SPECTRUM OF THE LAPLACIAN ON MANIFOLDS WITH SPIN(9) HOLONOMY

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ABSTRACT. We consider noncompact complete manifolds with Spin(9) holonomy and proved an one end result and a splitting type theorem under different conditions on the bottom of the spectrum. We proved that any harmonic functions with finite Dirichlet integral must be Cayley-harmonic, which allowed us to conclude an one end result. In the second part, we established a splitting type theorem by utilizing the Busemann function.

1. Introduction

In [13], the authors proved the following

Theorem. [13] *Let M be a complete Riemannian manifold with a parallel p-form ω. Assume that f is a harmonic function satisfying*

$$\int_{B_p(R)} |\nabla f|^2 = o(R^2)$$

as $R \to \infty$, then f satisfies

$$d * (df \wedge \omega) = 0.$$

Combining the above theorem with the fact that a quaternionic Kähler manifold supports a global parallel 4-form ω, the authors proved, by an explicit calculation involving ω, that a harmonic function with bounded Dirichlet integral is quaternionic-harmonic. Utilizing the quaternionic-harmonic condition they proved that, under an assumption on the bottom of the spectrum $\lambda_1(M)$, such a manifold must have exactly one infinite volume end. Since a manifold with holonomy group Spin(9) supports a global parallel 8-form Ω, by a careful and detail study of Ω, we proved that any harmonic functions with bounded Dirichlet integral is Cayley-harmonic. Similar to the work in [13], with a suitable lower bound assumption on $\lambda_1(M)$, an one infinite volume end result has been established by utilizing the Cayley-harmonicity condition. In the second part of this paper, we consider the case that $\lambda_1(M) = 121$ achieves its maximal value. By studying the Busemann function β on M and using the results in [10] and [12], we proved that either M has only one end or M must splits as $\mathbb{R} \times N$, where N is given by a level set of β.

2. Cayley hyperbolic space

We give a brief introduction on the Cayley numbers $\mathbb{O}$, and a description of the sectional curvature of the Cayley hyperbolic space $\mathbb{H}^2_{\mathbb{O}}$. The material presented here is adopted from [3], we refer the readers to there for further details. The Cayley numbers $\mathbb{O}$, is an 8-dimensional non-associative division algebra over the real numbers which satisfies the alternative law: $x(xy) = x^2y$, $(yx)x = yx^2$. It has a multiplicative identity 1 and a positive definite bilinear form $\langle , \rangle$ whose associated norm $||\cdot||$ satisfies $||ab|| = ||a|| \cdot ||b||$. Every element $a \in \mathbb{O}$ can be written as $a = \alpha 1 + a_0$, where α is real and $\langle a_0, 1 \rangle = 0$. The conjugation map $a \mapsto a^* = \alpha 1 - a_0$ is an anti-automorphism, that is $(ab)^* = b^*a^*$. Moreover, $aa^* = \langle a, a \rangle 1$ and $\langle a, b \rangle = \langle a^*, b^* \rangle$. $\mathbb{O}$ admits a canonical basis $\{1, e_0, \cdots, e_6\}$ such that $\langle e_i, e_j \rangle = \delta_{ij}$, $e_i^2 = -1$, $e_i e_j + e_j e_i = 0$ for $i \neq j$, and $e_i e_{i+1} = e_{i+3}$, if i is an integer mod 7. Obviously, we can extend the positive bilinear form from $\mathbb{O}$ to $\mathbb{O}^2$ by

$$\langle (a, b), (c, d) \rangle = \langle a, c \rangle + \langle b, d \rangle,$$

where $a, b, c, d \in \mathbb{O}$. For any point $x \in \mathbb{H}^2_{\mathbb{O}}$, we make the following identification $T_x \mathbb{H}^2_{\mathbb{O}} \simeq \mathbb{O}^2$. Let M be a Riemannian manifold with metric tensor $\langle , \rangle$. Let V be any tangent space to M. The curvature operator of M at V is a map

$$R : \Lambda^2(V) \to \Lambda^2(V) \subseteq \text{Hom}(V, V)$$

such that

$$R(x \wedge y)z + R(z \wedge x)y + R(y \wedge z)x = 0.$$

The above two properties implies R is a symmetric linear operator, that is

$$\langle R(x \wedge y)z, w \rangle = \langle R(x \wedge y), z \wedge w \rangle = \langle R(z \wedge w), x \wedge y \rangle$$

for any $x, y, z, w \in V$. For any $x, y \in V$ linearly independent, the sectional curvature of the 2-plane spanned by x and y is defined by

$$K_{x \wedge y} = \frac{\langle R(x \wedge y), x \wedge y \rangle}{||x \wedge y||^2}.$$

The sectional curvature $K_{(a,b) \wedge (c,d)}$ of the 2-plane $(a, b) \wedge (c, d)$ of $\mathbb{O}^2$ has the following properties:

(1) For any $a, b, c, d \in \mathbb{O}$ with $||(a, b)|| = ||(c, d)|| = 1$ and $\langle (a, b), (c, d) \rangle = 0$, we have

$$
\begin{aligned}
K_{(a,b) \wedge (c,d)} &= \alpha \Big\{ ||a \wedge c||^2 + ||b \wedge d||^2 + \frac{1}{4} ||a||^2 ||d||^2 + \frac{1}{4} ||b||^2 ||c||^2 \\
&+ \frac{1}{2} \langle ab, cd \rangle - \langle ad, cb \rangle \Big\}
\end{aligned}
$$

(2)
$$K_{(a,0) \wedge (b,0)} = \alpha \quad \text{if} \quad (a, 0) \wedge (b, 0) \neq 0.$$

(3)
$$K_{(a,0) \wedge (0,b)} = \frac{\alpha}{4} \quad \text{if} \quad (a, 0) \wedge (0, b) \neq 0.$$

(4)
$$\frac{|\alpha|}{4} \leq |K_{(a,b) \wedge (c,d)}| \leq |\alpha| \quad \text{if} \quad (a, b) \wedge (c, d) \neq 0.$$

In this article, we use the normalization that $\alpha = -4$, hence the sectional curvature of $\mathbb{H}^2_{\mathbb{O}}$ is pinched between -4 and -1. Let M be a complete noncompact Riemannian manifold with holonomy group Spin(9). It was proved in [3] that a manifold with holonomy group Spin(9) must be locally symmetric and its universal covering is either the Cayley projective plane or the Cayley hyperbolic space $\mathbb{H}^2_{\mathbb{O}}$. Since we are considering noncompact manifolds, its universal covering is $\mathbb{H}^2_{\mathbb{O}}$. We first compute the Laplacian of the distance function of $\mathbb{H}^2_{\mathbb{O}}$.

