AN $S_3$-SYMMETRIC LITTLEWOOD-RICHARDSON RULE

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1. Introduction

Fix Young shapes $\lambda, \mu, \nu \subseteq \Lambda := \ell \times k$. Viewing the Littlewood-Richardson coefficient $C_{\lambda,\mu,\nu}$ as the intersection number of generic translates of the Schubert varieties $X_{\lambda}, X_{\mu}, X_{\nu}$ in the Grassmannian of $\ell$-dimensional planes in $\mathbb{C}^{\ell+k}$, one has the obvious $S_3$-symmetries:

\begin{equation}
C_{\lambda,\mu,\nu} = C_{\mu,\nu,\lambda} = C_{\nu,\lambda,\mu} = C_{\mu,\lambda,\nu} = C_{\nu,\mu,\lambda} = C_{\lambda,\nu,\mu}.
\end{equation}

There is interest in the combinatorics of these symmetries. The Littlewood-Richardson coefficients appear in a variety of contexts in addition to the Schubert calculus context described above, such as the representation theory of the symmetric group/complex general linear groups; see, e.g., [Fu97]. Combinatorially, $C_{\lambda,\mu,\nu}$ arises as follows: Let $s_\lambda := s_\lambda(x_1, x_2, \ldots, x_\ell)$ be the Schur polynomial indexed by $\lambda$; see, e.g., [St99, Chapter 7]. These form a $\mathbb{Z}$-linear basis of the ring $\Lambda(x_1, \ldots, x_\ell)$ of symmetric polynomials in the variables $x_1, \ldots, x_\ell$ (when $\lambda$ ranges over all Young shapes with at most $\ell$ rows), and the Littlewood-Richardson coefficients are the structure constants with respect to this basis. Moreover, $C_{\lambda,\mu,\nu}$ is the coefficient of $s_\lambda s_\mu s_\nu$ in this basis, and thus $C_{\lambda,\mu,\nu}$ is clearly symmetric in the input data $\lambda, \mu, \nu$.

However, previously studied Littlewood-Richardson rules for $C_{\lambda,\mu,\nu}$ manifest at most three of the six symmetries, see, e.g., the original rule proposed by [LiRi34] as well as work of [BeZe91, KnTaWo01, VaPa05, HeKa06], and the references therein.

We construct a carton rule for $C_{\lambda,\mu,\nu}$ that transparently and uniformly explains all of the symmetries (1).

Figure 1 depicts a carton, i.e., a three-dimensional box with a grid drawn rectilinearly on the six faces of its surface. Along the "$\emptyset, T_{\lambda}, T_{\mu}$" face, the grid has $(|\mu|+1) \times (|\lambda|+1)$ vertices (including ones on the bounding edges of the face); the remainder of the grid is similarly determined. Define a carton filling to be an assignment of a Young diagram to each vertex of the grid so the shapes increase one box at a time while moving away from $\emptyset$, and so that, for any subgrid $\alpha - \beta$ the $\gamma - \delta$ Fomin growth assumptions hold:

(F1) if $\alpha$ is the unique shape containing $\gamma$ and contained in $\beta$, then $\delta = \alpha$;
(F2) otherwise there is a unique such shape other than $\alpha$, and this shape is $\delta$.

Notice that these conditions are symmetric in $\alpha$ and $\delta$.

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Initially, assign the shapes $\emptyset$ and $\Lambda$ to opposite corners, as in Figure 1. A standard Young tableau $T \in \text{SYT}(\sigma/\pi)$ of shape $\sigma/\pi$ is equivalent to a shape chain in Young’s lattice, e.g.,

$$
T = \begin{array}{ccc}
1 & 5 \\
2 & 3 \\
4
\end{array}
$$

by starting with the unfilled inner shape, and adding one box at a time to indicate the box containing 1, then 2, etc. Fix a choice of standard tableaux $T_\lambda, T_\mu, T_\nu$ of respective shapes $\lambda, \mu, \nu$. Initialize the indicated edges with the shape chains for these tableaux.

