POSITIVE QUATERNIONIC KÄHLER MANIFOLDS AND
SYMMETRY RANK: II

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Abstract. Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. If the isometry group $\text{Isom}(M)$ has rank at least $\frac{m}{2} + 3$, then $M$ is isometric to $\mathbb{H}P^m$ or $\text{Gr}_2(\mathbb{C}^{m+2})$. The lower bound for the rank is optimal if $m$ is even.

1. Introduction

A quaternionic Kähler manifold $M$ is an oriented Riemannian $4n$-manifold, $n \geq 2$, whose holonomy group is contained in $Sp(n)Sp(1) \subset SO(4n)$. If $n = 1$ we add the condition that $M$ is Einstein and self-dual. Equivalently, there exists a 3-dimensional subbundle $S$ of the endmorphism bundle, $\text{End}(TM, TM)$, locally generated by three anti-commuting almost complex structures $I, J, K = IJ$ so that the Levi-Civita connection preserves $S$. It is well-known [3] that a quaternionic Kähler manifold $M$ is always Einstein, and is necessarily locally hyperkähler if its Ricci tensor vanishes. A quaternionic Kähler manifold $M$ is called positive if it has positive scalar curvature. By [13] (for $n = 1$) and [20] (for $n \geq 2$, compare [16] [17]) a positive quaternionic Kähler manifold $M$ has a twistor space a complex Fano manifold. Hitchin [13] proved a positive quaternionic Kähler 4-manifold $M$ must be isometric to $\mathbb{C}P^2$ or $S^4$. Hitchin’s work was extended by Poon-Salamon [19] to dimension 8, which proves that a positive quaternionic Kähler 8-manifold $M$ must be isometric to $\mathbb{H}P^2$, $\text{Gr}_2(\mathbb{C}^4)$ or $G_2/\text{SO}(4)$.

This leads to the Salamon-Lebrun conjecture:

Every positive quaternionic Kähler manifold is a quaternionic symmetric space.

Very recently, the conjecture was further verified for $n = 3$ in [12], using the approach initiated in [20] [19] (compare [17]). For a positive quaternionic Kähler manifold $M$, Salamon [20] proved that the dimension of its isometry group is equal to the index of certain twisted Dirac operator, by the Atiyah-Singer index theorem, which is a characteristic number of $M$ coupled with the Kraines 4-form $\Omega$ (in analog with the Kähler form), and it was applied to prove that the isometry group of $M$ is large in lower dimensions (up to dimension 16).

By [17] a positive quaternionic Kähler 4n-manifold $M$ is simply connected and the second homotopy group $\pi_2(M)$ is a finite group or $\mathbb{Z}$, and $M$ is isometric to $\mathbb{H}P^n$ or $\text{Gr}_2(\mathbb{C}^{n+2})$ according to $\pi_3(M) = 0$ or $\mathbb{Z}$.

An interesting question is to study positive quaternionic Kähler manifold in terms of its isometry group. This approach dates back to the work [19] for $n = 2$ [12] for
$n = 3$ to proving the action is transitive, and [5] [18] for cohomogeneity one actions (and hence the isometry group must be very large). [4] classified positive quaternionic Kähler $4n$-manifolds with isometry rank $n + 1$, using an approach on hyper-Kähler quantizations. [6] establishes a connectedness theorem and using this tool the author proved that, a positive quaternionic Kähler $4n$-manifolds of symmetry rank $\geq n - 2$ must be either isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$, if $n \geq 10$.

In this paper we will combine Morse theory of the momentum map on quaternionic Kähler manifold [2] and the connectedness theorem in [6] to prove the following

**Theorem 1.1.** Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. Then the isometry group $\text{Isom}(M)$ has rank (denoted by $\text{rank}(M)$) at most $(m + 1)$, and $M$ is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$ if $\text{rank}(M) \geq \frac{m}{2} + 3$.

Notice that the fixed point set of an isometric circle action on a quaternionic Kähler manifold of dimension $4m$ is either a quaternionic Kähler submanifold or a Kähler manifold. In the latter case the fixed point set has dimension at most $2m$ (the middle dimension of the manifold). Moreover, if a fixed point component is contained in $\mu^{-1}(0)$ then it must be a quaternionic Kähler submanifold, and if it is in the complement $M - \mu^{-1}(0)$ then it is Kähler (see [2]).

**Theorem 1.2.** Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$ with an isometric $S^1$-action. Assume $m \geq 3$. If $N$ is a fixed point component of codimension $4$, then $M$ is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$.

