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CONGRUENCES BETWEEN ABELIAN PSEUDOMEASURES

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Dedicated to Professor Peter Roquette on his 80^{th} birthday

In this paper K is a totally real number field (finite over $\mathbb{Q}$), p a fixed odd prime number, and S a fixed finite set of non-archimedian primes of K containing all primes above p. Let K_S denote the maximal abelian extension of K which is unramified (at all non-archimedean primes) outside S and set $G_S = G(K_S/K)$. Serre's pseudomeasure $\lambda_K = \lambda_{K,S}$ has the property that $(1-g)\lambda_K$ is in the completed group ring $\mathbb{Z}_p[[G_S]]$ for all $g \in G_S$ [Se2].

Let L be a totally real Galois extension of K of degree p with group $\Sigma = G(L/K)$. Moreover, assume that the finite set S of places contains all the primes of K which ramify in L. Let L_S , $H_S = G(L_S/L)$ and λ_L be the corresponding objects over L with respect to the set of primes of L above S. Observe that L_S is a Galois extension of K, with group $\mathfrak{G}$, hence G_S is its maximal abelian factor group. Also, since H_S is a normal subgroup of $\mathfrak{G}$ of index p, the quotient $\Sigma = \mathfrak{G}/H_S$ acts on H_S by conjugation. This situation induces the transfer map $\mathrm{ver} : G_S \to H_S$ by means of which we can compare λ_K and λ_L.

Theorem.: *For $g_K \in G_S$ and $h_L = \mathrm{ver}(g_K) \in H_S$,*

$$\mathrm{ver}(\lambda_{g_K}) \equiv \lambda_{h_L} \mod T \ ,$$

where $\lambda_{g_K} = (1-g_K)\lambda_{K,S}$, $\lambda_{h_L} = (1-h_L)\lambda_{L,S}$, and where T is the ideal in the ring $\mathbb{Z}_p[[H_S]]^\Sigma$ of Σ-fixed points of $\mathbb{Z}_p[[H_S]]$ consisting of all Σ-traces $\sum_{\sigma\in\Sigma}\alpha^\sigma$, $\alpha \in \mathbb{Z}_p[[H_S]]$.

The proof follows from Deligne and Ribet [DR] by interpreting it on the Galois side as in [Se2]. Explicitly, the group $\varprojlim G_{\mathfrak{f}}$ of [DR, p.230], with $\mathfrak{f}$ running through the integral ideals of K with all prime factors in S, is identified with our G_S, via class field theory.

The theorem implies the "torsion congruences" of [RW, §3] in general, and thus the proof of the 'main conjecture of equivariant Iwasawa theory' of [RW2, §4] [1] is reduced to proving the integrality of the logarithmic pseudomeasure t of [RW, §2].

More precisely, let L_∞ be the cyclotomic $\mathbb{Z}_p$-extension of L and $\Gamma_L = G(L_\infty/L)$. If the element $g_K \in G_S$ (as above) has infinite order, then the image of $(1-h_L)$ under $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[\Gamma_L]]$ is not in $p\mathbb{Z}_p[[\Gamma_L]]$. Letting $\mathbb{Z}_p[[H_S]]_\bullet$ be the localization obtained

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[1]or the 'main conjecture of noncommutative Iwasawa theory for p-adic Lie groups of dimension 1' (see [K, §3])

by inverting the multiplicative set of elements of $\mathbb{Z}_p[[H_S]]$ whose image in $\mathbb{Z}_p[[\Gamma_L]]$ is not in $p\mathbb{Z}_p[[\Gamma_L]]$, the theorem reads

$$\mathrm{ver}(\lambda_{K,S}) \equiv \lambda_{L,S} \mod T_\bullet$$

with $T_\bullet$ the Σ-trace ideal in $\mathbb{Z}_p[[H_S]]_\bullet^\Sigma$. If M is totally real and Galois over K with $L_\infty \subset M \subset L_S$ and $[M : L_\infty]$ finite, then the "torsion congruence" is obtained by specializing $\mathfrak{G} \to G(M/K)$. Moreover, for p-extensions M/K the "torsion congruences" also imply the "logarithmic congruences" of [RW], so that we get a proof of many cases of the "main conjecture" of [RW2,FK] complementing the Heisenberg extensions of Kato [2].

Here is a short description of the individual sections to follow. In §1 we write λ_{g_K} as a limit element in $\lim\limits_{\overleftarrow{U}} \mathbb{Z}_p[G_S/U]/p^{m(U)}$ with U open in G_S and with certain integers $m(U)$. This allows us to study the claimed congruence on finite level, which is carried out in §2. The next section is some preparation concerning Hilbert modular forms that we need for the proof of the theorem, which in §4 is combined with the work of Deligne and Ribet to finish the proof. A final section briefly discusses a weaker version of the theorem when $p = 2$.

1. Approximations to pseudomeasures

We review the construction of pseudomeasures [Se2] in a more explicit form that will be essential for our purposes. We first fix notation.

For a coset x of an open subgroup U of G_S set $\delta^{(x)}(g) = 1$ or 0 according to the cases $g \in x$ or $g \notin x$. Then, for integers $k \geq 1$, define $\zeta_K(1-k, \delta^{(x)}) = \zeta_{K,S}(1-k, \delta^{(x)}) \in \mathbb{Q}$ to be the value at $1 - k$ of the partial ζ-function for the set of integral ideals $\mathfrak{a}$ of K prime to S with Artin symbol $g_\mathfrak{a}$ in x [3]. Note that the definition of $\zeta_K(1 - k, \delta^{(x)})$ extends linearly to locally constant functions ε on G_S with values in a $\mathbb{Q}$-vector space and gives values $\zeta_K(1 - k, \varepsilon)$ in that vector space.

Let $\mathcal{N} = \mathcal{N}_{K,p} : G_S \to \mathbb{Z}_p{}^\times$ be the continuous character whose value on $g_\mathfrak{a}$ for an integral ideal $\mathfrak{a}$ prime to S is its absolute norm $\mathcal{N}\mathfrak{a}$ [4]. For $g \in G_S$, $k \geq 1$ and ε a locally constant $\mathbb{Q}_p$-valued function on G_S we define, following [DR],

$$\Delta_g(1 - k, \varepsilon) = \zeta_K(1 - k, \varepsilon) - \mathcal{N}(g)^k \zeta_K(1 - k, \varepsilon_g) \in \mathbb{Q}_p\,,$$

where $\varepsilon_g(g') = \varepsilon(gg')$ for $g' \in G_S$.