Proposition 1. *Let $r(x) = r_p(x)$ be the distance function of $\mathbb{H}^2_{\mathbb{O}}$ from a fixed point p, then*

$$\triangle r = 14 \coth 2r + 8 \coth r.$$

Proof. Let $\gamma : [0, L] \to M$ be a normal geodesic from p to x. Let $e_1(t) = \gamma'(t)$ along γ. Let $\{e_A\}^{16}_{A=2}$ be a basis of $T_p\mathbb{H}^2_{\mathbb{O}}$ such that

(1)
$$\begin{cases} R_{1i1i} = -4, & 2 \le i \le 8 \\ R_{1\alpha1\alpha} = -1, & 9 \le \alpha \le 16. \end{cases}$$

We extend $\{e_A\}$ to be a local frame along $\gamma(t)$, $\{\gamma'(t) = e_1(t), e_2(t), \cdots, e_{16}(t)\}$ by parallel transporting along γ. Since $\mathbb{H}^2_{\mathbb{O}}$ is a symmetric space and thus locally symmetric, we have

$$\frac{\partial}{\partial t} R_{1A1A} = R_{1A1A,1} = 0, \ 2 \le A \le 16,$$

hence (1) is valid along γ. Let $X_A(t) = f_A(t)e_A(t)$ be the Jacobi field along γ with $X_A(0) = 0$, $X_A(L) = e_A(L)$. $f_A(t)$ satisfies the Jacobi equation

$$\frac{d^2}{dt^2} f_A(t) - c_A^2 f_A(t) = 0$$
$$f_A(0) = 0, \ f_A(p) = 1, \ 2 \le A \le 16.$$

where $c_i = 2$, $2 \le i \le 8$ and $c_\alpha = 1$, $9 \le \alpha \le 16$. Solving the above equation, we have

(2)
$$f_A(t) = \frac{\sinh(c_A t)}{\sinh(c_A L)}, \ 2 \le A \le 16.$$

Now, we can compute the Hessian of r at x

$$\begin{aligned} H(r)(e_A, e_A) &= \int_0^L \left(\left| \frac{dX_A}{dt} \right|^2 - \langle R(X_A, \gamma')\gamma', X_A \rangle \right) dt \\ &= \int_0^L \left(\left| \frac{df_A}{dt} \right|^2 + c_A^2 f^2 \right) dt \\ &= c_A \coth(c_A L). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \triangle r &= \sum_{A=2}^{16} H(r)(e_A, e_A) \\ &= 14 \coth 2r + 8 \coth r, \end{aligned}$$

where we have used the fact that $H(r)(e_1, e_1) = 0$. $\qquad \square$

Theorem 2. *Let M be a locally symmetric space with universal covering $\mathbb{H}_{\mathbb{O}}^2$. Then*

$$\lambda_1(M) \le 121,$$

and

$$\triangle_M r \le 14 \coth 2r + 8 \coth r,$$

in the sense of distribution.

Proof. Let $A(r), V(r)$ be the area and volume of the geodesic ball of radius r of $\mathbb{H}_{\mathbb{O}}^2$ respectively. By proposition 1, we have

$$\frac{A'(r)}{A(r)} = 14 \coth 2r + 8 \coth r,$$

hence

$$(3) \qquad \begin{aligned} V_M(p,r) \;&\le\; V(r) \\ &=\; \int_0^r A(t) dt \\ &\le\; C \int_0^r (\sinh 2t)^7 (\sinh t)^8 dt \\ &\le\; C_1 e^{22r}, \end{aligned}$$

where $V_M(p,r)$ is volume of the geodesic ball with radius r centered at p and for some constant C_1. On the other hand, it was shown in [10] that

$$V_M(p,r) \ge C_2 \exp\left(2\sqrt{\lambda_1(M)} r\right),$$

for any manifolds with positive spectrum. Combining the above inequality with (3), we conclude that $\lambda_1(M) \le 121$. For the second part, let $f(r) = 14 \coth 2r + 8 \coth r$. By proposition 1, we have

$$\triangle_M r(x) = f(r(x)),$$

for any $x \in M \setminus \mathrm{Cut}(p)$, where $\mathrm{Cut}(p)$ is the cut locus of p. For each direction $\theta \in S_p(M)$, let $R(\theta) = \sup_{t>0}\{t : r_p(\exp_p(t\theta)) = t\}$. Let $\phi \in C_0^\infty(M)$ be a non-negative smooth function with compact support, then

$$\begin{aligned} \int_M \phi f(r) \;&=\; \int_{S_p(M)} \int_0^{R(\theta)} \phi f(r)\, J(\theta, r)\, dr d\theta \\ &=\; \int_{S_p(M)} \int_0^{R(\theta)} \phi\, \frac{\partial J}{\partial r}\, dr d\theta \\ &=\; -\int_M \frac{\partial \phi}{\partial r} + \int_{S_p(M)} \phi(\theta, R(\theta)) J(\theta, R(\theta))\, d\theta \\ &\ge\; -\int_M \langle \nabla\phi, \nabla r\rangle \\ &=\; \int_M r \triangle\phi, \end{aligned}$$

where the second equality follows from the fact that $\triangle r = \frac{\partial}{\partial r}(\log J)$, for all $r < R(\theta)$ and the third equality follows from integration by parts, $\phi \ge 0$ and $J(\theta, 0) = 0$. Hence the second result follows. $\qquad\square$

Let us recall the definition of the Busemann function and some of its properties. Let M be a complete manifold and $\gamma : [0, +\infty) \to M$ be a geodesic ray. Let $\beta_\gamma^t(x) = t - r(\gamma(t), x)$, where $r(x, y)$ denotes the distance between x and y. Triangle inequality implies

$$|\beta_\gamma^t(x)| = |r(\gamma(t), \gamma(0)) - r(\gamma(t), x)| \leq r(\gamma(0), x),$$

and

$$\beta_\gamma^t(x) - \beta_\gamma^s(x) = t - s + r(\gamma(s), x) - r(\gamma(t), x) \geq 0,$$

if $t > s$. Hence $\{\beta_\gamma^t\}_{t \geq 0}$ is uniformly bounded on compact subsets of M and non-decreasing, it converges uniformly on any compact subsets of M. The Busemann function with respect to a geodesic ray γ is defined as

$$\beta(x) = \lim_{t \to +\infty} \beta_\gamma^t(x).$$

The following lemma is well-known, see [9] for instance.

Lemma 3.
$$|\nabla \beta| = 1$$

almost everywhere.

3. Manifolds with parallel forms

Let us first recall the Hodge star operator * and some of its basic properties. Let V^n be a n-dimensional oriented real inner product space, we have the Hodge star operator

$$* : \wedge^p V \to \wedge^{n-p} V,$$

for any $\theta \in \wedge^1 V, v \in V$, exterior multiplication and interior product operators

$$\varepsilon(\theta) : \quad \wedge^p V \to \wedge^{p+1} V$$
$$l(v) : \quad \wedge^p V \to \wedge^{p-1} V,$$

where $\varepsilon(\theta)\omega = \theta \wedge \omega$ and $(l(v)\omega)(\cdot) = \omega(v, \cdot)$ for any $\omega \in \wedge^p V$. Let $\theta, \theta' \in \wedge^1 V$ and $v, v' \in V$ be the dual of θ and θ' respectively with respect the inner product of V. For any $\eta \in \wedge^p V$, we have the following basic properties

(1) $** \eta = (-1)^{p(n-p)} \eta$
(2) $*\varepsilon(\theta)\eta = (-1)^p l(v) * \eta$
(3) $\varepsilon(\theta) * \eta = (-1)^{p-1} * l(v)\eta$
(4) $*\varepsilon(\theta) * \eta = (-1)^{(p-1)(n-p)} l(v)\eta$
(5) $l(v)\varepsilon(\theta')\eta + \varepsilon(\theta)l(v')\eta = 0$, where $v \perp v'$
(6) $l(v)\varepsilon(\theta)\eta + \varepsilon(\theta)l(v)\eta = \eta$

The following theorem is an over-determined system of equations satisfied by harmonic functions and generalized Corlette's argument to harmonic functions with finite Dirichlet integral on a complete manifold with a parallel p-form. This kind of result was first proved by Siu [14] for harmonic maps in his proof of the rigidity theorem for Kähler manifolds. Corlette [5] gave a more systematic approach for harmonic maps with finite energy from a finite volume quaternionic hyperbolic space or Cayley hyperbolic plane to a manifold with nonpositive curvature. In [7], the author generalized Siu's argument to harmonic functions with finite Dirichlet integral on Kähler manifolds.