The idea of proving Theorem 1.2 is as follows: by the assumption we know that the fixed point component $N \subset \mu^{-1}(0)$ since $4m - 4 \geq 2m + 1$ (cf. [2] Remark 3.2). Furthermore, by [8] the reduction $\mu^{-1}(0)/S^1$ has dimension at most $4m - 4$ (cf. [5]). We will prove in section 3 that $N = \mu^{-1}(0)$. This together with the equivariant Morse equality implies that $M = \mathbb{H}P^m$ if $b_2(M) = 0$ (cf. Lemma 4.1) and so Theorem 1.2 follows by [17].

With Theorem 1.2 in hand, the proof of Theorem 1.1 follows by induction on $m$ and Theorems 2.1 and 2.4.

Theorem 1.1 is optimal if $m$ is even since the rank of $\widetilde{Gr}_4(\mathbb{R}^{m+4})$ is $\frac{m}{2} + 2$. We conjecture that when $m$ is odd, the lower bound for the rank in Theorem 1.1 may be improved by 1, that is

**Conjecture 1.3.** Let $M$ be a positive quaternionic Kähler manifold of dimension $8m + 4$. Then $M$ is isometric to $\mathbb{H}P^{2m+1}$ or $Gr_2(\mathbb{C}^{2m+3})$ if $\text{rank}(M) \geq m + 3$.

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2. Preliminaries

In this section we recall some results on quaternionic Kähler manifolds needed in later sections.

Let \((M, g)\) be a quaternionic Kähler manifold of dimension \(4m\). Let \(F \to M\) be the principal \(Sp(m)\times Sp(1)\)-bundle over \(M\). Locally, \(F \to M\) can be lifted to a principal \(Sp(m)\times Sp(1)\)-bundle, i.e., the fiberwise double cover of \(F\). Let \(E, H\) be the locally defined bundles associated to the standard complex representation of \(Sp(m)\) and \(Sp(1)\) respectively. The complexified cotangent bundle \(T^*M_C\) is isomorphic to \(E \otimes C H\). The adjoint representations of \(Sp(m)\) and \(Sp(1)\) give two bundles \(S^2E\) and \(S^2H\) over \(M\), respectively. Given the inclusion of the holonomy algebra \(sp(m)\oplus sp(1)\) into \(so(4m)\), the bundle \(S^2E \oplus S^2H\) can be regarded as a subbundle of the bundle of 2-forms \(\Lambda^2T^*M_C\). The bundle \(S^2H\) has fiber the Lie algebra \(sp(1)\) and the local basis \(\{I, J, K\}\) corresponding to \(i, j, k \in sp(1)\) which are almost complex structures satisfying that \(IJ = -JI = K\).

The Kraines 4-form, \(\Omega\), associated to a quaternionic Kähler manifold \(M\), is a non-degenerate closed form which is defined by

\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3
\]

where \(\omega_1, \omega_2\) and \(\omega_3\) are the locally defined 2-forms associated to the almost complex structures \(I, J\) and \(K\). The form \(\Omega\) is globally defined and non-degenerate, namely \(\Omega^m\) is a constant non-zero multiple of the volume form. It is well-known that \(\Omega\) is parallel if and only if \(M\) has holonomy in \(Sp(m)Sp(1)\), if \(m \geq 2\). Moreover, by \([22]\), \(M\) has holonomy in \(Sp(m)Sp(1)\) if and only if \(\Omega\) is closed, provided \(m \geq 3\).

A quaternionic Kähler manifold may not have a global almost complex structure, e.g., the quaternionic projective space \(\mathbb{H}P^m\). If \(I, J, K\) are integrable and covariantly constant with respect to the metric, the holonomy group reduces to \(Sp(m)\), quaternionic Kähler manifold is hyperkähler. Wolf \([23]\) classified quaternionic symmetric spaces of compact type, they are \(\mathbb{H}P^m\), the complex Grassmannian \(Gr_2(\mathbb{C}^{m+2})\), and the oriented real Grassmannian \(Gr_4(\mathbb{R}^{m+4})\), and exactly one quaternionic symmetric space for each compact simple Lie algebra, \(G_2/\text{SO}(4), \ F_4/\text{Sp}(3)\text{Sp}(1), \ E_6/\text{SU}(6)\text{Sp}(1), \ E_7/\text{Spin}(12)\text{Sp}(1), \ E_8/E_7\text{Sp}(1)\).