Let $\text{CARTONS}_{\lambda, \mu, \nu}$ be all carton fillings with the above initial data.

**Main Theorem.** The Littlewood-Richardson coefficient $C_{\lambda, \mu, \nu}$ equals $\#\text{CARTONS}_{\lambda, \mu, \nu}$.

![Figure 1.](image)

This rule manifests bijections between $\text{CARTONS}_{\lambda, \mu, \nu}$ and $\text{CARTONS}_{\delta, \epsilon, \zeta}$ for permutations $(\delta, \epsilon, \zeta)$ of $(\lambda, \mu, \nu)$. We remark that it can be stated as counting lattice points of a 0, 1-polytope, and readily extends to the setting of [ThYo06, ThYo07].

The Theorem is proved in Section 2, starting from the jeu de taquin formulation of the Littlewood-Richardson rule, and using Fomin’s growth diagram ideas. In Figure 2 we give an example of the Main Theorem. An extended example is given in Section 3.

### 2. Tableau facts and proof of the Main Theorem

#### 2.1. Tableau sliding.

There is a partial order $\prec$ on the rectangle $\Lambda$ where $x \prec y$ if $x$ is weakly northwest of $y$. Given $T \in \text{SYT}(\nu/\lambda)$ consider $x \in \lambda$, maximal in $\prec$ subject to being less than some box of $\nu/\lambda$. Associate another standard tableau
Figure 2. The “front” three faces of a carton filling for \( C_{(2),(2,1),(1)} = 1 \). The choices of tableaux \( T_\lambda, T_\mu \) and \( T_\nu \) are as shown.

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= rectification(T). Placing these chains one atop another (starting with T's and ending with rectification(T)'s) gives a Fomin growth diagram, which in this case is given in Table 1. Growth diagrams satisfy (F1) and (F2); see Fomin’s [St99, Appendix 2] for more.

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Table 1. A growth diagram

Given a top row U and left column T, define infusion₁(U, T) to be the bottom row of the growth diagram, and infusion₂(U, T) to be the right column. We write infusion(U, T) for the ordered pair (infusion₁(U, T), infusion₂(U, T)). Growth diagrams are transpose symmetric, because (F1) and (F2) are. So one has the infusion involution:

\[
\text{infusion}(\text{infusion}(U, T)) = (U, T).
\]

If U is a straight shape, infusion₁(U, T) = rectification(T), while infusion₂(U, T) encodes the order in which squares were vacated in the jeu de taquin process.

Also, given \( T \in \text{SYT}(\nu/\lambda) \), consider \( x \in \Lambda \setminus \nu \) minimal subject to being larger than some element of \( \nu/\lambda \). The reverse jeu de taquin slide \( \text{revjdt}_x(T) \) of \( T \) into \( x \) is defined similarly to a jeu de taquin slide, except we move into \( x \) the largest of the labels among boxes in \( \nu/\lambda \) covered by \( x \). We define reverse rectification \( \text{revrectification}(T) \) similarly. The first fundamental theorem of jeu de taquin asserts \( (\text{rev})\text{rectification} \) is well-defined.

Fix \( T_\mu \in \text{SYT}(\mu) \). The number of \( T \in \text{SYT}(\nu/\lambda) \) such that rectification(T) = \( T_\mu \) equals \( C_{\lambda,\mu,\nu} = C_{\lambda,\mu} \nu \) where \( \nu \) is the straight shape obtained as the 180-degree rotation of \( \Lambda \setminus \nu \). This is Schützenberger’s jeu de taquin formulation of the Littlewood-Richardson rule, see [St99, Appendix 2].

By a slide we mean either kind of jeu de taquin slide. Consider two equivalence relations on a pair of tableaux \( T \) and \( U \). Tableaux are jeu de taquin equivalent if one can be obtained from the other by a sequence of slides. They are dual equivalent if any such sequence results in tableaux of the same shape [Ha92]. Facts: Tableaux of the same straight shape are dual equivalent. A common application of slides to dual equivalent tableaux produces dual equivalent tableaux. A pair of tableaux that are both jeu de taquin and dual equivalent must be equal.