We can now state

Theorem [(0.4) of [DR]].: *Let $\varepsilon_1, \varepsilon_2, \dots$ be a finite sequence of locally constant functions $G_S \to \mathbb{Q}_p$ so that $\sum_{k\geq1} \varepsilon_k(g')\mathcal{N}(g')^{k-1} \in \mathbb{Z}_p$ for all $g' \in G_S$. Then*

$$\sum_{k\geq1} \Delta_g(1 - k, \varepsilon_k) \in \mathbb{Z}_p \quad \text{for all } g \in G_S\,.$$

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[2] see [RW, 'Added in proof']

[3] so $\zeta_K(1 - k, \delta^{(x)}) = \zeta_S(x, 1 - k)$ in [Se2], and $= L(1 - k, \delta^{(x)})$ in [DR], up to identification

[4] i.e., $\mathcal{N}_p$ is the cyclotomic character, so $\mathcal{N}_p(g)$ is determined by the action of g on p-power roots of unity (see [Se2, (2.3)]

Call an open subgroup U of G_S *admissible*, if $\mathcal{N}(U) \subset 1 + p\mathbb{Z}_p$, and define $m(U) \geq 1$ by $\mathcal{N}(U) = 1 + p^{m(U)}\mathbb{Z}_p$.

Lemma 1.: *If U runs through the cofinal system of admissible open subgroups of G_S, then $\mathbb{Z}_p[[G_S]] = \varprojlim_U \mathbb{Z}_p[G_S/U]/p^{m(U)}\,\mathbb{Z}_p[G_S/U]$.*

Proof. The natural map

$$\mathbb{Z}_p[[G_S]] = \varprojlim_U \mathbb{Z}_p[G_S/U] \to \varprojlim_U \mathbb{Z}_p[G_S/U]/p^{m(U)}$$

is injective, since $m(U_1) \geq m(U_2)$ for $U_1 \leq U_2$ and since the $m(U)$'s are unbounded. In order to show surjectivity, it is sufficient to find a linearly ordered cofinal family $\{U'\}$ of open subgroups, because then it follows that the image of $\varprojlim_{U'} \mathbb{Z}_p[G_S/U']/p^{m(U')}$ is dense in the compact group $\varprojlim_{U'} \mathbb{Z}_p[G_S/U']/p^{m(U')}$, by taking successive approximations which are compatible with the projections. Now, G_S is finitely generated (over $\hat{\mathbb{Z}}$), as the inertia groups for the $\mathfrak{p} \in S$ are finitely generated and they together generate an open subgroup (the fixed field of which is the strict Hilbert class field of K). Thus G_S is a homomorphic image of a finite product $\prod \hat{\mathbb{Z}}$ and hence the closed subgroup $(G_S)^{n!}$ has index dividing the finite order of $\prod(\hat{\mathbb{Z}}/n!)$ and so is open. □

Proposition 2.: *For $g \in G_S$ there is a unique element $\lambda_g \in \mathbb{Z}_p[[G_S]]$, independent of k* [5]*, whose image in $\mathbb{Z}_p[G_S/U]/p^{m(U)}$ is*

$$\sum_{x \in G_S/U} \Delta_g(1-k, \delta^{(x)})\mathcal{N}(x)^{-k}x \mod p^{m(U)}\mathbb{Z}_p[G/U],$$

for all admissible U, where $\mathcal{N}$ here also denotes the homomorphism $G_S/U \to (\mathbb{Z}_p/p^{m(U)})^\times$ induced by our previous $\mathcal{N}$. Moreover, if λ is the pseudomeasure of [Se2], then

$$(1-g)\lambda = \lambda_g.$$

Note first that the displayed elements are well-defined by the definition of $m(U)$ and that, varying U, they determine a limit element $\lambda_g \in \mathbb{Z}_p[[G_S]]$, since Δ_g is a $\mathbb{Z}_p$-valued distribution (see [DR, (0.5)]).

We check that λ_g is independent of k. Fix U and a coset x. Choose a (set) map $\eta : G_S/U \to {\mathbb{Z}_p}^\times$ so that $\eta(g'U) \equiv \mathcal{N}^{k-1}(g') \mod p^{m(U)}$ for all $g' \in G_S$. Viewing η as a locally constant function on G_S, then

$$\Delta_g(0, \delta^{(x)}\eta) \equiv \Delta_g(1-k, \delta^{(x)}) \mod p^{m(U)}.$$

To see this, apply Theorem [(0.4) of [DR]], repeated above, with $\varepsilon_1 = p^{-m(U)}\delta^{(x)}\eta$, $\varepsilon_k = -p^{-m(U)}\delta^{(x)}$ (and the other ε's zero). Hence, with $\tilde{x} \in x$,

$$\begin{aligned}&\Delta_g(1-k, \delta^{(x)})\mathcal{N}\tilde{x}^{-k} \equiv \Delta_g(0, \delta^{(x)}\eta)\eta(x)^{-1}\mathcal{N}\tilde{x}^{-1} \\ &= \Delta_g(0, \eta(x)^{-1}\delta^{(x)}\eta)\mathcal{N}\tilde{x}^{-1} = \Delta_g(0, \delta^{(x)})\mathcal{N}\tilde{x}^{-1} \mod p^{m(U)}.\end{aligned}$$

We next check that our λ_g satisfies

$$\langle \varepsilon\mathcal{N}^k, \lambda_g\rangle = \Delta_g(1-k, \varepsilon)$$

[5] This allows us to take $k > 2$ to avoid difficulties with $K = \mathbb{Q}$.

(compare [Se2, (3.6)]). As above, choose $\eta : G_S/U \to \mathbb{Z}_p{}^\times$ so that now $\eta(yU) \equiv \mathcal{N}^k y$ mod $p^{m(U)}$. Then, by [Se2, (1.1)],

$$\begin{aligned}&\langle \varepsilon\mathcal{N}^k, \lambda_g\rangle \equiv \langle \varepsilon\eta, \lambda_g\rangle \equiv \textstyle\sum_x \varepsilon\eta(x)\Delta_g(1-k,\delta^{(x)})\mathcal{N}x^{-k}\\ &\equiv \textstyle\sum_x \varepsilon(x)\Delta_g(1-k,\delta^{(x)}) = \Delta_g(1-k, \sum_x \varepsilon(x)\delta^{(x)}) = \Delta_g(1-k,\varepsilon) \mod p^{m(U)}\ .\end{aligned}$$

By the argument following [Se2, (3.6)] it follows that $(1-g)\lambda$ is equal to our λ_g for all $g \in G_S$. □

2. Transfer

Let L/K be as in the introduction. We decorate objects which depend on L and are analogous to the ones of K appropriately, e.g. $\mathcal{N}_L, m_L, \ldots$; in particular we have the notion of admissible open subgroups of H_S. Note that if V is such an admissible open subgroup of H_S, then $\bigcap_{\sigma\in\Sigma} V^\sigma$ is also open and therefore the system of Σ-stable admissible open subgroups of H_S is a cofinal system of open subgroups of H_S.