Theorem 4. *([13]) Let M be a complete Riemannian manifold with a parallel p-form ω. Assume that f is a harmonic function satisfying*

$$\int_{B_p(R)} |\nabla f|^2 = o(R^2)$$

as $R \to \infty$, then f satisfies

$$d * (df \wedge \omega) = 0.$$

By taking a careful and closer look at the nature of the proof of the above theorem, we found out that the proof not only works for harmonic functions with finite Dirichlet integral but also L^2 harmonic 1-form. The key ingredient is that any L^2 harmonic 1-form is both closed and co-closed. We have the following:

Theorem 5. *Let M^n be a n dimensional complete Riemannian manifold with a parallel p-form ω. Assume that α is a L^2 harmonic 1-form, that is $\triangle \alpha = 0$ and*

$$\int_M |\alpha|^2 < +\infty.$$

Then α satisfies

$$d * (\alpha \wedge \omega) = 0.$$

Proof. We first show that

(4) $$*d * (\alpha \wedge \omega) = (-1)^{n-1} d * (\alpha \wedge *\omega).$$

For any $x \in M$, we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $\nabla_{e_i} e_j(x) = 0$. Let $\{\theta^i\}_{i=1}^n$ be the coframe. For any p-form ω, we have

$$d\omega = \varepsilon(\theta^i) \nabla_{e_i} \omega$$

at x and ω is parallel if and only if $\nabla_{e_i}\omega = 0, \forall i$. Let $\alpha = \sum_{i=1}^n a_i \theta^i$, and hence $\overline{\alpha} = \sum_{i=1}^n a_i e_i$ is the dual of α. We use the notation $d\alpha = \sum_{i,j=1}^n a_{i,j} \theta^j \wedge \theta^i$, where $a_{i,j} = \nabla_{e_j} a_i$. Since α is L^2 harmonic, it is both closed and co-closed, which are equivalent to the conditions that $a_{i,j} = a_{j,i}$ and $\sum_{i=1}^n a_{i,i} = 0$. The following calculations are all evaluated at x.

$$
\begin{aligned}
(5) \qquad d * (\alpha \wedge *\omega) &= d * \varepsilon(\alpha) * \omega \\
&= (-1)^{(p-1)(n-p)} d[l(\overline{\alpha})\omega] \\
&= (-1)^{(p-1)(n-p)} \sum_{i=1}^n \varepsilon(\theta^i) \nabla_{e_i}(l(\overline{\alpha})\omega) \\
&= (-1)^{(p-1)(n-p)} \sum_{i,j=1}^n \varepsilon(\theta^i) a_{j,i}(l(e_j)\omega),
\end{aligned}
$$

where the third equality follows from $\nabla_{e_i} e_j(x) = 0$ and the last equality follows from $\nabla \omega = 0$. On the other hand,

$$
\begin{aligned}
(6) \quad *d * (\alpha \wedge \omega) &= *d * \varepsilon(\alpha)\omega \\
&= * \sum_{i=1}^{n} \varepsilon(\theta^i)\nabla_{e_i}\left(*\varepsilon\left(\sum_{j=1}^{n} a_j \theta^j\right)\omega \right) \\
&= * \left(\sum_{i,j=1}^{n} a_{j,i}\varepsilon(\theta^i) * \left(\varepsilon(\theta^j)\omega\right) \right) \\
&= (-1)^{p(n-p-1)} \sum_{i,j=1}^{n} a_{i,j}l(e_i)\varepsilon(\theta^j)\omega \\
&= (-1)^{p(n-p-1)} \left(\sum_{i=1}^{n} a_{i,i}l(e_i)\varepsilon(\theta^i)\omega + \sum_{i\neq j}^{n} a_{i,j}l(e_i)\varepsilon(\theta^j)\omega \right) \\
&= (-1)^{p(n-p-1)} \Big(\sum_{i=1}^{n} a_{i,i}[\omega - \varepsilon(\theta^i)l(e_i)\omega] \\
&\quad - \sum_{i\neq j}^{n} a_{i,j}\varepsilon(\theta^j)l(e_i)\omega \Big) \\
&= (-1)^{p(n-p-1)+1} \sum_{i,j=1}^{n} a_{i,j}\varepsilon(\theta^i)(l(e_j)\omega),
\end{aligned}
$$

where the last equality follows from $a_{i,j} = a_{j,i}$ and $\sum_{i=1}^{n} a_{i,i} = 0$. (4) now follows from (5) and (6). Let

$$
\phi(x) = \begin{cases} 1 & \text{on} & B_p(R) \\ 0 & \text{on} & M \setminus B_p(2R) \end{cases}
$$

such that $|\nabla \phi| \leq C_1 R^{-1}$. Consider

$$
\begin{aligned}
(7) \quad \int_M \phi^2 |d * (\alpha \wedge \omega)|^2 &= \left| \int_M \phi^2 d * (\alpha \wedge \omega) \wedge *d * (\alpha \wedge \omega) \right| \\
&= \left| \int_M \phi^2 d * (\alpha \wedge \omega) \wedge d * (\alpha \wedge *\omega) \right| \\
&= \left| \int_M d\phi^2 \wedge *(\alpha \wedge \omega) \wedge d * (\alpha \wedge *\omega) \right| \\
&\leq 2 \left(\int_M |d\phi|^2 | * (\alpha \wedge \omega)| \right)^{1/2} \left(\int_M \phi^2 |d * (\alpha \wedge *\omega)|^2 \right)^{1/2} \\
&= 2 \left(\int_M |d\phi|^2 | * (\alpha \wedge \omega)| \right)^{1/2} \left(\int_M \phi^2 |d * (\alpha \wedge \omega)|^2 \right)^{1/2},
\end{aligned}
$$

where the second and the last equality follows from (4), the third equality follows from integration by parts and the fact that $d^2 = 0$. ω is parallel implies

$$
| * (\alpha \wedge \omega)| \leq C_2 |\alpha|,
$$

for some constant C_2. Combining the above with (7), we have

$$\int_{B_p(R)} |d*(\alpha \wedge \omega)|^2 \leq C_3 R^{-2} \int_{B_p(2R)} |\alpha|^2.$$

Let $R \to +\infty$, the result follows from the assumption that α is L^2 integrable. $\qquad \square$

Remark 6. Applications of the above theorem is available in the forthcoming paper [6].

4. Holonomy and Spin(9) invariant

We give a very brief introduction and list some basic principles about the holonomy group of a Riemannian manifold. We refer the readers to [2] and the references therein for further details. Most of the following introductory material are adopted from there. Let $p \in M$ and $\gamma : [0, l] \to M$ be a C^1-piecewise closed curve with $\gamma(0) = \gamma(l) = p$. Let $\tau(\gamma) : T_p M \to T_p M$ be the parallel transport along γ. Since parallel transport preserves inner product, $\tau(\gamma)$ is an element of $O(T_p M)$, the orthogonal group of $T_p M$. Since the inverse of a curve γ^{-1} and the composition of two curves $\gamma \cup \sigma$ satisfy $\tau(\gamma^{-1}) = (\tau(\gamma))^{-1}$ and $\tau(\gamma \cup \sigma) = \tau(\gamma) \circ \tau(\sigma)$. We can have the following definition:

Definition 7. *The holonomy group (or the holonomy representation of M at p) of a Riemannian manifold (M, g) at p is defined by*

$$Hol(p) = \{\tau(\gamma) : \gamma \in C^1\text{-piecewise closed curves of } M \text{ based at } p\},$$

the subgroup of the orthogonal group $O(T_p M)$.