Recall Schützenberger’s evacuation map. For \( T \in \text{SYT}(\lambda) \), let \( \hat{T} \) be obtained by erasing the entry 1 (in the northwest corner c) of \( T \) and subtracting 1 from the remaining entries. Let \( \Delta(T) = \text{jdt}_c(T) \). The evacuation \( \text{evac}(T) \in \text{SYT}(\lambda) \) is defined by the shape chain

\[
\emptyset = \text{shape}(\Delta^{|\lambda|}(T)) - \text{shape}(\Delta^{|\lambda|-1}(T)) - \ldots - \text{shape}(\Delta^{|\lambda|-|\lambda|}(T)) - T.
\]

This map is an involution: \( \text{evac}(\text{evac}(T)) = T \).
2.2. Proof of the rule. Given \( T \in \text{SYT}(\alpha) \) for a straight shape \( \alpha \), define \( \tilde{T} \in \text{SYT}(\text{rotate}(\alpha)) \) where \( \text{rotate}(\alpha) = \Lambda \setminus \alpha^\vee \) by computing \( \text{evac}(T) \in \text{SYT}(\alpha) \), replacing entry \( i \) with \( |\alpha| - i + 1 \) throughout and rotating the resulting tableau 180-degrees and placing it at the bottom right corner of \( \Lambda \).

We need the following well-known fact. The proof we give extends straightforwardly to the setup of \([\text{ThYo06, ThYo07}]\), allowing a “cominuscule” version of the Main Theorem.

**Lemma 2.1.** Suppose \( \alpha, \beta \) and \( \gamma \) are shapes where
\[
\text{SYT}(\beta), \text{SYT}(\gamma^\vee/\alpha) \text{ and rectification}(\text{SYT}(\gamma^\vee/\alpha)) = \text{SYT}(\beta).
\]
Then \( \text{revrectification}(\text{SYT}(\gamma^\vee/\alpha)) = \text{revrectification}(\text{SYT}(\beta)) = \tilde{T}_\beta \).

**Proof.** It suffices to show that \( \text{revrectification}(\text{SYT}(\beta)) = \tilde{T}_\beta \). We induct on \( |\beta| = n \).

Let \( U \) be the tableau obtained by removing the box labeled 1 from \( T_\beta \). So \( U \) is of skew shape \( \beta'/(1) \). Let \( V = \text{rectification}(U) \), a tableau of shape \( \kappa \subset \beta \). By induction, \( \text{rectification}(V) = \tilde{V} \). Now \( \text{revrectification}(\tilde{T}_\beta) \) and \( \text{revrectification}(V) \) agree except that the former has a label 1 in a box that does not appear in the latter. In \([\text{ThYo06, Proposition 4.6}]\), we showed that the reverse rectification of a tableau of shape \( \beta \) is necessarily of shape \( \text{rotate}(\beta) \). Thus, the location of 1 in \( \text{revrectification}(\tilde{T}_\beta) \) must be the box \( \kappa^\vee/\beta^\vee \). This is the 180-degree rotation of the box \( \beta'/\kappa \) in \( \Lambda \), the latter being the position of \( n \) in \( \text{evac}(\tilde{T}_\beta) \). Thus, 1 is located in the desired position in \( \text{revrectification}(\text{SYT}(\beta)) \). The rest of the statement follows from the fact that the rest of \( \text{revrectification}(\text{SYT}(\beta)) \) agrees with \( \tilde{V} \), and the entries of \( \text{evac}(\tilde{T}_\beta) \) other than \( n \) agree with \( \text{evac}(V) \).

**Corollary 2.2.** Fix a carton filling. The uninitialized corners of the “\( \emptyset, T_\lambda, T_\mu \)” and “\( \emptyset, T_\mu, T_\lambda \)” faces are \( \gamma^\vee \) and \( \mu^\vee \) respectively. The remaining “sixth” corner (the unique one not visible in Figure 1) is \( \lambda^\vee \). Thus, we can speak of the edges \( \lambda^\vee - \Lambda \), \( \mu^\vee - \Lambda \) and \( \gamma^\vee - \Lambda \). These represent the shape chains of \( \tilde{T}_\lambda, \tilde{T}_\mu \) and \( \tilde{T}_\nu \) respectively.