**Theorem 2.1** ([17]). (i) (Finiteness) For any \(m \in \mathbb{Z}_+\), there are, modulo isometries and rescalings, only finitely many positive quaternionic Kähler manifolds of dimension \(4m\).

(ii) (Strong rigidity) Let \((M, g)\) be a positive quaternionic Kähler manifold of dimension \(4m\). Then \(M\) is simply connected and

\[
\pi_2(M) = \begin{cases} 
0, & (M, g) = \mathbb{H}P^m \\
\mathbb{Z}, & (M, g) = Gr_2(\mathbb{C}^{m+2}) \\
\text{finite with 2-torsion}, & \text{otherwise}
\end{cases}
\]

A submanifold \(N\) in a quaternionic Kähler manifold is called a quaternionic submanifold if the locally defined almost complex structures \(I, J, K\) preserve the tangent bundle of \(N\).
Proposition 2.2 (9). Any quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic and quaternionic Kählerian.

Theorem 2.3 (6). Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. Assume $f = (f_1, f_2) : N \to M \times M$, where $N = N_1 \times N_2$ and $f_i : N_i \to M$ are quaternionic immersions of compact quaternionic Kähler manifolds of dimensions $4n_i$, $i = 1, 2$. Let $\Delta$ be the diagonal of $M \times M$. Set $n = n_1 + n_2$. Then:

(2.3.1) If $n \geq m$, then $f^{-1}(\Delta)$ is nonempty.
(2.3.2) If $n \geq m + 1$, then $f^{-1}(\Delta)$ is connected.
(2.3.3) If $f$ is an embedding, then for $i \leq n - m$ there is a natural isomorphism, $\pi_i(N_1, N_1 \cap N_2) \to \pi_i(M, N_2)$ and a surjection for $i = n - m + 1$.

As a direct corollary of (2.3.3) we have

Theorem 2.4 (6). Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. If $N \subset M$ is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \to M$ is $(2n - m + 1)$-connected.

3. Hyperkähler quotient and Quaternionic Kähler quotient

a. Hyperkähler quotient

Let $M$ be a hyperkähler manifold having a metric $g$ and covariantly constant complex structures $I, J, K$ which behave algebraically like quaternions:

$I^2 = J^2 = K^2 = -1; IJ = -JI = K$

Let $G$ be a compact Lie group acting on $M$ by isometries preserving the structures $I, J, K$. The group $G$ preserves the three Kähler forms $\omega_1, \omega_2, \omega_3$ corresponding to the complex structures $I, J, K$, so we may define moment maps $\mu_1, \mu_2, \mu_3$, respectively. These may be written as a single map

$\mu : M \to g^* \otimes \mathbb{R}^3$

Let

$\mu_+ = \mu_2 + i\mu_3 : M \to g^* \otimes \mathbb{C}$

where $g^*$ is the dual space of the Lie algebra of $G$.

By [14] $\mu_+$ is holomorphic, and so $N = \mu_+^{-1}(0)$ is a complex subvariety of $M$, with respect to the complex structure $I$. By definition, $\mu^{-1}(0) = N \cap \mu_1^{-1}(0)$. The hyperkähler quotient is the quotient space $\mu^{-1}(0)/G$, denoted by $M//G$. In particular, if $\mu_+^{-1}(0)$ is a manifold and the induced $G$-action is free, then the hyperkähler quotient $M//G$ is also a hyperKähler manifold. More generally, Dancer-Swann [5] proved that the hyperkähler quotient $M//G$ may be decomposed into the union of hyperkähler manifolds, according to the isotropy decomposition of the $G$-action on $M$. However, it is wide open if the decomposition of $M//G$ is a stratified topological space, as in the sympletic quotient case [21].
In this section we will consider the structure of this decomposition in the special case that $G = S^1$ and the action is semi-free, i.e., free outside the fixed point set.

Let us start with the standard example of isometric $S^1$-action on quaternionic linear space $\mathbb{H}^n$ defined by

$$\varphi_t(u) = e^{2\pi it}u; \quad t \in [0, 1)$$

where $i$ is one of the quaternionic units. With global quaternionic coordinates $\{u^\alpha\}$, $\alpha = 1, \ldots, n$, the standard flat metric on $\mathbb{H}^n$ may be written as:

$$ds^2 = \sum_\alpha d\bar{u}^\alpha \otimes du^\alpha$$

where $\bar{u}^\alpha$ is the quaternionic conjugate of $u^\alpha$.