On M, let us consider a tensor field α. If α is invariant by parallel transport, that is, for any $p, q \in M$ and any curve γ from p to q, we have

$$\tau^*(\gamma)(\alpha(p)) = \alpha(q),$$

where $\tau^*(\gamma)$ is the tensorial extension of the parallel transport $\tau(\gamma)$ along γ. By the above definition, $\alpha(p)$ at $T_p M$ is hence invariant by the tensorial extension of the holonomy representation $Hol(p) \subseteq O(T_p M)$. Conversely, given any tensor on $T_p M$, if α_0 is invariant under the tensorial extension of $Hol(p)$, we can construct a tensor field α on M by the formula $\tau^*(\gamma)(\alpha(p)) = \alpha(q)$. Since α_0 is invariant under the tensorial extension of $Hol(p)$, the above definition is independent of the choice of the curve γ and thus it is well-defined. Clearly, $\alpha(p) = \alpha_0$. By the above discussion, we have established a fundamental principle of holonomy group.

Proposition 8. *Let M be a Riemannian manifold and we consider a fixed type (r, s) tensors on M. Then the following three properties are equivalent:*

(1) *There exists a tensor field of type (r, s) which is invariant by parallel transport*
(2) *There exists $p \in M$ and a tensor α_0 of type (r, s) which is invariant by the tensorial extension of type (r, s) of the holonomy representation $Hol(p)$.*
(3) *There exists a tensor field α of type (r, s) which has zero covariant derivative.*

Proof. We have already established the equivalency of the first two statements in the discussion above. For the last statement, it can be seen easily via the formula

$$(D\alpha)(X_1, \cdots, X_s; X) = D_X(\alpha(X_1, \cdots, X_s)) - \sum_{i=1}^{s} \alpha(X_1, \cdots, D_X X_i, \cdots, X_s).$$

For any curve γ, let $X_1, \cdots, X_s$ be vector fields parallel along γ and $X = \gamma'$. Hence, the above equation becomes

$$(D\alpha)(X_1, \cdots, X_s; X) = D_X(\alpha(X_1, \cdots, X_s)).$$

Therefore, $D\alpha = 0$ is equivalent to $D_X(\alpha(X_1, \cdots, X_s))$, which implies $\alpha(X_1, \cdots, X_s)$ is constant along γ. Conversely, for any tangent vector $X(p)$, we can choose a curve γ such that $\gamma' = X(p)$. □

Let M be a manifold with holonomy group Spin(9). We are now ready to describe the parallel 8-form of M. The parallel 8-form of $\mathbb{H}^2_{\mathbb{O}}$ has been obtained by Brown and Gray in [3]. However, it is not easy to read off its properties for further applications because their 8-forms are defined via integration. In [1], the authors defined an 8-form Ω and showed that it is Spin(9) invariant. In [13] the authors used the explicit formula of the parallel 4-form of a quaternionic Kähler manifold and proved that any harmonic function with finite Dirichlet integral is quaternionic-harmonic. Similarly, we will combine the explicit formula of Ω in [1] with a result in [13] to conclude that any harmonic function with finite Dirichlet integral is Cayley-harmonic. We now give a brief description of the Spin(9) invariant 8-form Ω and we will follow the notations in [1]. For any point $p \in \mathbb{H}^2_{\mathbb{O}}$, we identify the tangent space at p to the ordered pair of Cayley numbers, $T_p(\mathbb{H}^2_{\mathbb{O}}) = \mathbb{O}^2 = \{(x, y) : x, y \in \mathbb{O}\}$. Let $\bar{e}_0 = 1, \bar{e}_1, \cdots \bar{e}_7$ be a basis of $\mathbb{O}$ as in [15]. For any $x \in \mathbb{O}$, we let $x^{(2)} = (x, 0)$ and $x^{(3)} = (0, x)$. Let $\{v_i\}_{i=0}^7$ be the dual 1-forms of $\{\bar{e}_i^{(2)}\}_{i=0}^7$ and $\{w_i\}_{i=0}^7$ be the dual 1-forms of $\{\bar{e}_i^{(3)}\}_{i=0}^7$. Equivalently, we have

$$v_i(\bar{e}_j^{(2)}) = \delta_{ij}, \quad v_i(\bar{e}_j^{(3)}) = 0$$
$$w_i(\bar{e}_j^{(2)}) = 0, \quad w_i(\bar{e}_j^{(3)}) = \delta_{ij},$$

for any $0 \leq i, j \leq 7$. Let $e_i = \bar{e}_{i-1}^{(2)}$ for $1 \leq i \leq 8$ and $e_j = \bar{e}_{j-9}^{(3)}$ for $9 \leq j \leq 16$ so that $\{e_i\}_{i=1}^{16}$ is an orthonormal basis of $T_p\mathbb{H}^2_{\mathbb{O}}$.

$$\omega_{ij} = v_{\sigma(i)} \wedge v_{\sigma(j)}, \quad \eta_{ij} = w_{\tau(i)} \wedge w_{\tau(j)},$$

for some permutation functions σ, τ defined in [1]. For our purpose, we do not need to know the explicit forms of σ, τ and so we ignore it here for the sake of simplicity. Now we are ready to write down the formula of Ω.

Theorem 9. [1] *With the above notations,*

$$\Omega = (-v_0 \wedge \cdots \wedge v_7 + w_0 \wedge \cdots \wedge w_7) + F(\omega_{ij}, \eta_{kl})$$

is Spin(9) invariant, where F is a linear combinations of 8-forms, each of which is wedge products of some combinations of ω_{ij}, η_{kl}.

We would like to point out that F was given explicitly in [1]. However, the above simplified form of Ω is enough for our application.

Theorem 10. *Let M be a manifold with holonomy group Spin(9). Assume that f is a harmonic function satisfying*

$$\int_{B_p(R)} |\nabla f|^2 = o(R^2),$$

as $R \to \infty$. Then with the above notations, we have

$$\sum_{i=1}^{8} f_{ii} = 0,$$

where $f_{ij} = Hess(f)(e_i, e_j)$.

Proof. Fix $x \in M$ and let $\{e_i\}_{i=1}^{16}$ be the orthonormal frame of $T_x M$ in the above discussion. By the above construction, $\{\theta^i\}_{i=1}^{16} = \{v_0, \cdots, v_7, w_0, \cdots, w_7\}$ is the orthonormal coframe. By theorem 9,

$$\Omega = (-v_0 \wedge \cdots \wedge v_7 + w_0 \wedge \cdots \wedge w_7) + F(\omega_{ij}, \eta_{kl})$$

is Spin(9) invariant. Since M has holonomy group Spin(9), proposition 8 implies that Ω can be extended to be a parallel form on M, which we still denote it by Ω. By theorem 4, we have

$$d * (df \wedge \Omega) = 0.$$

From (6), replacing $a_{i,j}$ by f_{ij}, the above equation is equivalent to

$$\sum_{i,j=1}^{16} f_{ij}\varepsilon(\theta^i)\left(l(e_j)\Omega\right) = 0.$$

Evaluate the above equation at x, we claim that the only terms containing $v_0 \wedge \cdots \wedge v_7$ are the followings

$$\sum_{i=1}^{8} f_{ii}\varepsilon(\theta^i)\left(l(e_i)(-v_0 \wedge \cdots \wedge v_7)\right) \quad = \quad -\sum_{i=1}^{8} f_{ii}v_0 \wedge \cdots \wedge v_7.$$