**Proof.** Think of the union of the faces “\( \emptyset, T_\lambda, T_\mu \)” with the adjacent face involving \( \Lambda \) and the “sixth corner” as a single growth diagram. This diagram computes \( \text{revrectification}(T_\lambda) \) and records the result along the edge involving \( \Lambda \) and diagonally opposite to the \( T_\lambda \) edge. But the lemma asserts this result is \( \tilde{T}_\lambda \) and hence the “sixth corner vertex” is assigned \( \lambda^\vee \). The other conclusions are proved similarly.

Corollary 2.2 affords us the convenience of referring to a face by its corner vertices. Note any carton filling gives a growth diagram on the face \( \emptyset - \mu - \gamma^\vee - \lambda \) for which the edge \( \lambda - \nu^\vee \) is a standard tableau of shape \( \nu^\vee/\lambda \) rectifying to \( T_\mu \). By the jeu de taquin Littlewood-Richardson rule, fillings of this face count \( C_{\lambda, \mu, \gamma} \). Hence it suffices to show that any such growth diagram for this face extends uniquely to a filling of the entire carton.

If such an extension exists, it is unique: \( \lambda - \nu^\vee \) and \( \Lambda - \gamma^\vee \) determine, by \((F1), (F2)\) and the Corollary, the face \( \lambda - \mu^\vee \) and \( \Lambda - \nu^\vee \). Similarly, these two faces determine the remaining faces (in order):
\[
\emptyset - \nu - \mu^\vee - \lambda \implies \nu - \lambda^\vee - \Lambda - \mu^\vee \implies \mu - \lambda^\vee - \Lambda - \nu^\vee \implies \emptyset - \nu - \lambda^\vee - \mu.
\]
We now show that with a filling of \( \emptyset - \mu - \nu - \lambda \) (together with the extra edges given in the Corollary) one can extend it to fill the carton, using the conclusions of Corollary 2.2.

\( \emptyset - \mu - \nu - \lambda \): This is given by labeling \( \lambda - \nu \) with \( T_{\nu/\lambda} \in \text{SYT}(\nu/\lambda) \) that witnesses \( C_{\lambda, \mu, \nu} = C_{\lambda, \mu} \), i.e., it rectifies to \( T_\mu \). By the infusion involution, the edge \( \mu - \nu \) representing \( T_{\nu/\mu} := \text{infusion}_2(T_{\lambda}, T_{\nu/\lambda}) \in \text{SYT}(\nu/\mu) \) witnesses \( C_{\mu, \lambda} \).

\( \lambda - \mu - \lambda - \nu \): Build the growth diagram using \( T_{\nu/\lambda} \) and \( \tilde{T}_\nu \) from the edges \( \lambda - \nu \) and \( \nu - \lambda \) respectively. The only boundary condition we need to check is that \( \text{infusion}_2(T_{\nu/\lambda}, \tilde{T}_\nu) = \tilde{T}_\mu \), which is true by the Lemma. The newly determined edge \( \lambda - \mu \) is \( \text{infusion}_1(T_{\nu/\lambda}, \tilde{T}_\nu) \). This is a tableau \( T_{\mu/\nu} \in \text{SYT}(\mu/\nu) \) which rectifies to \( T_\nu \), i.e., one that witnesses \( C_{\mu, \lambda} \).

\( \emptyset - \nu - \mu - \lambda \): Using the determined edges \( \lambda - \mu \) and \( \emptyset - \lambda \) we obtain a growth diagram with edge \( \emptyset - \nu \) representing \( \text{infusion}_1(T_{\lambda}, T_{\mu/\nu}) \), which equals \( T_\nu \), as desired. By the infusion involution, \( \nu - \mu \) represents a tableau \( T_{\mu/\nu} \in \text{SYT}(\mu/\nu) \) that witnesses \( C_{\nu, \mu} \).

