMO-convergence. In fact, the (integer-valued) VMO degree of F at a point p is *defined* as

$$\text{VMO-deg}(F, p) := \deg(F_\varepsilon, p) \quad \text{for } \varepsilon < \varepsilon_0,$$

up to some domain considerations. As mentioned before, the $H^1(\mathbb{R}^2)$ argument of Coifman and Semmes can be viewed as a prototype of the Brezis–Nirenberg theory in a relatively simple case; for more on the VMO degree theory and related topics, see *e.g.* [Bre], [BN95], [BN96], and [BBM].

- (2) The localization condition that f be compactly supported in Theorem 1.5.2 is far from sharp. An inspection of the proof of Lemma 3.3 shows that it is sufficient to demand BMO-convergence of the series defining the Zak transform near the unit square, which would in turn be implied by uniform convergence of the series. This latter can easily be guaranteed by simply requiring the mild decay condition

$$\sum_{\ell \in \mathbb{Z}} \|f\chi_{[\ell, \ell+1]}\|_{L^\infty} < \infty,$$

for instance. This observation is due to the author, C. Heil, and A. Powell, arising from a joint discussion. However, as noted, this decay condition is stronger than is necessary to guarantee $Zf \in \text{VMO}$.

- (3) Theorem 3.1 is of mild interest in its own right. The spaces $S_{p,q}$ above were chosen *ad hoc* for the setting of the Balian–Low Theorem; one might hope to prove an embedding result for spaces with analogous L^r -based regularity conditions, $r \neq 2$.

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