Lemma 3.: (1) *If V is an admissible open subgroup of H_S and U is an admissible open subgroup of G_S contained in* $\mathrm{ver}^{-1}(V)$*, then* $m_K(U) \geq m_L(V) - 1$.

(2) *Let y be a coset of a Σ-stable admissible open subgroup of H_S. If $h \in H_S$ is fixed by Σ, then* $\Delta_h(1-k, \delta_L^{(y^\sigma)}) = \Delta_h(1-k,\delta_L^{(y)})$ *, where* $\Delta_h = \Delta_{L,h}$*. In particular,* λ_{h_L} *is fixed by* Σ*.*

The first assertion uses $\mathcal{N}_L(\mathrm{ver}(g)) = \mathcal{N}_K(g)^p$ for $g \in G_S$. Now $U \leq \mathrm{ver}^{-1}(V)$ implies $\mathrm{ver}(U) \leq V$, hence $\mathcal{N}_L(V) \supset \mathcal{N}_L(\mathrm{ver}(U)) = \mathcal{N}_K(U)^p$, i.e., $1 + p^{m_L(V)}\mathbb{Z}_p \supset (1+p^{m_K(U)}\mathbb{Z}_p)^p = 1 + p^{m_K(U)+1}\mathbb{Z}_p$. Thus $m_K(U) + 1 \geq m_L(V)$.

For the second assertion it suffices to show that $\zeta_L(1-k,\delta_L^{(y^\sigma)}) = \zeta_L(1-k,\delta_L^{(y)})$ for all y, because $(\delta_L^{(y^\sigma)})_h = \delta_L^{(h^{-1}y^\sigma)} = \delta_L^{((h^{-1}y)^\sigma)}$ and $\delta_L^{(h^{-1}y)} = (\delta_L^{(y)})_h$. Now view $\delta_L^{(y)}$ as a complex valued function on H_S/V and write it as a $\mathbb{C}$-linear combination of the (abelian) characters χ of H_S/V. It suffices to check whether $\zeta_L(1-k,\chi) = \zeta_L(1-k,\chi^\sigma)$, with $\chi^\sigma(h) = \chi(h^{\sigma^{-1}}) = \chi(\sigma h\sigma^{-1})$. But this follows from the compatibility of the Artin L-functions with induction, because $\mathrm{ind}_{H_S/V}^{\mathfrak{G}/V}\chi = \mathrm{ind}_{H_S/V}^{\mathfrak{G}/V}\chi^\sigma$.

This finishes the proof. □

Let N be the kernel of $\mathrm{ver} : G_S \to H_S$. A Σ-stable admissible open subgroup V of H_S gives rise to the transfer map $G_S/U \to H_S/V$ whenever $U \leq \mathrm{ver}^{-1}(V)$. These transfer maps combined yield the right hand map in the commutative square

$$\begin{array}{ccc}\mathbb{Z}_p[[G_S]] & \to & \varprojlim_{U\geq N} \mathbb{Z}_p[G_S/U]/p^{m_K(U)}\\ \mathrm{ver}\downarrow & & \downarrow\\ \mathbb{Z}_p[[H_S]] & \xrightarrow{\simeq} & \varprojlim_{V,\,\Sigma-\mathrm{stable}} \mathbb{Z}_p[H_S/V]/p^{m_L(V)-1}\end{array}\ ,$$

explicitly sending $(x_U)_U$ to $(y_V)_V$ with y_V the image of x_U under $\mathbb{Z}_p[G_S/U]/p^{m_K(U)} \xrightarrow{\mathrm{ver}} \mathbb{Z}_l[H_S/V]/p^{m_K(U)} \to \mathbb{Z}_p[H_S/V]/p^{m_L(V)-1}$ whenever $U \leq \mathrm{ver}^{-1}(V)$. The bottom arrow is an isomorphism by the proof of Lemma 1.

We recall that a locally constant function ε_L on H_S is *even*, if $\varepsilon_L(c_w h) = \varepsilon_L(h)$ for all $h \in H_S$ and all "Frobenius elements" c_w at the archimedean primes w of L (so

$c_w \in H_S$ is the restriction of complex conjugation with respect to an embedding $L_S \hookrightarrow \mathbb{C}$ inducing w on L).

Set $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$.

PROPOSITION 4.: *A sufficient condition for the Theorem in the introduction to hold is the following:*

$$\Delta_{h_L}(1-k, \varepsilon_L) \equiv \Delta_{g_K}(1-pk, \varepsilon_L \circ \mathrm{ver}) \mod p\mathbb{Z}_p$$

for all even locally constant $\mathbb{Z}_{(p)}$-valued functions ε_L on H_S satisfying $\varepsilon_L^\sigma = \varepsilon_L$ $(\forall \sigma \in \Sigma)$ with $\varepsilon_L^\sigma(h) = \varepsilon_L(h^{\sigma^{-1}})$.

Proof. Look at the coordinates of

$$\lambda_{h_L} \text{ and } \mathrm{ver}(\lambda_{g_K}) \text{ in } \mathbb{Z}_p[H_S/V]/p^{m_L(V)-1}$$

for a Σ-stable admissible open subgroup $V \leq H_S$ containing the group C generated by all elements c_w. Note that $\mathrm{ver}(\lambda_{g_K})$ is then the image under 'ver' of the U-coordinate of λ_{g_K}, where $U = \mathrm{ver}^{-1}(V) \leq G_S$ contains N. These coordinates are the images of

(i) $\sum_{y \in H_S/V} \Delta_{h_L}(1-k, \delta_L^{(y)}) \mathcal{N}_L(y)^{-k} y$,

respectively

(ii) $\sum_{x \in G_S/U} \Delta_{g_K}(1-pk, \delta_K^{(x)}) \mathcal{N}_K(x)^{-pk} \mathrm{ver}(x)$

in $(\mathbb{Z}_p[H_S/V]/p^{m_L(V)-1})^\Sigma$ by Proposition 2 (recall that it asserts independence of λ_g from k).

