ON HOFMANN’S BILINEAR ESTIMATE

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Abstract. Using the framework of a previous article joint with Axelsson and McIntosh, we extend to systems two results of S. Hofmann for real symmetric equations and their perturbations going back to a work of B. Dahlberg for Laplace’s equation on Lipschitz domains. The first one is a certain bilinear estimate for a class of weak solutions and the second is a criterion which allows to identify the domain of the generator of the semi-group yielding such solutions.

1. Introduction

S. Hofmann proved in [10] that weak solutions of

\[ \text{div}_{t,x} A(x) \nabla_{t,x} U(t,x) = \sum_{i,j=0}^{n} \partial_i A_{i,j}(x) \partial_j U(t,x) = 0 \]

on the upper half space \( \mathbb{R}_{+}^{1+n} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n ; t > 0\} \), \( n \geq 1 \), where the matrix \( A = (A_{i,j}(x))_{i,j=0}^{n} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{1+n})) \) is assumed to be \( t \)-independent and within some small \( L^\infty \) neighborhood of a real symmetric strictly elliptic \( t \)-independent matrix, obey the following bilinear estimate

\[ \left| \int_{\mathbb{R}_{+}^{1+n}} \nabla_{t,x} U \cdot \nabla dt \, dx \right| \leq C \| U_0 \|_2 \left( \| t \nabla v \| + \| N_* v \|_2 \right) \]

for all \( C^{1+n} \)-valued field \( v \) such that the right-hand side is finite. See below for the definition of the square-function \( \| \| \| \) and the non-tangential maximal operator \( N_* \).

The trace of \( U \) at \( t = 0 \) is assumed to be in the sense of non-tangential convergence a.e. and in \( L^2(\mathbb{R}^n) \).

In addition, he proves that the solution operator \( U_0 \rightarrow U(t, \cdot) \) defines a bounded \( C_0 \) semi-group on \( L^2(\mathbb{R}^n) \) whose infinitesimal generator \( A \) has domain \( W^{1,2}(\mathbb{R}^n) \) with \( \| Af \|_2 \sim \| \nabla f \|_2 \).

Such results were first proved by B. Dahlberg [8] for harmonic functions on a Lipschitz domain. A version of the bilinear estimate for Clifford-valued monogenic functions was proved by Li-McIntosh-Semmes [15]. A short proof of Dahlberg’s estimate for harmonic functions and some applications appear in Mitrea’s work [16]. \( L^p \) versions are recently discussed by Varopoulos [19].

Hofmann’s arguments for variable coefficients rely on the deep results of [1], and in particular Theorem 1.11 there where the boundedness and invertibility of the layer potentials are obtained from a \( T(b) \) theorem, Rellich estimates in the case of real

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symmetric matrices and perturbation. This also generalizes somehow the case where
\( A_{i,i} = A_{i,0} = 0 \) for \( i = 1, \ldots, n \) corresponding to the Kato square root problem.

The recent works [3, 4], pursuing ideas in [2], allow us to extend this further to
systems, making clear in particular that specificities of real symmetric coefficients and
their perturbations and of equations - in particular the De Giorgi-Nash-Moser esti-
mates - are not needed: it only depends on whether the Dirichlet problem is solvable.
We use the solution operator constructed in [2] and the proof using \( P_1 - Q_1 \) techniques
of Coifman-Meyer from [7] makes transparent the para-product like character of this
bilinear estimate. We also establish a necessary and sufficient condition telling when
the domain of the infinitesimal generator \( A \) of the Dirichlet semi-group is \( W^{1,2} \).

We apologize to the reader for the necessary conciseness of this note and suggests
he (or she) has (at least) the references [2, 3, 4] handy. In Section 2, we try to extract
from them the relevant information. The proof or the bilinear estimate for variable
coefficients systems is in Section 3. Section 4 contains the discussion on the domain
of the Dirichlet semi-group.

2. Setting

We begin by giving a precise definition of well-posedness of the Dirichlet problem
for systems. Throughout this note, we use the notation \( X \approx Y \) and \( X \lesssim Y \) for
estimates to mean that there exists a constant \( C > 0 \), independent of the variables in
the estimate, such that \( X/C \approx Y \leq CX \) and \( X \approx CY \), respectively.

We write \( (t, x) \) for the standard coordinates for \( \mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n \), \( t \) standing for
the vertical or normal coordinate. For vectors \( v = (v^\alpha_i)_{1 \leq \alpha \leq m} \in \mathbb{C}^{(1+n)\times m} \), we write
\( v = (v_0)_{0 \leq \alpha \leq m} \) for the normal and tangential parts of \( v \), i.e.
\( v_0 = (v^\alpha_i)_{1 \leq \alpha \leq m} \) whereas \( v_i = (v^\alpha_i)_{1 \leq \alpha \leq m} \).