Since $d * (df \wedge \Omega) = 0$, we conclude that

$$\sum_{i=1}^{8} f_{ii} = 0$$

at x. To prove the claim, since $l(e_j)F(\omega_{ab}, \eta_{cd})$ kills off a v_{j-1} term if $1 \leq j \leq 8$ or a w_{j-9} term $9 \leq j \leq 16$ of $F(\omega_{ab}, \eta_{cd})$. On the other hand, when $\varepsilon(\theta^i)$ acts on $l(e_j)F(\omega_{ab}, \eta_{cd})$, it adds a v_{i-1} term if $1 \leq i \leq 8$ or a w_{i-9} term $9 \leq i \leq 16$ to $l(e_j)F(\omega_{ab}, \eta_{cd})$. By the above discussion, since

$$\omega_{ab} = v_{\sigma(a)} \wedge v_{\sigma(b)}$$

and

$$\eta_{ab} = w_{\tau(a)} \wedge w_{\tau(b)},$$

it is easy to see that $\varepsilon(\theta^i)\left(l(e_j)F(\omega_{ab}, \eta_{cd})\right)$ does not contain any terms of the form $v_0 \wedge \cdots \wedge v_7$ and $w_0 \wedge \cdots \wedge w_7$, for any $1 \leq i, j \leq 16$. This proved the claim and the result follows. $\qquad \square$

5. Manifolds with positive spectrum

We summarize some useful properties of manifolds with positive spectrum. We refer the readers to [10] for a more detailed description on this subject. Let M be a manifold with positive spectrum $\lambda_1(M) > 0$. By the variational principle, it is equivalent to the following condition:

$$\lambda_1(M) \int_M \phi^2 \leq \int_M |\nabla\phi|^2,$$

for any compactly supported smooth function $\phi \in C_c^\infty(M)$. Since $\lambda_1(M) > 0$, M must be nonparabolic and it implies M must have at least one nonparabolic end. $\lambda_1(M) > 0$ also implies an end E of M is nonparabolic if and only if it has infinite volume. Assume that M has at least two infinite volume ends, E_1, E_2. Let $B_p(R)$ be the geodesic ball with radius R centered at p. We write $B(R) = B_p(R)$ when there is no ambiguity. We construct a sequence of harmonic functions $\{f_R\}$ by solving the following equation

$$\begin{cases} \triangle f_R = 0 & \text{on} & B(R) \\ f_R = 1 & \text{on} & \partial B(R) \cap E_1 \\ f_R = 0 & \text{on} & \partial B(R) \setminus E_1 \end{cases}$$

By the theory of [8], $\{f_R\}$ converges (by passing to a subsequence if necessary) to a nonconstant harmonic function f with finite Dirichlet integral on M as $R \to +\infty$. Maximum principle implies that $0 \leq f \leq 1$. By the construction, it is clear that $\sup_M f = \sup_{E_1} f = 1$ and $\inf_M f = \inf_{E_2} f = 0$. We will need the following lemmas:

Lemma 11. [10] *With the above notations, f as constructed above. Then*

(1)

$$\int_{E_1(R+1)\setminus E_1(R)} (1-f)^2 \leq C\exp(-2\sqrt{\lambda_1(M)}R)$$

$$\int_{E(R+1)\setminus E(R)} f^2 \leq C\exp(-2\sqrt{\lambda_1(M)}R)$$

for some constant C depends on f, $\lambda_1(M)$ and the dimension of M, where E is any other end different from E_1.

(2)

$$\int_{E(R+1)\setminus E(R)} |\nabla f|^2 \leq C\exp(-2\sqrt{\lambda_1(M)}R),$$

for R sufficiently large, where E is any end of M.

Lemma 12. [12] *For the function f constructed above, let $\inf f < a < b < \sup f$,*

$$l(t) = \{x \in M : f(x) = t\}$$

and

$$\mathcal{L}(a,b) = \{x \in M : a < f(x) < b\}.$$

Then

$$\int_{\mathcal{L}(a,b)} |\nabla f|^2 = (b-a) \int_{l(b)} |\nabla f|$$

and

$$\int_{l(b)} |\nabla f| = \int_{l(t)} |\nabla f|,$$

for any $t \in (\inf f, \sup f)$.

6. An one end result

Theorem 13. *Let M be a complete noncompact 16-dimensional manifold with holonomy group Spin(9). Assume that the lowest spectrum satisfies $\lambda_1(M) \geq \frac{216}{7}$. Then M has only one end with infinite volume.*

Proof. Suppose that M has at least two infinite volume ends, E_1, E_2. Since $\lambda_1(M) > 0$, E_1, E_2 must be nonparabolic. Let f be the harmonic function constructed as in the previous section. Let $e_1 = \frac{\nabla f}{|\nabla f|}$ and $\{e_1, \cdots, e_8, e_9, \cdots, e_{16}\}$ be a local orthonormal frame as in theorem 10 such that $e_1 f = |\nabla f|$, $e_\alpha f = 0$, $2 \leq \alpha \leq 16$ at a point x and

$$\sum_{i=1}^{8} f_{ii} = 0,$$

hence we have

$$
\begin{aligned}
\sum_{i,j=1}^{16} f_{ij}^2 &\geq f_{11}^2 + \sum_{i=2}^{8} f_{ii}^2 + 2\sum_{i=2}^{16} f_{1j}^2 \\
&\geq f_{11}^2 + \frac{1}{7}\left(\sum_{i=2}^{8} f_{ii}\right)^2 + 2\sum_{i=2}^{16} f_{1j}^2 \\
&\geq \frac{8}{7}\sum_{j=1}^{8} f_{1j}^2 \\
&= \frac{8}{7}|\nabla|\nabla f||^2
\end{aligned}
$$

at x. Combining the above inequality with Bochner formula gives us

$$
\begin{aligned}
\frac{1}{2}\triangle|\nabla f|^2 &= \sum_{i,j=1}^{16} f_{ij}^2 + \mathrm{Ric}(\nabla f, \nabla f) \\
&\geq \frac{8}{7}|\nabla|\nabla f|^2|^2 - 36|\nabla f|^2.
\end{aligned}
$$

Let $g = |\nabla f|^{6/7}$, the above inequality becomes

(8) $$\triangle g \geq -\frac{216}{7}g.$$

The variational principle of $\lambda_1(M)$ implies that for any compactly supported smooth function $\phi \in C_c^\infty(M)$, we have

$$
\begin{aligned}
\frac{216}{7}\int_M \phi^2 g^2 &\leq \int_M |\nabla(\phi g)|^2 \\
&= \int_M \left(|\nabla\phi|^2 g^2 + |\nabla g|^2\phi^2 + \frac{1}{2}\langle\nabla\phi^2, \nabla g^2\rangle\right) \\
&= \int_M |\nabla\phi|^2 g^2 - \int_M \phi^2 g\triangle g.
\end{aligned}
$$

Combining the above with (8), we have

(9)
$$0 \leq \int_M \phi^2 g \left(\triangle g + \frac{216}{7} g \right)$$
$$\leq \int_M |\nabla \phi^2| g^2.$$