We show that the sums in (i),(ii) are congruent modulo $T(V)$, where $T(V)$ is the Σ-trace ideal in $(\mathbb{Z}_p[H_S/V]/p^{m_L(V)-1})^\Sigma$, by distinguishing two cases:

(1) *y is fixed by Σ.* Then $\delta_L^{(y)}$ is an ε_L as appearing in the proposition and so $\Delta_{h_L}(1-k, \delta_L^{(y)}) \equiv \Delta_{g_K}(1-pk, \delta_L^{(y)} \circ \mathrm{ver}) \mod p$. Now, if $y = \mathrm{ver}(x)$, then, because $\mathrm{ver} : U/N \to V$ is an isomorphism, x is uniquely determined by y and $\mathcal{N}_L(y)^{-k} = \mathcal{N}_L(\mathrm{ver}(x))^{-k} = \mathcal{N}_K(x)^{-pk}$. Moreover, $\delta_L^{(y)} \circ \mathrm{ver} = \delta_K^{(x)}$. Hence the corresponding summands in (i) and (ii) cancel out modulo $T(V)$, since $p\alpha$ is a Σ-trace whenever α is Σ-invariant. However, if $y \notin \mathrm{im}\,(\mathrm{ver})$, then $\delta_L^{(y)} \circ \mathrm{ver} = 0$, hence the y-summand vanishes modulo $T(V)$.

(2) *y is not fixed by Σ.* By 2. of Lemma 3, $\Delta_{h_L}(1-k, \delta_L^{(y)}) = \Delta_{h_L}(1-k, \delta_L^{(y^\sigma)})$, whence the Σ-orbit of y yields the sum $\Delta_{h_L}(1-k, \delta_L^{(y)}) \mathcal{N}_L(y)^{-k} \sum_{\sigma \in \Sigma} y^\sigma$ which is in $T(V)$.

Now subtracting type (ii) sums from type (i) sums for all Σ-stable admissible open $V \geq C$ gives a compatible system of elements in $\varprojlim_{V \geq C} T(V) \subset \varprojlim_{V \geq C} \mathbb{Z}_p[H_S/V]/p^{m_L(V)-1}$.

Set $H_S^+ = H_S/C$; so $H_S^+ = G(L_S^+/L)$ where L_S^+ is the maximal totally real subfield of L_S. Since $T(V_1) \to T(V)$ is surjective whenever $V_1 \leq V$, we get a limit $s^+ \in T^+ \subset \mathbb{Z}_p[[H_S^+]]$. Thus the proposition follows from

LEMMA 5.: *Suppose that $s \stackrel{\text{def}}{=} \lambda_{h_L} - \mathrm{ver}(\lambda_{g_K}) \in \mathbb{Z}_p[[H_S]]^\Sigma$ has image s^+ under $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$ in the Σ-trace ideal T^+ in $\mathbb{Z}_p[[H_S^+]]^\Sigma$. Then $s \in T$.*

Proof. We know, from [Se2, (3.12)], that the Frobenius elements $c_v \in G_S$ for the real primes v of K satisfy $c_v^2 = 1$, $c_v \lambda_{g_K} = \lambda_{g_K}$, and that they generate the kernel of $G_S \to G_S^+$. Put $c_K = \prod_v (1 + c_v)$.

The analogous properties hold for the c_w for the real primes w of L, and we can form c_L. Moreover,

$$c_L \equiv \mathrm{ver}(c_K) \mod T\,.$$

To see this, expand c_L in a sum of products of c_w's and consider the Σ-action on the summands. The sum of each orbit of length p is in T and the products fixed by Σ add up to $\mathrm{ver}(c_K)$, because $\mathrm{ver}(c_v) = \prod_{w|v} c_w$ for every v.

Now $s^+ \in T^+$ and the surjectivity of $T \to T^+$ mean that $s^+ = t^+$ for some $t \in T$, hence $s - t$ is in the kernel of $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$ which is generated by all $1 - c_w$ as a $\mathbb{Z}_p[[H_S]]$-module. Then $c_L(s-t) = 0$, implying $c_L s \in T$ because $c_L \in \mathbb{Z}_p[[H_S]]^\Sigma$ and T is an ideal of $\mathbb{Z}_p[[H_S]]^\Sigma$.

Moreover $c_L \lambda_{h_L} = 2^{[L:\mathbb{Q}]}\lambda_{h_L}$ and

$$c_L \mathrm{ver}(\lambda_{g_K}) \equiv \mathrm{ver}(c_K)\mathrm{ver}(\lambda_{g_K}) = \mathrm{ver}(c_K\lambda_{g_K}) = 2^{[K:\mathbb{Q}]}\mathrm{ver}(\lambda_{g_K}) \mod T\ .$$

Since $2^p \equiv 2 \mod p$, it follows that $2^{[K:\mathbb{Q}]} s \equiv c_L(\lambda_{h_L} - \mathrm{ver}(\lambda_{g_K})) = c_L s \equiv 0 \mod T$, from which the lemma follows as $p \in T$ is odd. □

3. q-EXPANSIONS

Let $[K:\mathbb{Q}] = r$, let $\mathfrak{f}$ be an integral ideal with all prime factors in S, and denote the usual Hilbert upper half plane associated to K by $\mathfrak{H} = \{\tau \in K \otimes \mathbb{C} : \Im(\tau) \gg 0\}$.

If k is an even positive integer, we define, as usual, the action of $\mathrm{GL}^+(2, K\otimes\mathbb{R})$ of matrices with totally positive determinant on functions $F : \mathfrak{H} \to \mathbb{C}$ by

$$(F_{|k}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right))(\tau) = \mathcal{N}(ad-bc)^{k/2}\mathcal{N}(c\tau+d)^{-k}F(\frac{a\tau+b}{c\tau+d})\,,$$

with $\mathcal{N} : K \otimes \mathbb{C} \to \mathbb{C}$ denoting the norm.