For systems, gradient and divergence act as \( (\nabla_{t,x} U)^\alpha_i = \partial_i U^\alpha \) and \( (\text{div}_{t,x} F)^\alpha_i = \sum_{i=1}^n \partial_i F^\alpha_i \),
with corresponding tangential versions \( \nabla_{t,x} U = (\nabla_{t,x} U)^\alpha_i \) and \( \text{div}_{t,x} F = \sum_{i=1}^n \partial_i F^\alpha_i \).
With \( \text{curl}_{t,x} F_i = 0 \), we understand \( \partial_j F^\alpha_i = \partial_i F^\alpha_j \), for all \( i, j = 1, \ldots, n \),
\( \alpha = 1, \ldots, m \).

We consider divergence form second order elliptic systems
\begin{equation}
\sum_{i,j=0}^n \sum_{\alpha=1}^m \partial_i A_{ij}^{\alpha \beta}(x) \partial_j U^\beta(t,x) = 0, \quad \alpha = 1, \ldots, m,
\end{equation}
on the half space \( \mathbb{R}^{1+n}_+ := \{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t > 0 \} \), \( n \geq 1 \), where the matrix
\( A = (A_{ij}(x))_{i,j=0,\ldots,n} \in L_\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^{1+n\times m})) \)
is assumed to be \( t \)-independent with complex coefficients, and strictly accretive on
\( N(\text{curl}_1) \) in the sense that there exists \( \kappa > 0 \) such that
\begin{equation}
\sum_{i,j=0}^n \sum_{\alpha,\beta=1}^m \int_{\mathbb{R}^n} \text{Re}(A_{ij}^{\alpha \beta}(x) F_j^\beta(x) \overline{F_i^\alpha(x)}) dx \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbb{R}^n} |F_i^\alpha(x)|^2 dx,
\end{equation}
for all \( f \in N(\text{curl}_1) := \{ g \in L_2(\mathbb{R}^n; \mathbb{C}^{1+n\times m}) : \text{curl}_1(g) = 0 \} \). This is nothing but
ellipticity in the sense of Gårding. See the discussion in [5]. By changing \( m \) to \( 2m \) we
could assume that the coefficients are real-valued. But this does not simplify matters
and we need the complex hermitean structure of our \( L_2 \) space anyway.
Definition 2.1. The Dirichlet problem (Dir-A) is said to be well-posed if for each $u \in L_2(\mathbb{R}^n; C^m)$, there is a unique function

$$U_t(x) = U(t, x) \in C^1(\mathbb{R}^+; L_2(\mathbb{R}^n; C^m))$$

such that $\nabla_x U \in C^0(\mathbb{R}^+; L_2(\mathbb{R}^n; C^m))$, where $U$ satisfies (2) for $t > 0$, $\lim_{t \to 0} U_t = u$, $\lim_{t \to -\infty} U_t = 0$, $\lim_{t \to -\infty} \nabla_{t,x} U_t = 0$ in $L_2$ norm, and $\int_{t_0}^{t_1} \nabla_x U_t \, ds$ converges in $L_2$ when $t_0 \to 0$ and $t_1 \to \infty$. More precisely, by $U$ satisfying (2), we mean that $\int_{t_0}^{t_1} ((A \nabla_{s,x} U_s)_t, \nabla_x v) \, ds = -((A \nabla_{t,x} U_t)_0, v)$ for all $v \in C^0(\mathbb{R}^n; C^m)$.

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [10]. However, a sufficient condition is provided in [5] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem[1] (see Section [4]).

Remark 2.2. In the case of block matrices, i.e. $A_{ii,j}^\alpha(x) = 0 = A_{i,0}^\alpha(x)$, $1 \leq i \leq n$, $1 \leq \alpha, \beta \leq m$, the second order system (2) can be solved using semi-group theory: $V(t, \cdot) = e^{-tL^{1/2}} I_0$ for $L = -A_{i0}^{1/2} \text{div}_x A_{i1} \nabla_x$ acting as an unbounded operator on $L_2(\mathbb{R}^n, C^{m \alpha})$ (See below for the notation). This solution satisfies $V_t = V(t, \cdot) \in C^0(\mathbb{R}^+; L_2(\mathbb{R}^n; C^m)) \cap C^1(\mathbb{R}_+, D(L^{1/2}))$, $\lim_{t \to 0} V_t = u$, $\lim_{t \to -\infty} V_t = 0$ in $L_2$ norm, and (2) holds in the strong sense in $\mathbb{R}^n$ for all $t > 0$ (and in the sense of distributions in $\mathbb{R}^+_\alpha$). Hence, the two notions of solvability are not a priori equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for $L_2(\mathbb{R}^n, C^{m \alpha})$ with $\|L^{1/2} f\|_2 \sim \|\nabla f\|_2$. See [6] where this is explicitly proved when $A_{00} \neq I$.