\( \nu - \lambda - \lambda - \mu \): This time, we grow the face using the edges \( \nu - \mu \) and \( \mu - \lambda \). The edge \( \lambda - \nu \) is thus \( \text{infusion}_2(T_{\mu/\nu}, \tilde{T}_\mu) \) which indeed equals \( \tilde{T}_\lambda \), by the Lemma. The newly determined edge \( \nu - \lambda \) is \( T_{\lambda/\nu} := \text{infusion}_1(T_{\mu/\nu}, \tilde{T}_\mu) \in \text{SYT}(\lambda/\nu) \) that witnesses \( C_{\lambda, \nu} \).

\( \mu - \lambda - \lambda - \nu \): Growing the face using \( \mu - \nu \) and \( \nu - \lambda \) we find that the edge \( \lambda - \lambda \) equals \( \text{infusion}_2(T_{\nu/\mu}, \tilde{T}_\nu) \), which is the already determined \( \tilde{T}_\lambda \). The newly determined edge \( \mu - \lambda \) is \( T_{\lambda/\mu} := \text{infusion}_1(T_{\nu/\mu}, \tilde{T}_\nu) \in \text{SYT}(\lambda/\mu) \) that witnesses \( C_{\lambda, \mu} \).

\( \emptyset - \nu - \lambda - \mu \): We grow this final face using \( \emptyset - \nu \) and \( \nu - \lambda \), but need to make two consistency checks. First the edge \( \emptyset - \mu \) is \( \text{infusion}_1(T_{\nu}, T_{\lambda/\nu}) \) which clearly is \( T_\mu \).

It remains to check that

\[
(2) \quad \text{infusion}_2(T_{\nu}, T_{\lambda/\nu}) = T_{\lambda/\mu}.
\]

(Notice that \( \text{infusion}_2(T_{\nu}, T_{\lambda/\nu}) \) is a filling of \( \lambda/\mu \) that rectifies to \( T_\nu \), just as \( T_{\lambda/\mu} \) does. However it is not clear a priori that they are the same.)

To prove (2), we need a definition. For tableaux \( A \) and \( B \) of respective (skew) shapes \( \alpha \) and \( \gamma/\alpha \) let \( A \ast B \) be their concatenation as a (nonstandard) tableau. If \( C \) is a tableau of shape \( \Lambda/\gamma \) and \( \alpha \) is a straight shape, \( A \ast B \ast C \) is a layered tableau of shape \( \Lambda \).

**Example 2.3.** Let \( \alpha = (2, 1) \) and \( \gamma = (4, 2, 1) \). We have

\[
A = \begin{array}{ccc}
1 & 2 \\
3 & 1 \\
4 & & \\
\end{array}, \quad B = \begin{array}{ccc}
2 & 3 \\
1 & & \\
\end{array}, \quad C = \begin{array}{ccc}
\underline{2} & \underline{3} \\
\underline{1} & \underline{4} \\
\end{array}, \quad \text{and} \quad A \ast B \ast C = \begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
4 & 1 & 5 \\
\end{array}.
\]

We have used boldface and underlining to distinguish the entries from \( A, B \) and \( C \).
Let $I_1$ and $I_2$ be operators on layered tableaux defined by
\[ I_1 : A \ast B \ast C \mapsto \text{infusion}_1(A, B) \ast \text{infusion}_2(A, B) \ast C \]
and
\[ I_2 : A \ast B \ast C \mapsto A \ast \text{infusion}_1(B, C) \ast \text{infusion}_2(B, C). \]
The infusion involution says $I_1^2$ and $I_2^2$ are the identity operator. In fact, the following crucial "braid identity" holds, showing $I_1$ and $I_2$ generate a representation of $S_3$:

**Proposition 2.4.** $I_1 \circ I_2 \circ I_1 = I_2 \circ I_1 \circ I_2.$

In view of the Proposition, (2) follows by setting $\alpha = \lambda$, $\gamma = \nu^\vee$, $A = T_\lambda$, $B = T_\lambda \nu^\vee / \nu$ and $C = \tilde{T}_\nu$, since the assertion merely says the “middle” tableau in

(3) \[ I_1 \circ I_2 \circ I_1 \circ I_2(T_\lambda \star T_\nu^\vee / \lambda \star \tilde{T}_\nu) \text{ and } I_2 \circ I_1 \circ I_2(T_\lambda \star T_\nu^\vee / \lambda \star \tilde{T}_\nu) \]

are the same, whereas we even have equality of the two layered tableaux.