Set

$$\Gamma_{00}(\mathfrak{f}) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,K) : a,d \in 1+\mathfrak{f}\,,\ b \in \mathfrak{D}^{-1}\,,\ c \in \mathfrak{f}\mathfrak{D}\}$$

where $\mathfrak{D}$ is the different of K. A Hilbert modular form F of weight k on $\Gamma_{00}(\mathfrak{f})$ is a holomorphic function $\mathfrak{H} \to \mathbb{C}$ [6] satisfying $F_{|k}M = F$ for all $M \in \Gamma_{00}(\mathfrak{f})$. Denote the space of these by $M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ (see [DR, (5.7)]). Such an F can be expanded as a Fourier series

$$c(0) + \sum_{\substack{\mu\in\mathfrak{o}_K \\ \mu\gg 0}} c(\mu)q^\mu \quad \text{with} \quad q^\mu = e^{2\pi i \mathrm{tr}(\mu\tau)} \quad {}^{7},$$

called the standard q-expansion of F, i.e., the q-expansion at the cusp $\infty = \frac{1}{0}$.

Lemma 6.: *Let $\beta \in \mathfrak{o}_K$ be totally positive with $\mathfrak{f} \subset \beta\mathfrak{o}_K$. There is a Hecke operator U_β on $M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ so that, if $F \in M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ has standard q-expansion as above, then $F_{|k}U_\beta \in M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ has standard q-expansion $c(0) + \sum_{\mu\gg 0} c(\beta\mu)q^\mu$.*

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[6] and holomorphic at infinity, if $K = \mathbb{Q}$

[7] $\mathfrak{o}_K$ is the ring of integers in K; from now on μ will always be in $\mathfrak{o}_K$

Following [AL, §§2,3] for the proof, let $B = \left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right)$ and set $\Omega = B\Gamma_{00}(\mathfrak{f})B^{-1} \cap \Gamma_{00}(\mathfrak{f})$. The matrices $S_\xi = \left(\begin{smallmatrix}1 & \xi\\ 0 & 1\end{smallmatrix}\right)$, with ξ running through a set of coset representatives of $\beta\mathfrak{D}^{-1}$ in $\mathfrak{D}^{-1}$, satisfy $\Gamma_{00}(\mathfrak{f}) = \dot{\bigcup}_\xi \Omega S_\xi$, because $\mathfrak{f} \subset \beta\mathfrak{o}_K$.

Define U_β on $M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ by

$$F_{|k}U_\beta = \mathcal{N}(\beta)^{\frac{k}{2}-1} \sum_\xi F_{|k}B^{-1}S_\xi\,, \text{ with } \mathcal{N} \text{ as above in this section.}$$

Then $F_{|k}B^{-1}$ is modular on $B\Gamma_{00}(\mathfrak{f})B^{-1}$, hence on Ω. The usual averaging argument then shows that $F_{|k}U_\beta$ is modular on $\Gamma_{00}(\mathfrak{f})$. Now,

$$\begin{aligned}
&(F_{|k}U_\beta)(\tau) = \mathcal{N}(\beta)^{-1}\textstyle\sum_\xi F(\beta^{-1}\tau + \beta^{-1}\xi)\\
&= \mathcal{N}(\beta)^{-1}\textstyle\sum_\xi \Big(c(0) + \sum_{\mu\gg 0} c(\mu) e^{2\pi i \mathrm{tr}(\mu\,\frac{\tau+\xi}{\beta})}\Big)\\
&= \mathcal{N}(\beta)^{-1}[\mathfrak{D}^{-1} : \beta\mathfrak{D}^{-1}]c(0) + \textstyle\sum_{\mu\gg0} c(\mu)\Big(\mathcal{N}(\beta)^{-1}\sum_\xi e^{2\pi i \mathrm{tr}_{K/\mathbb{Q}}(\mu\xi/\beta)}\Big)e^{2\pi i\mathrm{tr}(\mu\tau/\beta)}\\
&= c(0) + \textstyle\sum_{\substack{\mu\gg0\\ \mu\in\beta\mathfrak{o}_K}} c(\mu)e^{2\pi i\mathrm{tr}(\mu\tau/\beta)}
\end{aligned}$$

as $\mathcal{N}(\beta) = [\mathfrak{D}^{-1} : \beta\mathfrak{D}^{-1}]$ and as $\xi \mapsto e^{2\pi i \mathrm{tr}_{K/\mathbb{Q}}(\mu\xi/\beta)}$ is a character on $\mathfrak{D}^{-1}/\beta\mathfrak{D}^{-1}$. Since β is totally positive, the proof of the lemma is complete. □

We next discuss restriction of Hilbert modular forms from L to K.

The containment $K \subset L$ induces natural maps $\mathfrak{H}_K \xrightarrow{*} \mathfrak{H}_L$ and $\mathrm{SL}(2, K\otimes\mathbb{R}) \xrightarrow{*} \mathrm{SL}(2, L\otimes\mathbb{R})$. For a holomorphic $F : \mathfrak{H}_L \to \mathbb{C}$ define the restriction $\mathrm{res}\, F : \mathfrak{H}_k \to \mathbb{C}$ of F to be the holomorphic function satisfying $(\mathrm{res}\, F)(\tau) = F(\tau^*)$. Then

$$(\star) \qquad (\mathrm{res}\, F)_{|pk}M = \mathrm{res}\,(F_{|k}M^*) \text{ for } M \in \mathrm{SL}(2, K\otimes\mathbb{R})\,.$$

The q-expansion at a cusp determined by a finite idèle $\alpha \in \hat{K}^\times$ is discussed in [DR, bottom of p.229 and (5.8)].

Lemma 7.: *Let $F \in M_k(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_L), \mathbb{C})$ and let $c(0) + \sum_{\substack{\nu\gg0\\ \nu\in\mathfrak{o}_L}} c(\nu)q_L^\nu$ be its standard q-expansion (with $q_L^\nu = e^{2\pi i\mathrm{tr}_L(\nu\tau)}$). Let $\alpha \in \hat{K}^\times$. Then*

1. *$\mathrm{res}\, F \in M_{pk}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ has standard q-expansion $c(0) + \sum_{\substack{\mu\gg0\\ \mu\in\mathfrak{o}_K}} c_*(\mu)q_K^\mu$ with $c_*(\mu) = \sum_{\substack{\nu\gg0,\,\nu\in\mathfrak{o}_L\\ \mathrm{tr}_{L/K}(\nu)=\mu}} c(\nu)$ (and $q_K^\mu = e^{2\pi i\mathrm{tr}_K(\mu\tau)}$),*
2. *the constant term of $\mathrm{res}\, F$ at the cusp determined by α equals the constant term of F at the cusp determined by $\alpha^* \in \hat{L}^\times$.*

Assertion 1. follows from observing that $\mathrm{tr}_L(\nu\tau^*) = \mathrm{tr}_K(\mathrm{tr}_{L/K}(\nu)\tau)$ for $\nu \in L$, $\tau \in K\otimes\mathbb{C}$, and substituting this into the definition.