The following result is Corollary 3.4 of [5] (which, as we recall, furnishes a different proof of results obtained by combining [11] and [9] in the case of real symmetric matrices equations $(m = 1)$).

Theorem 2.3. Let $A \in L_\infty(\mathbb{R}^n; \mathcal{L}(C^{1+n+m}))$ be a $t$-independent, complex matrix function which is strictly accretive on $\mathcal{N}(\text{curl}_1)$ and assume that (Dir-A) is well-posed. Then any function $U_t(x) = U(t, x) \in C^1(\mathbb{R}^+; L_2(\mathbb{R}^n; C^m))$ solving (2), with properties as in Definition 2.1, has estimates

$$\int_{\mathbb{R}^n} |u|^2 \, dx \approx \sup_{t > 0} \int_{\mathbb{R}^n} |U_t|^2 \, dx \approx \int_{\mathbb{R}^n} |\nabla_x U|^2 \, dx \approx \|t \nabla_{t,x} U\|^2,$$

where $u = U|_{\mathbb{R}^n}$. If furthermore $A$ is real (not necessarily symmetric) and $m = 1$, then Moser’s local boundedness estimate [17] gives the pointwise estimate $\tilde{N}_s(U)(x) \approx N_s(U)(x)$, where the standard non-tangential maximal function is $N_s(U)(x) := \sup_{|y - x| < c} |U(t, y)|$, for fixed $0 < c < \infty$.

We use the square-function norm

$$\|F_t\|^2 := \int_0^\infty \|F_t\|^2 \, dt = \int_{\mathbb{R}_+^{1+n}} |F(t, x)|^2 \, dt \, dx \over t$$

We showed with A. Axelsson [2] that well-posedness in the sense of Definition 2.1 is equivalent to well-posedness in the class of weak solutions $U \in W^{1,2}_{loc}(\mathbb{R}^+_{1+n})$ of (2) such that $U \in C^0(\mathbb{R}^+; L_2(\mathbb{R}^n; C^{m \alpha}))$ and $\|t \nabla_{t,x} U\| < \infty$.\footnote{We showed with A. Axelsson [2] that well-posedness in the sense of Definition 2.1 is equivalent to well-posedness in the class of weak solutions $U \in W^{1,2}_{loc}(\mathbb{R}^+_{1+n})$ of (2) such that $U \in C^0(\mathbb{R}^+; L_2(\mathbb{R}^n; C^{m \alpha}))$ and $\|t \nabla_{t,x} U\| < \infty$.}
and the following version $\tilde{N}_*(F)$ of the modified non-tangential maximal function introduced in [12]

$$\tilde{N}_*(F)(x) := \sup_{t > 0} t^{-(1+n)/2} \|F\|_{L^2(Q(t,x))},$$

where $Q(t,x) := [(1-c_0)t, (1+c_0)t] \times B(x;c_1t)$, for some fixed constants $c_0 \in (0,1)$, $c_1 > 0$.

Next is Theorem 3.2 of [5], specialized to the Dirichlet problem.

**Theorem 2.4.** The set of matrices $A$ for which $(\text{Dir}-A)$ is well-posed is an open subset of $L_\infty(\mathbb{R}^n;\mathcal{L}(C^{(1+n)m}))$. Furthermore, it contains

(i) all Hermitean matrices $A(x) = A(x)^*$ (and in particular all real symmetric matrices),

(ii) all block matrices where $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \leq i \leq n$, $1 \leq \alpha, \beta \leq m$, and

(iii) all constant matrices $A(x) = A$.

More importantly is the solution algorithm using an “infinitesimal generator” $T_A$.

Write $v \in C^{(1+n)m}$ as $v = [v_0, v_\parallel]^t$, where $v_0 \in C^m$ and $v_\parallel \in C^{nm}$, and introduce the auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_0 & A_0 \\ 0 & I \end{bmatrix}, \quad \Delta := \begin{bmatrix} 1 & 0 \\ A_0 & A_\parallel \end{bmatrix}, \quad \text{if } A = \begin{bmatrix} A_0 & A_\parallel \\ A_\parallel & A_\parallel \end{bmatrix}$$

in the normal/tangential splitting of $C^{(1+n)m}$. The strict accretivity of $A$ on $N(\text{curl}_\parallel)$, as in [3], implies the pointwise strict accretivity of the diagonal block $A_{00}$. Hence $A_{00}$ is invertible, and consequently $\overline{A}$ is invertible [This is not necessarily true for $A$]. We define

$$T_A = \overline{A}^{-1}DA$$

as an unbounded operator on $L^2(\mathbb{R}^n, C^{(1+n)m})$ with $D$ the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{bmatrix} 0 & \text{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

**Proposition 2.5.** Let $A \in L_\infty(\mathbb{R}^n;\mathcal{L}(C^{(1+n)m}))$ be a $t$-independent, complex matrix function which is strictly accretive on $N(\text{curl}_\parallel)$.