**Proof of Proposition 2.4:** By the Lemma it follows that
\[ \tilde{C} \ast \overline{B} \ast \tilde{A} := I_1 \circ I_2 \circ I_1(A \ast B \ast C), \text{ and } \]
\[ \tilde{C} \ast \overline{B} \ast \tilde{A} := I_2 \circ I_1 \circ I_2(A \ast B \ast C), \]

where the shapes of $\tilde{C}$, $\overline{B}$ and $\tilde{A}$ are respectively the 180 degree rotations of the shapes of $A$, $B$ and $C$, and we know the fillings of $\tilde{C}$ and $\tilde{A}$ in terms of evacuation. We thus also know the shapes of $\overline{B}$ and $\tilde{B}$ are the same. It remains to show $\overline{B}$ equals $\tilde{B}$.

We first study (4). Let $B' = \text{infusion}_1(A, B)$, and $A' = \text{infusion}_2(A, B)$. Now
\[ I_1 \circ I_2(B' \ast A' \ast C) = \tilde{C} \ast \overline{B} \ast \tilde{A}, \]
and the skew tableau $\overline{B} \ast \tilde{A}$ is obtained by treating $B' \ast A'$ as a single standard tableau by valuing an entry $i$ of $A'$ as $\vert B' \vert + i$, evacuating, rotating the result 180 degrees and finally sending the entry $i$ from $A'$ (respectively $B'$) to $\vert A' \vert - i + 1$ (respectively $\vert B' \vert - i + 1$).

**Example 2.5.** Continuing the previous example,
\[ B' \ast A' = \begin{array}{|c|c|}
|1| & 2 \\
|4| & 3 \\
\end{array} \] and $\text{evac}(B' \ast A') = A'' \ast B'' = \begin{array}{|c|c|}
|1| & 3 \\
|2| & 1 \\
|4| & 2 \\
\end{array}$ $\rightarrow \overline{B} \ast \tilde{A} = \begin{array}{|c|c|}
|3| & 1 \\
|4| & 2 \\
|1| & 2 \\
\end{array}$

Let us focus on $\text{evac}(B' \ast A') = A'' \ast B''$ and momentarily ignore the subsequent rotation and complementation of entries. Fomin has shown in [St99, Appendix 2] that a fruitful way to think of $\text{evac}(X)$ is with triangular growth diagrams obtained by placing the shape chain of $X$, then $\Delta(X), \Delta^2(X), \ldots$ on top of one another (slanted left to right); see Figure 3.

We begin with the data $B' \ast A'$ along the left side of the diagram. This is given as a concatenation of two shape chains, one from $\emptyset - \mu$ and then one from $\mu - \gamma$ where $\mu$ is the shape of $B'$. Applying (F1) and (F2) from Section 1, we obtain $A'' \ast B'' = \text{evac}(B' \ast A')$, given along the righthand side as a pair of chains connected at $\alpha$. It follows from the construction of growth diagrams that the thick lines represent $\text{evac}(B')$ and $A = \text{evac}(A'')$ (the latter being a consequence of the Lemma).
These thick lines together with the vertex $\gamma$ define a rectangular growth diagram, so:

(7) \quad \text{infusion}(A, B'') = (\text{evac}(B'), A').

In particular, the shape of $\text{rectification}(B'') = \mu$. On the other hand, by definition

(8) \quad \text{infusion}(A, B) = (B', A'),

and $B$ rectifies to a tableau of shape $\mu$ also.