The Killing vector field $X$ of the above action is $\mathbb{H}$-valued:

$$X^\alpha(u) = iu^\alpha$$

which is triholomorphic.

Consider the $\mathbb{H}$-valued 2-form

$$\omega = \sum_\alpha d\bar{u}^\alpha \wedge du^\alpha$$

Observe that $\omega$ is purely imaginary since $\omega + \bar{\omega} = 0$. Note that $\omega = \omega_1i + \omega_2j + \omega_3k$, where $\omega_1$ is as above.

It is easy to see that the moment map (cf. [7])

$$\mu^X = \sum_\alpha \bar{u}^\alpha iu^\alpha : \mathbb{H}^n \to i\mathbb{R}^3$$

Write $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$. The zero set of the holomorphic moment map $\mu^{-1}_+(0)$ is the complex algebraic variety of complex dimension $(2n - 1)$:

$$\{(a, b) \in \mathbb{C}^n \oplus j\mathbb{C}^n : \langle a, \bar{b} \rangle = \sum_\alpha a^\alpha \bar{b}^\alpha = 0\}$$

In particular, if $n = 1$, then $\mu^{-1}_+(0)$ is a reducible algebraic curve with two irreducible components the standard complex lines.

The hyperkähler quotient $\mathbb{H}^n//S^1$ is an open cone over a $(4n - 5)$-dimensional manifold $W$:

$$W = \{(a, b) \in \mathbb{C}^n \oplus \mathbb{C}^n : |a|^2 = |b|^2 = 1, \langle a, \bar{b} \rangle = \sum_\alpha a^\alpha \bar{b}^\alpha = 0\}/S^1$$

In particular, $\mathbb{H}^n//S^1 = \{0\}$, a single point.

**Theorem 3.1.** Let $M^{4n}$ be a hyperkähler manifold with an isometric effective $S^1$-action preserving the hyperkähler structure. Let $\mu$ be the hyperkähler moment map. If $Y \subset M^{S^1} \cap \mu^{-1}(0)$ is a connected fixed point component of dimension $4m - 4$, then $Y \subset M//S^1$ is a connected component.

Before we start the proof, let us give the analog of the classical Darboux-Weinstein theorem for complex symplectic manifolds. Recall that a complex symplectic manifold $W$ is a complex manifold with a complex value symplectic form $\omega^c = \omega_2 + i\omega_3$, where $\omega_2, \omega_3$ are two real value symplectic forms.
Lemma 3.2 (Relative equivariant Darboux theorem). Let $G$ be a compact Lie group acting on a complex symplectic manifold $W$. Let $\omega_0^c$ and $\omega_1^c$ be two $G$-equivariant complex symplectic forms on $W$. Assume they coincide on a closed $G$-invariant complex symplectic submanifold $V$. Then there exists a $G$-invariant neighborhood $U_0$ of $V$ in $W$ and a $G$-equivariant map

$$\psi : U_0 \to W$$

such that

$$\psi|_V = \text{Id}_V \text{ and } \psi^* \omega_1 = \omega_0$$

Proof. We apply the path method due to Moser. Consider the form $\omega_t^c = \omega_0^c + t(\omega_1^c - \omega_0^c)$. This is a closed complex value 2-form which is non-degenerate on $V$ and thus on a small $G$-invariant tubular neighborhood $U$ of $V$.

Since $\omega_0^c$ and $\omega_1^c$ are closed, and so is $\omega_0^c - \omega_1^c$, and thus we may find a complex value 1-form $\beta$ on a neighborhood of $V$ such that $d\beta = \omega_0^c - \omega_1^c$. Indeed, $U$ is $G$-equivariant diffeomorphic to an open neighborhood of the zero section of a $G$-equivariant vector bundle on $V$, hence retracts on $V$. By applying the Poincaré lemma to $\omega_0^c - \omega_1^c$ we get the $G$-invariant 1-form $\beta$, that can be chosen so that $\beta_x = 0$ for all $x \in V$.

The complex value closed 2-form $\omega_t^c$ being non-degenerate, it defines a time dependent $G$-invariant vector field $X_t$ such that the contraction $i_{X_t}\omega_t^c = \beta$. Note that $X_t = 0$ on $V$. Its flow $\varphi_t$ keeps $V$ fixed, and thus one can find a $G$-invariant neighborhood $U_0$ of $V$ where $\varphi_t$ is defined and such that $\varphi_1(U_0) \subset U$.