For 2., the constant terms in question are those of

$$(\mathrm{res}\, F)_\alpha = (\mathrm{res}\, F)_{|pk}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right) \quad\text{and}\quad F_{\alpha^*} = F_{|k}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)^*,$$

respectively, by [DR, p.229]. By 1., F_{α^*} and $\mathrm{res}\, F_{\alpha^*}$ have the same constant term in their respective standard q-expansion, so it suffices to show $(\mathrm{res}\, F)_\alpha = \mathrm{res}\, F_{\alpha^*}$. For that, decompose $M = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right) \in \mathrm{SL}(2, \hat{K})$ as $M = M_1M_2$ according to $\mathrm{SL}(2, \hat{K}) =$

$\widehat{\Gamma_{00}(\mathfrak{f})} \cdot \mathrm{SL}(2,K)$, hence $M^* = M_1^* M_2^*$ according to $\mathrm{SL}(2,\hat{L}) = \widehat{\Gamma_{00}(\mathfrak{f}\mathfrak{o}_L)} \cdot \mathrm{SL}(2,L)$. Then

$$(\mathrm{res}\,F)_\alpha = (\mathrm{res}\,F)_{|pk} M = (\mathrm{res}\,F)_{|pk} M_2 \overset{(\star)}{=} \mathrm{res}\,(F_{|k} M_2^*) = \mathrm{res}\,(F_{|k} M^*) = \mathrm{res}\,F_{\alpha^*}\,,$$

with equation $\overset{(\star)}{=}$ referring to the formula displayed prior to Lemma 7. □

4. Proof of the main result

We use the notation of the previous section, except that we now also use $\mathcal{N}$ for the norm map $K \to \mathbb{Q}$ and any norm map derived from it [8], as in [DR,§2].

We attach an Eisenstein series of every even weight k to even locally constant $\mathbb{C}$-valued functions ε via [DR, (6.1)].

Proposition 8.: *Let ε be an even locally constant $\mathbb{C}$-valued function on G_S.*

(1) *There is an integral ideal $\mathfrak{f}$ in K with all its prime factors in S and a modular form $G_{k,\varepsilon} \in M_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ with standard q-expansion*

$$2^{-r}\zeta_K(1-k,\varepsilon) + \sum_{\substack{\mu \gg 0 \\ \mu \in \mathfrak{o}_K}} \Big(\sum_{\substack{\mu \in \mathfrak{a} \subset \mathfrak{o}_K \\ \mathfrak{a}\text{ prime to } S}} \varepsilon(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{k-1} \Big) q^\mu$$

where $\varepsilon(\mathfrak{a}) = \varepsilon(g_\mathfrak{a})$ with $g_\mathfrak{a} \in G_S$ the Artin symbol of $\mathfrak{a}$.

(2) *Its q-expansion at the cusp determined by $\alpha \in \hat{K}^\times$ has constant term*

$$\mathcal{N}((\alpha))^k 2^{-r} \zeta_K(1-k, \varepsilon_a)\,,$$

where (α) is the ideal generated by α and $a \in G_S$ is the image of α under the map

$$\text{(2a)} \qquad \hat{K}^\times \xrightarrow{j} G = G(K^{\mathrm{ab}}/K) \twoheadrightarrow G_S$$

with j taken from [DR, (2.22)] and the identification $G = G(K^{\mathrm{ab}}/K)$ as in [DR, p.240], via the Artin symbol on integral ideals prime to $\mathfrak{f}$.

(3) *$\mathcal{N}((\alpha)) = \mathcal{N}(\alpha_p) \cdot \mathcal{N}_p(a)$ where $\alpha_p \in K \otimes \mathbb{Q}_p$ is the p-component of $\alpha \in \hat{K}^\times$ and $\mathcal{N}_p(a) = \mathcal{N}_{K,p}(a)$, as in §1.*

For 1. choose an open subgroup U of G_S so that ε is constant on each coset of G_S/U. Let $\mathfrak{f}$ be an integral ideal which is a multiple of the conductor of the field fixed by U acting on K_S and with all its prime factors in S. Then the Artin symbol maps the strict ideal class group $G_\mathfrak{f}$ onto G_S/U. Viewing $G_\mathfrak{f}$ as the group of invertible elements of $A_\mathfrak{f}$, as in [DR, (2.6)], makes ε a map on $G = \varprojlim_{\mathfrak{f}' \subset \mathfrak{f}} G_{\mathfrak{f}'}$. Finally extend ε to I by zero [9]. In particular, if $\mathfrak{a}$ is an integral ideal prime to S, then $\varepsilon(\mathfrak{a}) = \varepsilon(g_\mathfrak{a})$. Moreover, by [DR, (2.3) and (2.4)], $\varepsilon(\mathfrak{a}) = 0$ for every (fractional) ideal $\mathfrak{a}$ of K which is not integral and prime to S.

Now, with this ε, [DR, (6.2)] gives the standard q-expansion of $G_{k,\varepsilon}$:

$$2^{-r}\zeta_K(1-k,\varepsilon) + \sum_{\substack{\mu \gg 0 \\ \mu \in \mathfrak{o}_K}} \Big(\sum_{\mathfrak{x} \subset \mathfrak{o}_K} \varepsilon(\mu\mathfrak{x}^{-1}) \mathcal{N}(\mu\mathfrak{x}^{-1})^{k-1} \Big) q^\mu\,,$$

[8] hence consistent with our usage in §3

[9] for the definition of I see [DR, §2]

where we have chosen the ideal $\mathfrak{B}$ of [DR] to be $\mathfrak{o}_K$. Set $\mathfrak{a} = \mu\mathfrak{x}^{-1}$, so $\mu \in \mathfrak{a}$, and we may assume that $\mathfrak{a}$ is integral and prime to S, because otherwise ε will be zero on $\mathfrak{a}$. Thus the above μ th coefficient is turned into $\sum_{\substack{\mu\in\mathfrak{a}\subset\mathfrak{o}_K \\ \mathfrak{a}\text{ prime to }S}} \varepsilon(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k-1}$.