By (7) and (8) combined with the aforementioned results of [Ha92] we see $B$ and $B''$ are dual equivalent, as they both are obtained by an application of the same sequence of slides (encoded by $A'$) to a pair of tableaux of the same straight shape. Moreover, by (7) $B''$ is jeu de taquin equivalent to $\text{evac}(B') = \text{evac}(\text{rectification}(B))$. Carrying out the analogous “reverse” analysis on $I_2 \circ I_1 \circ I_2(A \star B \star C)$ we let $C^\circ = \text{infusion}_1(B, C)$ and $B^\circ = \text{infusion}_2(B, C)$ and study $I_2 \circ I_1(A \star C^\circ \star B^\circ) = C^\circ \star B^\circ$. Parallel to (6) we consider the “reverse evacuation” $B^\circ \star C^\circ$ of $C^\circ \star B^\circ$ using reverse jeu de taquin slides. Then we similarly conclude $B^\circ$ is dual equivalent to $B$ (and thus $B''$). Also $B^\circ$ is jeu de taquin equivalent to the reverse evacuation of $\text{revrectification}(B)$. Thus by the lemma, $B''$ and $B^\circ$ are jeu de taquin equivalent. Hence $B'' = B^\circ$ and so $B = \hat{B}$, as required.

Example 2.6. Revisiting Example 2.3, $C^\circ \star B^\circ = \begin{array}{c} 2 \ 3 \\ 4 \ 5 \ 3 \end{array}$ with reverse evacuation

$\begin{array}{c} 3 \ 4 \\ 1 \ 2 \ 3 \\ 2 \ 1 \ 2 \ 4 \ 3 \end{array}$

and $B^\circ$ (boldface in the latter tableau) is also $B''$ from Example 2.5. Rotation and complementation gives $\hat{B}$, which is $\overline{B}$ from Example 2.5, as desired. □
3. An extended example of the main theorem

Let $\Lambda = 3 \times 4 = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array}$, $\lambda = (2, 1) = \begin{array}{c}
\hline
& \\
& \\
& \\
\hline
\end{array}$, $\mu = (3, 1) = \begin{array}{c}
\hline
& \\
& \\
\hline
\end{array}$, $\nu = (3, 2) = \begin{array}{c}
\hline
& \\
\hline
\end{array}$. Therefore $\lambda^\vee = (4, 3, 2), \mu^\vee = (4, 3, 1)$ and $\nu^\vee = (4, 2, 1)$. Also

$T_\lambda = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
\end{array}$, $T_\mu = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}$, $T_\nu = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
\end{array}$

$\tilde{T}_\lambda = \begin{array}{c}
1 & \\
2 & 3 \\
\end{array}$, $\tilde{T}_\mu = \begin{array}{c}
1 & \\
2 & 3 \\
\end{array}$, $\tilde{T}_\nu = \begin{array}{c}
1 & \\
2 & 3 \\
\end{array}$

Here $C_{\lambda, \mu, \nu} = 1$, and we now give the unique carton filling. We begin with $T_{\nu^\vee/\lambda} = \begin{array}{c}
2 & 3 \\
4 & \\
\end{array}$ Then we have the following sides of the carton, described as growth diagrams with the obvious identifications of boundaries (the partitions correspond to shapes placed on the vertices of the carton and the diagrams have been oriented to be consistent with Figure 1):

<table>
<thead>
<tr>
<th>(2, 1) = $\lambda$</th>
<th>$(2, 1, 1)$</th>
<th>$(3, 1, 1)$</th>
<th>$(4, 1, 1)$</th>
<th>$(4, 2, 1)$ = $\nu^\vee$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)</td>
<td>(2, 1)</td>
<td>(3, 1)</td>
<td>(4, 1)</td>
<td>(4, 2)</td>
</tr>
<tr>
<td>(1)</td>
<td>(1, 1)</td>
<td>(2, 1)</td>
<td>(3, 1)</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(3, 1) = $\mu$</td>
</tr>
</tbody>
</table>