Therefore,

$$\frac{d}{dt}[\varphi_t^*\omega_t^c] = \varphi_t^*[d\omega_t^c + \mathfrak{L}_{X_t}\omega_t^c] = \varphi_t^*[\omega_1^c - \omega_0^c + \omega_0^c - \omega_1^c] = 0$$

because $\mathfrak{L}_{X_t}\omega_t^c = di_{X_t}\omega_t^c + i_{X_t}d\omega_t^c = d\beta$ using the Cartan formula and the definition of $\varphi_t$. The form $\varphi_t^*\omega_t^c$ does not depend on $t$, and it equals $\omega_0^c$ for $t = 0$. Put $\psi = \varphi_1$ the desired result follows.

Now we are ready to prove

Proof of Theorem 3.1. Recall that $\omega^c = \omega_2 + i\omega_3$ defines a complex symplectic structure on $M$. Choose an open ball $V$ in $Y$ with restricted complex symplectic structure. The complex normal bundle of $V$ in $M$ is a trivial complex vector bundle $V \times \mathbb{C}^2$. The $S^1$-action on the bundle is the product action of a trivial action on $V$ and an effective complex linear action on $\mathbb{C}^2$, saying, $t \cdot (z_1, z_2) = (t^p z_1, t^q z_2)$ for some $p, q \in \mathbb{Z}$. Since the hyperkähler metric is $S^1$-invariant, the normal exponential map $\varphi = \exp_{\omega^c} : V \times \mathbb{C}^2 \to M$ defines an $S^1$-equivariant diffeomorphism from an $S^1$-invariant tubular neighborhood $U_0$ of $V \times \{0\}$ in $V \times \mathbb{C}^2$ to an $S^1$-invariant tubular neighborhood $U_0$ of $V$ in $M$. Note that $pq = -1$ since the action is effective and preserves the hyperkähler structure.

Consider the $S^1$-invariant complex symplectic form $\omega_0^c = \omega^c|_V \times (dz_1 \wedge dz_2)$ on $V \times \mathbb{C}^2$. The pullback form $(\varphi^{-1})^*\omega_0^c$ and $\omega^c|_{U_0}$ are both $S^1$-invariant complex symplectic forms on $U_0$ which coincide on $V$. By Lemma 3.2 there exists an $S^1$-equivariant diffeomorphism $\psi$ such that $\psi^*\omega^c|_{U_0} = (\varphi^{-1})^*\omega_0^c$. Therefore, $(\psi \circ \varphi)^*\omega^c|_{U_0} = \omega_0^c$.

The moment map $\mu^+|_{U_0}$ of $(U_0, \omega^c)$ with the restricted $S^1$-action may be identified with the moment map of $(V_0, \omega_0^c)$ with the product action. Thus,

$$\mu^+_V(0) \cap U_0 \cong (V \times \{0\}) \times L \cap V_0$$
where $L \subset \mathbb{C}^2$ is the zero locus of the moment map of $(\mathbb{C}^2, dz_1 \wedge dz_2)$ with the above mentioned $S^1$-action. By the paragraph before Theorem 3.1 we already knew that $L$ is the reducible curve given by $z_1z_2 = 0$. Therefore, $\mu_+^{-1}(0) \cap U_0$ is the union of two $S^1$-invariant complex submanifolds, $N_1, N_2 \subset M$ (with respect to the complex structure $I$), and the $S^1$-action on $N_1$ (resp. $N_2$) has a real codimension 2 fixed point set (e.g., $N_1 \cong (V \times \{0\}) \times \{(z_1, 0) : z_1 \in \mathbb{C}\} \cap V_0$). Obviously, both $N_1$ and $N_2$ have induced Kähler structures (w.r.t. $I$), and the restriction of $\mu_1$ on $N_1$ (resp. $N_2$) equals the moment map of the restricted $S^1$-action on $N_1$ (resp. $N_2$) with the induced Kähler structure.

By definition, $\mu_1^{-1}(0) \cap U_0 = \mu_2^{-1}(0) \cap U_0 \cap \mu_3^{-1}(0) = (\mu_1|_{N_1})^{-1}(0) \cup (\mu_1|_{N_2})^{-1}(0)$. Consider the moment map $\mu_1$ of the Kähler manifold $(N_1, \omega_1)$ with the restricted $S^1$-action. In a small tubular neighborhood $W_1$ of the fixed point component $V \subset N_1$ of codimension 2, by $[10]$ $(\mu_1|_{N_1})^{-1}(0)$ is a conic bundle over the fixed point set $V$. Since $(\mu_1|_{N_1})^{-1}(0)$ has dimension $4m - 4$, where $\dim(N_1) = 4m - 2$, thus, $(\mu_1|_{N_1})^{-1}(0) \cap W_1 = V$. Similarly, $(\mu_1|_{N_2})^{-1}(0) \cap W_2 = V$, where $W_2$ is a small tubular neighborhood of $V$ in $N_2$. Therefore, $\mu^{-1}(0) \cap U_0 \cap V$ for a possibly smaller tubular neighborhood $U_0$ of $V$ in $M$. By definition of $\mathcal{M} / S^1$ we conclude the desired result.