We choose $\phi = \psi \cdot \chi$ to be the product of two compactly smooth functions. For any $\varepsilon \in (0, 1/2)$, we construct ψ, χ as follows

$$\chi(x) = \begin{cases} 0 & \text{on} & \mathcal{L}(0, \varepsilon/2) \cup \mathcal{L}(1 - \varepsilon/2, 1) \\ (\log 2)^{-1}(\log f - \log(\varepsilon/2)) & \text{on} & \mathcal{L}(\varepsilon/2, \varepsilon) \cap (M \setminus E_1) \\ (\log 2)^{-1}(\log(1 - f) - \log(\varepsilon/2)) & \text{on} & \mathcal{L}(1 - \varepsilon, 1 - \varepsilon/2) \cap E_1 \\ 1 & \text{otherwise} \end{cases}.$$

$$\psi = \begin{cases} 1 & \text{on} & B(R - 1) \\ R - r & \text{on} & B(R) \setminus B(R - 1) \\ 0 & \text{on} & M \setminus B(R) \end{cases}.$$

Then applying the right hand side of (9), we have

(10)
$$\int_M |\nabla \phi|^2 g^2 \leq 2 \int_M |\nabla \psi|^2 \chi^2 |\nabla f|^{\frac{12}{7}} + 2 \int_M |\nabla \chi|^2 \psi^2 |\nabla f|^{\frac{12}{7}}.$$

M is Einstein and the Ricci curvature satisfies $\text{Ric}_M = -36$ under our normalization. The local gradient estimate of Cheng-Yau [4] (see also [11]) implies that

$$|\nabla f| \leq Cf,$$

for some constant C. The above inequality implies that

(11)
$$|\nabla f| \leq C|1 - f|,$$

by replacing f with $1 - f$. On E_1, the first term of (10) can be estimated by

(12)
$$\int_{E_1} |\nabla \psi|^2 \chi^2 |\nabla f|^{\frac{12}{7}} \leq \left(\int_\Omega |\nabla f|^2 \right)^{6/7} \left(\int_\Omega 1 \right)^{1/7},$$

where $\Omega = E_1 \cap (B(R) \setminus B(R-1)) \cap (\mathcal{L}(1 - \varepsilon, 1 - \varepsilon/2) \cup \mathcal{L}(\varepsilon/2, \varepsilon))$. Since $0 < \varepsilon < 1/2$, $\varepsilon/2 \leq 1 - f$ on Ω and we have

$$\int_\Omega 1 \leq \int_\Omega \left(\frac{2(1 - f)}{\varepsilon} \right)^2$$
$$\leq C_1 \varepsilon^{-2} \exp(-2\sqrt{\lambda_1(M)} R),$$

where the last inequality follows from lemma 11. Combining lemma 11, the above inequality and (12), we conclude that

(13)
$$\int_{E_1} |\nabla \psi|^2 \chi^2 |\nabla f|^{12/7} \leq C_2 \varepsilon^{-2/7} \exp(-2\sqrt{\lambda_1(M)} R).$$

The second term of (10) can be estimated by

$$
\int_{E_1} |\nabla \chi|^2 \psi^2 |\nabla f|^{12/7} \leq (\log 2)^{-2} \int_{\mathcal{L}(1-\varepsilon,1-\varepsilon/2) \cap E_1 \cap B(R)} |\nabla f|^{12/7} |\nabla \log(1-f)|^2
$$

$$
= (\log 2)^{-2} \int_{\mathcal{L}(1-\varepsilon,1-\varepsilon/2) \cap E_1 \cap B(R)} |\nabla f|^{2+12/7}(1-f)^{-2}
$$

$$
\leq C_3 \int_{\mathcal{L}(1-\varepsilon,1-\varepsilon/2) \cap E_1 \cap B(R)} |\nabla f|^2 (1-f)^{-2/7},
$$

where the last inequality follows from (11). Co-area formula and lemma 12 give us

$$
\int_{\mathcal{L}(1-\varepsilon,1-\varepsilon/2) \cap E_1 \cap B(R)} |\nabla f|^2 (1-f)^{-2/7} = \int_{1-\varepsilon}^{1-\varepsilon/2} (1-t)^{-2/7} \int_{l(t) \cap E_1 \cap B(R)} |\nabla f| dA dt
$$

$$
\leq \int_{l(b)} |\nabla f| dA \int_{1-\varepsilon}^{1-\varepsilon/2} (1-t)^{-2/7} dt
$$

$$
\leq C_4 \varepsilon^{5/7} \int_{l(b)} |\nabla f| dA.
$$

Therefore, combing the above inequalities, (10) becomes

$$
(14) \qquad \int_{E_1} |\nabla \phi|^2 g^2 \leq C_5 \left(\varepsilon^{-2/7} \exp(-2\sqrt{\lambda_1(M)}R) + \varepsilon^{5/7} \right).
$$

Applying the same argument to $1 - f$ instead of f to the rest of the ends of M, we have

$$
(15) \qquad \int_{M \setminus E_1} |\nabla \phi|^2 g^2 \leq C_5 \left(\varepsilon^{-2/7} \exp(-2\sqrt{\lambda_1(M)}R) + \varepsilon^{5/7} \right).
$$

Combining (9), (14) and (15), letting $R \to +\infty$ and $\varepsilon \to 0$, we conclude that

$$
\triangle g = -\frac{216}{7} g,
$$

and hence all the inequalities in proving (8) are indeed equalities. In particular, $(f_{\alpha\beta})$ is diagonal and there exists a function μ such that

$$
(16) \qquad (f_{\alpha\beta}) = \begin{pmatrix} -7\mu & & \\ & D_1 & \\ & & D_2 \end{pmatrix},
$$

where $D_1 = \mu I$ and D_2 is the 8×8 zero matrix. Since $f_{1\alpha} = 0$ for any $\alpha \neq 1$, $|\nabla f|$ is constant along the level set of f. In particular, the level sets of $|\nabla f|$ and f coincide. Suppose $|\nabla f|(x) = 0$, by considering $f + c$, we may assume that $f(x) = 0$. The regularity theory of harmonic functions asserts that f locally in a neighborhood of x behaves like a homogeneous harmonic polynomial in $\mathbb{R}^n$ with the origin at x. This is impossible since the level sets of $|\nabla f|$ and f coincide. Hence $|\nabla f| \neq 0$ on M and M is diffeomorphic to $\mathbb{R} \times N$, where N is given by the level set of f. N is compact since we have assumed that M has at least two ends. Fix a level set N of f. We choose a local orthonormal frame $\{e_i\}_{\alpha=2}^{16}$ of N and $e_1 = \frac{\nabla f}{|\nabla f|}$. Let $\gamma(t)$ be the integral curve of e_1 and $\{e_\alpha(t)\}_{\alpha=2}^{16}$ be the parallel transport of $\{e_\alpha\}_{\alpha=2}^{16}$ along γ. $\langle \nabla_{e_1} e_1, e_1 \rangle = 0 = \langle \nabla_{e_1} e_1, e_\alpha \rangle$ for any $\alpha \geq 2$ implies

$$
\nabla_{e_1} e_1 = 0,
$$

and hence γ is a geodesic. The second fundamental form of the level set of f satisfies the following equations

$$
(17) \qquad \begin{aligned}
f_{\alpha\beta} &= e_\alpha e_\beta f - (\nabla_{e_\alpha} e_\beta) f \\
&= \langle -(\nabla_{e_\alpha} e_\beta), e_1 \rangle f_1 \\
&= h_{\alpha\beta} f_1
\end{aligned}
$$

$$
(18) \qquad \nabla_{e_\alpha} e_1 = \sum_{\beta=2}^{16} h_{\alpha\beta} e_\beta
$$

where $h_{\alpha\beta} = \langle -(\nabla_{e_\alpha} e_\beta), e_1 \rangle$ is the second fundamental form of N. We now compute the curvature of M