Table 2. $\emptyset - \mu - \nu^\vee - \lambda$

<table>
<thead>
<tr>
<th>(4, 3, 1) = $\mu^\vee$</th>
<th>$(4, 3, 2)$</th>
<th>$(4, 3, 3)$</th>
<th>$(4, 4, 3)$</th>
<th>$(4, 4, 4)$ = $\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4, 2, 1)$</td>
<td>$(4, 2, 2)$</td>
<td>$(4, 3, 2)$</td>
<td>$(4, 4, 2)$</td>
<td>$(4, 4, 3)$</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>(4, 2, 1)</td>
<td>(4, 3, 1)</td>
<td>(4, 4, 1)</td>
<td>(4, 4, 2)</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>(3, 2, 1)</td>
<td>(3, 3, 1)</td>
<td>(4, 3, 1)</td>
<td>(4, 3, 2)</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>(2, 2, 1)</td>
<td>(3, 2, 1)</td>
<td>(4, 2, 1)</td>
<td>(4, 2, 2)</td>
</tr>
<tr>
<td>(2, 1) = $\lambda$</td>
<td>(2, 1, 1)</td>
<td>(3, 1, 1)</td>
<td>(4, 1, 1)</td>
<td>(4, 2, 1) = $\nu^\vee$</td>
</tr>
</tbody>
</table>

Table 3. $\lambda - \mu^\vee - \Lambda - \nu^\vee$

<table>
<thead>
<tr>
<th>(4, 3, 1) = $\mu^\vee$</th>
<th>$(4, 2, 1)$</th>
<th>$(4, 2)$</th>
<th>(3, 2)</th>
<th>(2, 2)</th>
<th>(2, 1) = $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4, 2, 1)$</td>
<td>$(4, 1, 1)$</td>
<td>$(4, 1)$</td>
<td>(3, 1)</td>
<td>(2, 1)</td>
<td>(2)</td>
</tr>
<tr>
<td>(3, 2, 1)</td>
<td>(3, 1, 1)</td>
<td>(3, 1)</td>
<td>(2, 1)</td>
<td>(1, 1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(3, 2) = $\nu$</td>
<td>(3, 1)</td>
<td>(3)</td>
<td>(2)</td>
<td>(1)</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Table 4. $\emptyset - \nu - \mu^\vee - \lambda$
$$\begin{array}{ccccccc}
(4,3,1) = \mu^\vee & (4,3,2) & (4,3,3) & (4,4,3) & (4,4,4) = \Lambda \\
(4,2,1) & (4,2,2) & (4,3,2) & (4,4,2) & (4,4,3) \\
(3,2,1) & (3,2,2) & (3,3,2) & (4,3,2) & (4,4,2) \\
(3,2) = \nu & (3,2,1) & (3,3,1) & (4,3,1) & (4,3,2) = \Lambda^\vee \\
\end{array}$$

Table 5. $\nu - \lambda^\vee - \Lambda - \mu^\vee$

$$\begin{array}{ccccccc}
(4,4,4) = \Lambda & (4,4,3) & (4,4,2) & (4,3,2) & (4,2,2) & (4,2,1) = \nu^\vee \\
(4,4,3) & (4,4,2) & (4,3,1) & (4,2,1) & (4,2) \\
(4,3,3) & (4,3,2) & (4,3,1) & (3,3,1) & (3,2,1) & (3,2) \\
(4,3,2) = \lambda^\vee & (4,2,2) & (4,2,1) & (3,2,1) & (3,1,1) & (3,1) = \mu \\
\end{array}$$

Table 6. $\lambda^\vee - \Lambda - \nu^\vee - \mu$

$$\begin{array}{ccccccc}
(3,2) = \nu & (3,2,1) & (3,3,1) & (4,3,1) & (4,3,2) = \lambda^\vee \\
(3,1) & (3,1,1) & (3,2,1) & (4,2,1) & (4,2,2) \\
(3) & (3,1) & (3,2) & (4,2) & (4,2,1) \\
(2) & (2,1) & (2,2) & (3,2) & (3,2,1) \\
(1) & (1,1) & (2,1) & (3,1) & (3,1,1) \\
\emptyset & (1) & (2) & (3) & (3,1) = \mu \\
\end{array}$$

Table 7. $\emptyset - \nu - \lambda^\vee - \mu$

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References


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