b. Quaternionic Kähler quotient

Let $M$ be a quaternionic Kähler manifold with non-zero scalar curvature. If $G$ acts on $M$ by isometries, there is a well-defined moment map, which is a section $\mu \in \Gamma(S^2H \otimes g^*)$ solving the equation

$$\langle \nabla \mu, X \rangle = \sum_{i=1}^{3} I_i \bar{X} \otimes I_i$$

for each $X \in g$; where $\bar{X} = g(X, \cdot)$ denote the 1-form dual to $X$ with respect to the Riemannian metric. Equivalently, the above equation may be written in the following form similar to the symplectic case

$$d\mu(X) = i_X \Omega$$

A nontrivial feature for quaternionic quotient is, the section $\mu$ is uniquely determined if the scalar curvature is nonzero. Moreover, only the preimage of the zero section of the moment map, $\mu^{-1}(0)$, is well-defined.

**Theorem 3.3** ([8]). Let $M^{4n}$ be a quaternionic Kähler manifold with nonzero scalar curvature acted on isometrically by $S^1$. If $S^1$ acts freely on $\mu^{-1}(0)$ then $\mu^{-1}(0)/S^1$ is a quaternionic Kähler manifold of dimension $4(n - 1)$.

Since the proof of the Galicki-Lawson’s theorem is local, so if the circle action is free on a piece of the manifold, the same result applies to the moment map on this piece.

Let $f = ||\mu||^2$. By [2] the critical set of $f$ is the union of the zero set $f^{-1}(0) = \mu^{-1}(0)$ and the fixed point set of the circle action. Moreover, the zero set $\mu^{-1}(0)$ is connected, and a fixed point component is either contained in $\mu^{-1}(0)$ or does not intersect with
\[ \mu^{-1}(0). \] Following [2] the Morse function \( f \) is called equivariantly perfect over \( \mathbb{Q} \) if the equivariant Morse equalities hold, that is if
\[ \hat{P}_t(M) = \hat{P}_t(\mu^{-1}(0)) + \sum t^{\lambda_F} \hat{P}_t(F) \]
where the sum ranges over the set of connected components outside \( \mu^{-1}(0) \) of the fixed point set, \( \lambda_F \) is the index of \( F \), and \( \hat{P}_t \) is the equivariant Poincaré polynomial for the equivariant cohomology with coefficients in \( \mathbb{Q} \).

**Theorem 3.4** ([2]). Let \( M^{4n} \) be a quaternionic Kähler manifold acted on isometrically by \( S^1 \). Then the Morse function \( \|\mu\|^2 \) is equivariantly perfect over \( \mathbb{Q} \).

**Proposition 3.5** ([2]). Let \( M^{4n} \) be a positive quaternionic Kähler manifold acted on isometrically by \( S^1 \). Then every connected component of the fixed point set, not contained in \( \mu^{-1}(0) \), is a Kähler submanifold of \( M - \mu^{-1}(0) \) of real dimension less than or equal to \( 2n \) whose Morse index is at least \( 2n \), with respect to the function \( f \).

For each quaternionic Kähler manifold \( M \) with non-zero scalar curvature, following [22], let \( u(M) \) denote the \( H^*/\{\pm 1\} \)-bundle over \( M \):
\[ u(M) = F \times_{Sp(n)Sp(1)} (H^*/\{\pm 1\}) \]
where \( F \) is the principal \( Sp(n)Sp(1) \)-bundle over \( M \). Let \( \pi : u(M) \to M \) denote the bundle projection. Obviously, if \( G \) is acts on \( M \) by isometries, \( G \) can be lifted to a \( G \)-action on \( u(M) \). It is proved in [22] that, if the scalar curvature is positive, \( u(M) \) has a hyperkähler structure which is preserved by the lifted \( G \)-action. Let \( \hat{\mu} \) denote the moment map of the lifted \( G \)-action on \( u(M) \). By [22] Lemma 4.4 \( \hat{\mu} = \mu \circ \pi \).