For 2., [DR, (6.2)] shows that $\mathcal{N}((\alpha))^k 2^{-r}\zeta_K(1-k,\varepsilon_c)$ is the constant term of the q-expansion at the cusp determined by $\alpha \in \hat{K}^\times$, with $c = j(\alpha)$. Our extension of ε to G has been such that, for $g \in G$, $\varepsilon_c(g) = \varepsilon(cg) = \varepsilon(\overline{cg}) = \varepsilon_{\overline{c}}(\overline{g})$ with $\overline{c}, \overline{g}$ the images of c, g in $\varprojlim_{\mathfrak{f}'\subset\mathfrak{f},\text{ in }S} G_{\mathfrak{f}'}$, where '$\mathfrak{f}'$ in S' means that every prime factor of $\mathfrak{f}'$ is in S. Hence the commutative square

$$\begin{array}{ccc} G & \longrightarrow & \varprojlim_{\mathfrak{f}'\subset\mathfrak{f},\text{ in }S} G_{\mathfrak{f}'} \\ \| & & \| \\ G(K^{\mathrm{ab}}/K) & \longrightarrow & G_S \end{array}$$

shows $\overline{c} = a$, up to identification.

For 3., we get from [DR, (2.12),(2.16)] that the norm of $c = j(\alpha)$ is $\mathcal{N}(\alpha)^{-1}\mathcal{N}((\alpha))$. Thus the p-component of $\mathcal{N}(c) \in \hat{\mathbb{Z}}^\times$ in $\mathbb{Z}_p{}^\times$ is $\mathcal{N}(\alpha_p)^{-1}\mathcal{N}((\alpha))$ since $\mathcal{N}((\alpha)) \in \mathbb{Q}^\times$. On the other hand, the p-component of $\mathcal{N}(c)$ is $\mathcal{N}_p(a)$ by the commutative diagram

$$\begin{array}{ccccc} c \in G & \longrightarrow & \varprojlim_n G_{p^n\mathfrak{o}_K} & \xrightarrow{\mathrm{N}_{K/\mathbb{Q}}} & \varprojlim_n G_{p^n\mathbb{Z}} \\ \downarrow\!\!\!\!\!\!\top \quad \downarrow & & & & \simeq\downarrow \\ a \in G_S & \longrightarrow & \varprojlim_n G(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) & \xrightarrow{\simeq} & \varprojlim_n (\mathbb{Z}/p^n)^\times \end{array}$$

with the left map as in *(2a)* and μ_{p^n} the p^n th roots of unity. Here, the map $G \to \varprojlim_n (\mathbb{Z}/p^n)^\times = \mathbb{Z}_p{}^\times$ around the top row takes c to the p-component of $\mathcal{N}(c)$, which thus is $\mathcal{N}_p(a)$.

The proof of the proposition is complete. □

Lemma 9.: [10] *Let k be an even positive integer and ε_L an even locally constant $\mathbb{Z}_{(p)}$-valued function on H_S. There is an integral ideal $\mathfrak{f} \subset p\mathfrak{o}_K$ with all prime factors in S, so that*

$$E = (\operatorname{res} G_{k,\varepsilon_L})_{|pk} U_p - G_{pk,\varepsilon_L\circ\mathrm{ver}} \quad \text{is in} \quad M_{pk}(\Gamma_{00}(\mathfrak{f}), \mathbb{C}) \quad {}^{11}.$$

If $\varepsilon_L^\sigma = \varepsilon_L$ for all $\sigma \in \Sigma$, then the constant term of the standard q-expansion of E is

$$2^{-pr}\zeta_L(1-k,\varepsilon_L) - 2^{-r}\zeta_K(1-pk,\varepsilon_L\circ\mathrm{ver})$$

and all non-constant coefficients are in $p\mathbb{Z}_{(p)}$.

Choose an $\mathfrak{f} \subset p\mathfrak{o}_K$ by Proposition 8 so that $G_{pk,\varepsilon_L\circ\mathrm{ver}} \in M_{pk}(\Gamma_{00}(\mathfrak{f}),\mathbb{C})$ and $G_{k,\varepsilon_L} \in M_k(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_L),\mathbb{C})$. Using Lemmas 6 and 7, the standard q-expansion of $G_{pk,\varepsilon_L\circ\mathrm{ver}}$ is

$$2^{-r}\zeta_K(1-pk,\varepsilon_L\circ\mathrm{ver}) + \sum_{\substack{\mu\gg 0\\ \mu\in\mathfrak{o}_K}} \Big(\sum_{\substack{\mu\in\mathfrak{a}\subset\mathfrak{o}_K \\ \mathfrak{a}\text{ prime to }S}} \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)\mathcal{N}_K(\mathfrak{a})^{pk-1}\Big) q_K^\mu$$

[10] compare [Ty]

[11] with U_p as in Lemma 6

because $(\varepsilon_L \circ \mathrm{ver})(\mathfrak{a}) = (\varepsilon_L \circ \mathrm{ver})(g_\mathfrak{a}) = \varepsilon_L(\mathrm{ver}(g_\mathfrak{a})) = \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)$ (see [Se1, VII,8]) , and that of $(\mathrm{res}\, G_{k,\varepsilon_L})_{|pk}U_p$ is

$$2^{-pr}\zeta_L(1-k,\varepsilon_L) + \sum_{\substack{\mu \gg 0 \\ \mu \in \mathfrak{o}_K}} \Big(\sum_{\substack{(\mathfrak{b},\nu)\ \mathrm{so}\ \nu\in\mathfrak{b}\subset\mathfrak{o}_L\ ,\ \nu\gg 0 \\ \mathfrak{b}\ \mathrm{prime\ to}\ S\ ,\ \mathrm{tr}_{L/K}(\nu)=p\mu}} \varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-1}\Big) q_K^\mu \ .$$

Hence, the μth coefficient of E is

$$\sum_{(\mathfrak{b},\nu)} \varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-1} - \sum_{\mathfrak{a}} \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)\mathcal{N}_K(\mathfrak{a})^{pk-1}$$

with $(\mathfrak{b},\nu)$ so that $\nu \gg 0, \nu \in \mathfrak{b} \subset \mathfrak{o}_L, \mathfrak{b}$ prime to S, $\mathrm{tr}_{L/K}(\nu) = p\mu$ and $\mathfrak{a} \subset \mathfrak{o}_K$ prime to S.