$$
(19) \qquad \begin{aligned}
\langle R(e_1, e_\alpha) e_1, e_\alpha \rangle &= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1 - \nabla_{e_\alpha} \nabla_{e_1} e_1 - \nabla_{[e_1, e_\alpha]} e_1, e_\alpha \rangle \\
&= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle - \langle \nabla_{[e_1, e_\alpha]} e_1, e_\alpha \rangle \\
&= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle - \langle \nabla_{\nabla_{e_1} e_\alpha - \nabla_{e_\alpha} e_1}, e_\alpha \rangle \\
&= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle + \sum_{\beta=2}^{16} \langle \nabla_{e_\alpha} e_1, e_\beta \rangle \langle \nabla_{e_\beta} e_1, e_\alpha \rangle \\
&= \sum_{\beta=2}^{16} \langle e_1(h_{\alpha\beta}) e_\beta, e_\alpha \rangle + \sum_{\beta=2}^{16} h_{\alpha\beta}^2 \\
&= e_1(h_{\alpha\alpha}) + h_{\alpha\alpha}^2,
\end{aligned}
$$

where we have used (17), (18) and the fact that $(f_{\alpha\beta})$ is diagonal. In particular, since M is covered by $\mathbb{H}_{\mathbb{O}}^2$, (16) and (19) implies the sectional curvature

$$
K_M(e_1, e_k) = K_{\mathbb{H}_{\mathbb{O}}^2}(e_1, e_k) = 0,
$$

for any $k \geq 9$. It contradicts to the fact that the sectional curvature of $\mathbb{H}_{\mathbb{O}}^2$ is pinched between -4 and -1. Therefore, M has only one infinite volume end. $\qquad \square$

7. Splitting type theorem

Theorem 14. *Let M be a complete noncompact 16-dimensional manifold with holonomy group Spin(9). Assume that the lowest spectrum of M achieves the maximal value, that is $\lambda_1(M) = 121$. Then either*

(1) M has only one end; or
(2) M is diffeomorphic to $\mathbb{R} \times N$ with metric

$$
ds_M^2 = dt^2 + e^{-4t} \sum_{k=2}^{8} \omega_k^2 + e^{-2t} \sum_{k=9}^{16} \omega_k^2,
$$

where $\{\omega_2, \cdots, \omega_{16}\}$ is an orthonormal basis for a compact manifold N given by a compact quotient of the horosphere of the universal cover $\tilde{M}$ of M.

Proof. Since $\lambda_1(M) > 0$, M is nonparabolic and hence M has at least one nonparabolic end. Assume that M has at least two ends. Theorem 13 implies that M must have a parabolic end. Let E_1 be a nonparabolic end and E_2 be a parabolic end with respect to $B_p(R_0)$, the geodesic ball with radius R_0 centered at p. In other words, E_1, E_2 are two unbounded component of $M \setminus B_p(R_0)$. Let $\bar{\gamma} : [0, +\infty) \to M$

be a geodesic ray with $\overline{\gamma}(0) = p$ and $\overline{\gamma}([R_0, +\infty)) \subseteq E_2$, for some $a > 0$. Let $\beta(x) = \lim_{t \to \infty}(t - r(x, \overline{\gamma}(t)))$ be the Busemann function with respect to $\overline{\gamma}$. Theorem 2 gives us

$$\triangle r(x, \overline{\gamma}(t)) \leq 14 \coth(2r(x, \overline{\gamma}(t))) + 8 \coth r(x, \overline{\gamma}(t)),$$

which implies

$$\triangle \beta \geq -22,$$

in the sense of distribution. Let $f = \exp(11\beta)$, we compute

$$\begin{aligned}
\triangle f &= 11f\triangle\beta + 121f|\nabla\beta|^2 \\
&\geq -121f.
\end{aligned}$$

Using the variation principle of $\lambda_1(M) = 121$, for any $\phi \in C_c^\infty(M)$ nonnegative smooth function with compact support, we have

$$\begin{aligned}
121\int_M \phi^2 f^2 &\leq \int_M |\nabla(\phi f)|^2 \\
&= \int_M |\nabla\phi|^2 f^2 + \frac{1}{2}\int_M \langle\nabla\phi^2, \nabla f^2\rangle + \int_M \phi^2|\nabla f|^2 \\
&= \int_M |\nabla\phi|^2 f^2 - \int_M \phi^2 f\triangle f,
\end{aligned}$$

thus

$$(20) \qquad \int_M \phi^2 f(\triangle f + 121f) \leq \int_M |\nabla\phi|^2 f^2.$$

Follow the argument in [9], if we choose the following cut-off function

$$\phi(x) = \begin{cases} 1 & \text{on} \quad B_p(R) \\ \frac{2R - r(x)}{R} & \text{on} \quad B_p(2R) \setminus B_p(R) \\ 0 & \text{on} \quad M \setminus B_p(R) \end{cases},$$

then the right hand side of (20) converges to zero as $R \to +\infty$. Indeed,

$$(21) \quad \int_M |\nabla\phi^2|f^2 \leq R^{-2}\int_{(B_p(2R)\setminus B_p(R))\cap E_2} f^2 + R^{-2}\int_{(B_p(2R)\setminus B_p(R))\setminus E_2} f^2.$$

For an end E, let $V_R(E)$ be the volume of the set $B_p(R) \cap E$ and let $k - 1 \leq R < k$. The first term on the right hand side of (21) can be estimated by

$$\begin{aligned}
(22) \qquad \int_{(B_p(2R)\setminus B_p(R))\cap E_2} f^2 &\leq \sum_{i=1}^k \int_{(B_p(R+i)\setminus B_p(R+i-1))\cap E_2} f^2 \\
&\leq \sum_{i=1}^k e^{22(R+i)}\left(V_{E_2}(R+i) \setminus V_{E_2}(R+i-1)\right) \\
&\leq C_1 \sum_{i=1}^k e^{22(R+i)}e^{-22(R+i-1)} \\
&\leq C_2 R,
\end{aligned}$$

where the second inequality follows from $|\beta(x)| \leq r(x,p)$ and the third inequality follows from the volume estimate on a parabolic end E of [10],

$$V_\infty(E) - V_R(E) \leq C \exp(-2\sqrt{\lambda(E)}R) \ \text{ if } \lambda(E) > 0.$$

On the other hand, let τ be the geodesic ray given in lemma 3. For any $x \in M \setminus (B_p(R_0) \cup E_2)$, then τ must intersect $B_p(R_0)$. Let y to be the first point on τ that intersects $B_p(R_0)$, it is easy to see that ([9])

$$
\begin{aligned}
\beta(y) - \beta(x) &\geq r(x,y) \\
&\geq r(x,p) - r(y,p),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\beta(x) &\leq -r(x,p) + r(y,p) + \beta(y) \\
&\leq -r(x,p) + 2r(y,p) \\
&\leq -r(x,p) + 2R_0,
\end{aligned}
$$

and hence the second term of the right hand side of (21) can now be estimated by

$$
\begin{aligned}
(23)\int_{(B_p(2R)\setminus B_p(R))\setminus E_2} f^2 &\leq \sum_{i=1}^{k} \int_{(B_p(R+i)\setminus B_p(R+i-1))\setminus E_2} f^2 \\
&\leq \sum_{i=1}^{k} \int_{(B_p(R+i)\setminus B_p(R+i-1))\setminus E_2} \exp(44R_0 - 22r(x,p)) \\
&\leq \sum_{i=1}^{k} C_3 e^{-22(R+i-1)} V(B_p(R_0+i)) \\
&\leq \sum_{i=1}^{k} C_3 e^{-22(R+i-1)} e^{22(R+i)} \\
&\leq C_4 R.
\end{aligned}
$$