**Lemma 3.6.** Let \( M \) be a positive quaternionic Kähler manifold of dimension \( 4n \). Assume that \( S^1 \) acts on \( M \) effectively by isometries. Let \( \mu \in \Gamma(S^2H) \) be its moment map. If \( N \subset \mu^{-1}(0) \) is a fixed point component of codimension 4, then \( N = \mu^{-1}(0) \).

**Proof.** Let \( u(M) \) be as above. By Proposition 4.2 of [5], at the fixed point \( x \in N \), the isotropy representation of \( S^1 \) in \( SO(3) \cong \text{Aut}(u(M)_x) \) factors through a finite group and hence the representation is trivial, where \( \text{Aut}(u(M)_x) \) is the isomorphism group of the fiber at \( x \) preserving the quaternionic structure, and \( u(M)_x = \pi^{-1}(x) \) is the fiber of the bundle at \( x \). Therefore, \( \pi^{-1}(N) \) is also a fixed point component of the lifted \( S^1 \)-action on \( u(M) \) of codimension 4.

By [22] Lemma 4.4 we see that \( \pi^{-1}(N) \subset \hat{\mu}^{-1}(0) \), where \( \hat{\mu} \) is the moment map for the lifted \( S^1 \)-action on \( u(M) \). Now \( S^1 \) acts on the normal slice of \( \pi^{-1}(N) \) in \( u(M) \) through a representation in \( Sp(1) \). For dimension reasoning, this representation is faithful, otherwise, a finite order subgroup of \( S^1 \) acts trivially on the whole manifold \( u(M) \) and so on \( M \), a contradiction to the effectiveness of the action from our assumption. Therefore, \( S^1 \) acts semi-freely on a neighborhood of \( \pi^{-1}(N) \) in \( u(M) \). By now we may apply Theorem 3.1 to show that \( \pi^{-1}(N) \) is a connected component of \( \hat{\mu}^{-1}(0) \). Since the moment map \( \hat{\mu} \) projects to the moment map \( \mu \), we conclude \( N \) is also a connected component of \( \mu^{-1}(0) \). By [2] \( \mu^{-1}(0) \) is connected, thus \( N = \mu^{-1}(0) \), the desired result follows. \( \square \)
4. Proof of Theorem 1.2

Theorem 1.2 follows readily from the following Lemma and Theorem 2.1, where the dimension bound \( m \geq 3 \) implies that the fixed point component of codimension 4 has to be contained in \( \mu^{-1}(0) \), by Proposition 3.5.

**Lemma 4.1.** Let \( M \) be a positive quaternionic Kähler \( 4n \)-manifold with an isometric \( S^1 \)-action where \( m \geq 3 \). Let \( \mu \) be the moment map. Assuming \( b_2(M) = 0 \). If \( N \subset \mu^{-1}(0) \) is a fixed point component of dimension \( 4m - 4 \) of the circle action, then \( M \) is isometric to \( \mathbb{H}^{Pm} \).

**Proof.** By Lemma 3.5 \( \mu^{-1}(0) = N \), therefore \( S^1 \) acts trivially on \( \mu^{-1}(0) \).

By Theorem 3.4

\[
\hat{P}_t(M) = \hat{P}_t(N) + \sum_F t^{\lambda_F} \hat{P}_t(F)
\]

where \( F \) runs over fixed point components outside \( N \), and \( \lambda_F \) the Morse index of \( F \). By Proposition 3.5 the Morse index \( \lambda_F \geq 2n \) and are all even numbers. Thus the inclusion \( N \to M \) is a \((2n - 1)\)-equivalence.

By [2] Lemma 2.2 \( \hat{P}_t(M) = P_t(M)P_t(BS^1) \). Since \( S^1 \) acts trivially on \( F \) and \( N \), we get that \( \hat{P}_t(F) = P_t(F)P_t(BS^1) \) and \( \hat{P}_t(N) = P_t(N)P_t(BS^1) \). The above identity reduces to

\[
P_t(M) - P_t(N) = \sum_F t^{\lambda_F} P_t(F)
\]

Observe that the last two terms of the left hand side, according to increasing degree in \( t \), is \( b_2(M)t^{4m-2} + t^{4m} \) by Poincaré duality.