The group Σ acts on the pairs $(\mathfrak{b},\nu)$ by $(\mathfrak{b},\nu)^\sigma = (\mathfrak{b}^\sigma,\nu^\sigma)$. If Σ moves $(\mathfrak{b},\nu)$, then the orbit sum $\sum_\sigma \varepsilon_L(\mathfrak{b}^\sigma)\mathcal{N}_L(\mathfrak{b}^\sigma)^{k-1} = p\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-1}$ because $\varepsilon_L(\mathfrak{b}^\sigma) = \varepsilon_L^{\sigma^{-1}}(\mathfrak{b}) = \varepsilon_L(\mathfrak{b})$ and $\mathcal{N}_L(\mathfrak{b}^\sigma) = \mathcal{N}_L(\mathfrak{b})$.

However, if Σ fixes $(\mathfrak{b},\nu)$, then $\nu \in K$, $\mathrm{tr}_{L/K}(\nu) = p\mu$, so $\nu = \mu$, and $\mathfrak{b}^\sigma = \mathfrak{b}$, so $\mathfrak{b} = \mathfrak{a}\mathfrak{o}_L$ for a unique integral ideal $\mathfrak{a}$ of K prime to S, since S contains all primes which are ramified in L. Thus $(\mathfrak{b},\nu) = (\mathfrak{a}\mathfrak{o}_L,\mu)$.

The above claim on E now follows from

$$\begin{aligned}&\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-1} = \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)\mathcal{N}_L(\mathfrak{a}\mathfrak{o}_L)^{k-1} = \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)\mathcal{N}_K(\mathfrak{a})^{p(k-1)} \\ &\equiv \varepsilon_L(\mathfrak{a}\mathfrak{o}_L)\mathcal{N}_K(\mathfrak{a})^{pk-1} \mod p\ ,\ \text{ by } \mathcal{N}_K(\mathfrak{a})^{p-1} \equiv 1 \mod p\,. \qquad \square\end{aligned}$$

We finally turn to the Proof of the Theorem stated in the introduction. We check the sufficient conditions for every ε_L as in Proposition 4. These are the ε_L appearing in Lemma 9. With E as in Lemma 9 and $\alpha \in \hat{K}^\times$, let E_α be the q-expansion of E at the cusp determined by α and let $E(\alpha) = \mathcal{N}_K(\alpha_p)^{-pk}E_\alpha$.

Since, by [DR, (2.23)], the map j in $(2a)$ is surjective, there is an idèle $\gamma \in \hat{K}^\times$ which maps to $g_K \in G_S$ by $(2a)$. According to Lemma 9, $E(1) = E_1$ has non-constant coefficients in $p\mathbb{Z}_{(p)}$. Then, by [DR, (0.3) and *Variant: Forms on* $\Gamma_{00}(\mathfrak{f})$ at the end of §5], $E(1) - E(\gamma)$ has constant coefficient in $p\mathbb{Z}_p$. This coefficient is, by Lemmas 6,7,9 and Proposition 8,

$$\begin{aligned}&2^{-pr}\zeta_L(1-k,\varepsilon_L) - 2^{-r}\zeta_K(1-pk,\varepsilon_L\circ\mathrm{ver}) - \\ &\quad \mathcal{N}_K(\gamma_p)^{-pk}\mathcal{N}_K((\gamma))^{pk}\Big[2^{-pr}\zeta_L(1-k,(\varepsilon_L)_{h_L}) - 2^{-r}\zeta_K(1-pk,(\varepsilon_L\circ\mathrm{ver})_{g_K})\Big] \\ &= 2^{-pr}\Big[\zeta_L(1-k,\varepsilon_L) - \mathcal{N}_K(g_K)^{pk}\zeta_L(1-k,(\varepsilon_L)_{h_L})\Big] - \\ &\quad 2^{-r}\Big[\zeta_K(1-pk,\varepsilon_L\circ\mathrm{ver}) - \mathcal{N}_K(g_K)^{pk}\zeta_K(1-pk,(\varepsilon_L\circ\mathrm{ver})_{g_K})\Big] \\ &= 2^{-pr}\Delta_{h_L}(1-k,\varepsilon_L) - 2^{-r}\Delta_{g_K}(1-pk,\varepsilon_L\circ\mathrm{ver}) \ \equiv \\ &\quad 2^{-r}\Big(\Delta_{h_L}(1-k,\varepsilon_L) - \Delta_{g_K}(1-pk,\varepsilon_L\circ\mathrm{ver})\Big) \mod p\end{aligned}$$

where we have used that $\gamma^* \in \hat{L}^\times$ has image $\mathrm{ver}(g_K) = h_L$ under the map $(2a)_L$ as well as $\mathcal{N}_K(g_K)^p = \mathcal{N}_L(h_L)$.

Thus, Proposition 4 finishes the proof. $\square$

5. About $p = 2$

For $p = 2$ the theorem needs to be reformulated because of the "extra" 2-adic divisibilities of [DR]. In view of Lemma 8, we define

$$\tilde{\zeta}_{K,S}(1-k,\varepsilon) = 2^{-r}\zeta_{K,S}(1-k,\varepsilon)\,,$$

whence $\tilde{\Delta}_{g_K}(1-k,\varepsilon) = 2^{-r}\Delta_{g_K}(1-k,\varepsilon)$ takes values in $\mathbb{Z}_2$ for $\mathbb{Z}_2$-valued ε, since an admissible subgroup never admits conductor (1) (see [Ri, §3]). Hence $\tilde{\lambda}_{g_K} = 2^{-r}\lambda_{g_K}$ is in $\mathbb{Z}_2[[G_S]]$ (by e.g. Proposition 2). Following the proof of the theorem now shows that

$$\text{the image of } \operatorname{ver}(\tilde{\lambda}_{g_K}) - \tilde{\lambda}_{h_L} \text{ under } \mathbb{Z}_2[[H_S]] \to \mathbb{Z}_2[[H_S^+]] \text{ is in } T^+\,,$$

in the notation of Lemma 5. But the proof of Lemma 5 does not work anymore. One imagines that the methods of [DR], which gave the extra 2-adic divisibilities in the first place, would also sharpen the conclusion displayed above.

Remark. Actually, we can do the same modification for odd p. The equivariant "main conjecture" of [RW2] is unaffected because $[\mathcal{Q}G_\infty, 2]$ is then in the kernel of $\partial : K_1(\mathcal{Q}(\mathbb{Z}_p[[G_\infty]])) \to K_0T(\mathbb{Z}_p[[G_\infty]])$ (see equation ($\Im$) on p. 550 of [RW2]).

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