Combining (21), (22) and (23), we conclude that the right hand side of (20) converges to zero as $R \to +\infty$. Since f is non-negative, (20) now implies

$$\triangle f + 121 f = 0,$$

and all inequalities in the proving (20) are indeed equalities and in particular,

$$(24) \qquad \triangle\beta = -22, \ \ |\nabla\beta| = 1,$$

and β is smooth by the regularity of the above equation. The above equation implies M is diffeomorphic to $\mathbb{R} \times N$, where N is diffeomorphic to the level set of β. N is compact since otherwise M would have only one end, contradicts to our assumption that M has two ends. Let N_0 be the level set of β with $x \in N_0$. Let $e_1 = \nabla\beta(x) = \tau'(x)$ be the unit normal direction of N_0 at x, where τ was the geodesic ray given in lemma 3. Let $\gamma(t)$ be the integral curve of $\nabla\beta$ with $\gamma(0) = x \in N_0$ and $e_1(t) = \gamma'(t)$. We pick a local orthonormal frame $\{e_i\}_{A=2}^{16}$ of N_0 around x as in the proof of proposition 1 such that

$$
\begin{aligned}
R_{1i1i}(x) &= -4, \ 2 \leq i \leq 8 \\
R_{1\alpha1\alpha}(x) &= -1, \ 9 \leq \alpha \leq 16,
\end{aligned}
$$

at x. We extend the frame to a local orthonormal frame $\{e_A(t)\}_{A=2}^{16}$ along γ by parallel transport. $e_1\langle e_1, e_\alpha \rangle = 0 = e_1\langle e_1, e_1 \rangle$ implies $\nabla_{e_1} e_1 = 0$, thus $\gamma(t)$ is a normal geodesic with $\gamma'(0) = \tau'(x)$. Therefore $\gamma \equiv \tau$. As in the proof of proposition 1, we have

$$\begin{aligned} R_{1i1i}(\gamma(t)) &= -4, \ 2 \le i \le 8 \\ R_{1\alpha1\alpha}(\gamma(t)) &= -1, \ 9 \le \alpha \le 16, \end{aligned}$$

along γ. Bochner formula gives us

(25)
$$\begin{aligned} 0 &= \frac{1}{2}\triangle|\nabla\beta|^2 \\ &= \sum_{i,j=1}^{16} \beta_{ij}^2 + \mathrm{Ric}(\nabla\beta, \nabla\beta) + \langle \nabla\beta, \nabla\triangle\beta \rangle \\ &= \sum_{i,j=1}^{16} \beta_{ij}^2 - 36. \end{aligned}$$

The proof of proposition 1 with $e_1 = \gamma'(0) = \tau'(x)$ implies

$$\beta_{11} = 0, \quad \sum_{i=2}^{8} \beta_{ii} \ \geq \ -14, \quad \sum_{\alpha=9}^{16} \beta_{\alpha\alpha} \geq -8,$$

where the first equality comes from the fact that β is linear along τ (lemma 3). Combining the above with (24) implies

(26)
$$\sum_{i=2}^{8} \beta_{ii} \ = \ -14, \quad \sum_{\alpha=9}^{16} \beta_{\alpha\alpha} = -8.$$

Combining (25) and (26), we have

$$\begin{aligned} 36 &= \sum_{A,B=1}^{16} \beta_{AB}^2 \\ &\geq \sum_{i=2}^{8} \beta_{ii}^2 + \sum_{\alpha=9}^{16} \beta_{\alpha\alpha}^2 \\ &\geq \frac{1}{7}\left(\sum_{i=2}^{8} \beta_{ii}\right)^2 + \frac{1}{8}\left(\sum_{\alpha=9}^{16} \beta_{\alpha\alpha}\right)^2 \\ &= 36. \end{aligned}$$

Therefore all inequalities in the above proof are indeed equalities β_{AB} is diagonal and

(27)
$$\beta_{AB} = -c_A \delta_{AB},$$

where

$$c_A = \begin{cases} 0 & A = 1 \\ 2 & 2 \le A \le 8 \\ 1 & 9 \le A \le 16 \end{cases}.$$

The second fundamental form of the each level set $N_t = \{x \in M : \beta(x) = t\}$ with respect to the normal vector $\nabla\beta$ can now be calculated

$$
\begin{aligned}
h_{\sigma\tau} &= \langle -\nabla_{e_\sigma} e_\tau, e_1 \rangle \\
&= \langle -\nabla_{e_\sigma} e_\tau, \nabla\beta \rangle \\
&= -(\nabla_{e_\sigma} e_\tau)\beta \\
&= \beta_{\sigma\tau},
\end{aligned}
$$

where $2 \le \sigma, \tau \le 16$ and the last equality follows from the fact that N_t is a level set of β. In particular, we have

$$
(28) \qquad \nabla_{e_\sigma} e_1 = \sum_{\tau=2}^{16} \beta_{\sigma\tau} e_\tau.
$$

For any $p \in N_0$, let $\gamma(t)$ be the integral curve of $\nabla\beta$ with $\gamma(0) = p$. Define $\overline{\psi}_t(p) = \gamma(t)$, and it induces a map $\psi_t : N_0 \to N_t$. As we have already seen that the integral curve of $\nabla\beta$ is a normal geodesic, $\sigma(t) = \psi_t(\cdot)$ is always a normal geodesic and thus ψ_t is a geodesic flow on M, therefore $d\psi_t(X)$ is a Jacobi field along each integral curve. Let $\overline{e}_k$ be the restriction of e_k on N_0, $1 \le k \le 16$. We claim that $d\psi_t(\overline{e}_i) = V_i(t)$, where

$$
V_A(t) = e^{-c_A t} e_A(t), \quad 2 \le A \le 16.
$$

By the uniqueness of Jacobi field, it is sufficient to show that $V_A(t)$ satisfies the Jacobi equation with the same initial conditions as $d\psi_t(\overline{e}_A)$. We have

$$
\begin{aligned}
\nabla_{\gamma'} \nabla_{\gamma'} V_A &= -c_A^2 e_A \\
&= R_{1A1A} e_A \\
&= R(\gamma', V)\gamma',
\end{aligned}
$$

since $R_{AB} = R_{1A1B}$ is diagonal. On the other hand, $V_A(0) = \overline{e}_A = d\psi_0(\overline{e}_A)$ and (28) implies

$$
\begin{aligned}
\nabla_{\gamma'}(d\psi_t(\overline{e}_A))(0) &= \nabla_{\overline{e}_A} e_1(0) \\
&= \sum_{\tau=2}^{16} \beta_{A\tau} \overline{e}_\tau \\
&= -c_A \overline{e}_A,
\end{aligned}
$$

since we can view e_1 and $d\psi_t(\overline{e}_A)$ as tangent vectors of a map from a rectangle. Therefore $V_A'(0) = -c_A \overline{e}_A = \nabla_{\gamma'}(d\psi_t(\overline{e}_A))(0)$. In conclusion, each N_t can be viewed as a copy of N_0 and M is diffeomorphic $\mathbb{R} \times N_0$ with the metric

$$
ds_M^2 = dt^2 + e^{-4t} \sum_{k=2}^{8} \omega_k^2 + e^{-2t} \sum_{k=9}^{16} \omega_k^2,
$$

where $\{\omega_k\}_{k=2}^{16}$ is the coframe of $\{\overline{e}_2, \cdots, \overline{e}_{16}\}$.

$\square$

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