If \( F \) is a fixed point component outside \( \mu^{-1}(0) \) such that \( \dim R F > 0 \), we claim that \( \dim R F + \lambda_F \leq 4m - 4 \). Otherwise, by the above identity \( \dim R F + \lambda_F = 4m - 2 \) is impossible, and if the even integer \( \dim R F + \lambda_F = 4m \), we conclude that the coefficients of \( t^{4m-2} \) of the right hand side is also non-zero, since \( F \) must be a compact Kähler manifold (by Proposition 3.4) and so \( P_t(F) \) has nonzero coefficient at every even degree not larger than the dimension. A contradiction.

By the above, the identity also implies that there is no isolated fixed point outside \( \mu^{-1}(0) \) with Morse index \( 4m - 2 \), and there is an isolated fixed point with Morse index \( 4m \).

Put all together, by Morse theory, up to homotopy equivalence,

\[
M \simeq N \cup_i e^{\lambda_i} \cup e^{4m}
\]

where \( 2m \leq \lambda_i \leq \dim R F + \lambda_F \leq 4m - 4 \), and \( e^i \) denotes cell of dimension \( i \). Therefore \( H^{4m-2}(M, N) = 0 \). By duality \( H_2(M - N) \cong H^{4m-2}(M, N) = 0 \). Since the codimension of \( N \) is 4, it follows that \( H_2(M) \cong H_2(M - N) = 0 \). Therefore by Theorem 2.1 \( M = \mathbb{H}^{Pm} \). The desired result follows. \( \square \)
5. Proof of Theorem 1.1

Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. We call the rank of the isometry group $\text{Isom}(M)$ the symmetry rank of $M$, denoted by $\text{rank}(M)$. By [6] we know that $\text{rank}(M) \leq m + 1$.

Proof of Theorem 1.1. Let $r = \text{rank}(M)$. Consider the isometric $T^r$-action on $M$. Note that the $T^r$-action on $M$ must have non-empty fixed point set since the Euler characteristic $\chi(M) > 0$ by [20]. Consider the isotropy representation of $T^r$ at a fixed point $x \in M$, which must be a representation through the local linear holonomy $Sp(m)Sp(1)$ representation at $T_xM \cong \mathbb{H}^m$. If there is a stratum (a fixed point set of an isotropy group of rank $\geq 1$) of codimension 4, then it must be contained in $\mu^{-1}(0)$ if $m \geq 3$ (by [2] or Proposition 3.6). By Theorem 2.1 and Lemma 4.1 the desired result follows. Thus we can assume that at $x$, the isotropy representation does not have any codimension 4 linear subspace fixed by some rank 1 subgroup of $T^r$. Let $N$ be a maximal dimensional submanifold of $M$ passing through $x$ fixed by a circle subgroup of $T^r$.

Case (i): If $m \equiv 0 \pmod{2}$.

By the above assumption $4m - 8 \geq \dim N \geq 2m + 4$ since $\text{rank}(N) = r - 1 \geq \frac{m}{2} + 2$, by Lemma 2.1 of [6]. Note that $N$ is a quaternionic Kähler manifold since $N \subset \mu^{-1}(0)$. By Theorem 2.4 we see that $\pi_2(N) \cong \pi_2(M)$. By Theorem 2.1 it suffices to prove $\pi_2(N) = 0$ or $\mathbb{Z}$. By induction we may consider $T^r$-action on $N$, and applying Lemma 4.1 once again. Finally it suffices to consider the case where a 16-dimensional quaternionic Kähler submanifold of $M$, $M^{16}$, with an effective isometric action by torus of rank $\geq 5$. In this case there is a quaternionic Kähler submanifold $M^{12} \subset M^{16}$ and fixed by a circle group (cf. [6]). By Lemma 4.1, Theorem 2.1 and Theorem 2.4 $M^{16} = \mathbb{H}P^4$ or $\text{Gr}_2(\mathbb{C}^6)$, the desired result follows.

Case (ii): If $m \equiv 1 \pmod{2}$.

Similar to the above $\dim N \geq 2m + 6$ for the same reasoning. By Theorem 2.4 $\pi_2(N) \cong \pi_2(M)$. The same argument by induction reduces the problem to the case of a quaternionic Kähler submanifold of dimension 20, $M^{20}$, with an effective isometric torus action of rank $\geq 6$. Once again the argument in [6] shows that $M^{20}$ has a quaternionic submanifold $M^{16}$ of rank $\geq 5$. By (i) we see that $M^{16} = \mathbb{H}P^4$ or $\text{Gr}_2(\mathbb{C}^6)$. By Theorem 2.1 and Theorem 2.4 again we complete the proof. □

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