(1) The operator $T_A$ has quadratic estimates and a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n, C^{(1+n)m})$. In particular, for any holomorphic function $\psi$ on the left and right open half planes, with $z\psi(z)$ and $z^{-1}\psi(z)$ qualitatively bounded, one has

$$\|\psi(tT_A)f\| \lesssim \|f\|_2.$$  

(2) The Dirichlet problem $(\text{Dir}-A)$ is well-posed if and only if the operator

$$S : R(\chi_+(T_A)) \to L^2(\mathbb{R}^n, C^m), f \mapsto f_0$$

is invertible. Here, $\chi_+ = 1$ on the right open half plane and 0 on the left open half plane.

Item (1) is [5, Corollary 3.6] (and see [4] for an explicit direct proof) and item (2) can be found in [5, Section 4, proof of Theorem 2.2].
Lemma 2.6. Assume that (Dir-A) is well-posed. Let \( u_0 \in L_2(\mathbb{R}^n, C^m) \). Then the solution \( U \) of (Dir-A) in the sense of Definition 2.1 is given by
\[
U(t, \cdot) = (e^{-tT_A} f)_0, \quad f = S^{-1} u_0 \in \mathcal{R}(\chi_+(T_A))
\]
and furthermore
\[
\nabla_{t,x} U(t, \cdot) = \partial_t e^{-tT_A} f.
\]


3. The bilinear estimate

We are now in position to state and prove the generalisation of Hofmann’s result.

Theorem 3.1. Assume that (Dir-A) is well-posed. Let \( u_0 \in L_2(\mathbb{R}^n, C^m) \) and \( U \) be the solution to (Dir-A) in the sense of Definition 2.1. Then for all \( v: \mathbb{R}^{1+n} \to C^{(1+n)m} \) such that the right-hand side is finite,
\[
\left\| \int \int_{\mathbb{R}^{1+n}} \nabla_{t,x} U \cdot \nabla \, dt \, dx \right\| \leq C \| u_0 \|_2 (\| t \nabla_{t,x} v \| + \| N_* v \|_2).
\]

The pointwise values of \( v(t, x) \) in the non-tangential control \( N_* v \) can be slightly improved to \( L^1 \) averages on balls having radii \( \sim t \) for each fixed \( t \). See the end of proof.

Proof. It follows from the previous result that there exists \( f \in \mathcal{R}(\chi_+(T_A)) \) such that
\[
U(t, \cdot) = (e^{-tT_A} f)_0 \quad \text{and} \quad \nabla_{t,x} U(t, \cdot) = \partial_t f = -T_A e^{-tT_A} f, \quad F = e^{-tT_A} f.
\]

Integrating by parts with respect to \( t \), we find
\[
\int \int_{\mathbb{R}^{1+n}} \nabla U \cdot \nabla \, dt \, dx = - \int \int_{\mathbb{R}^{1+n}} t \partial_t F \cdot \overline{\partial_t v} \, dt \, dx - \int \int_{\mathbb{R}^{1+n}} t \partial_t^2 F \cdot \nabla \, dt \, dx.
\]
The boundary term vanishes because \( t \partial_t F \) goes to 0 in \( L_2 \) when \( t \to 0, \infty \) (this uses \( f \in \mathcal{R}(\chi_+(T_A)) \)) and \( \sup_{t > 0} \| v(t, \cdot) \|_2 < \infty \) from \( \| N_* v \|_2 < \infty \).

For the first term, we use Cauchy-Schwarz inequality and that \( \| t \partial_t F \| \lesssim \| u_0 \|_2 \) from Theorem 2.2.

For the second term, we use the following identity: \( T_A = \overline{A}^{-1} DB \overline{A} \) with \( B = A \overline{A}^{-1} \) which, by [5] Proposition 3.2], is strictly accretive on \( N(\text{curl}) \), and observe that
\[
t^2 \partial_t^2 F = \overline{A}^{-1} (tDB)^2 e^{-tDB} (\overline{A} f)
\]
\[
= \overline{A}^{-1} (tDB)(I + (tDB)^2)^{-1} \psi(tDB) (\overline{A} f)
\]
\[
= \overline{A}^{-1} (tDB)(I + (tDB)^2)^{-1} A \psi(tA)(f)
\]
with
\[
\psi(z) = z(1 + z^2) e^{-(\text{sgn} \text{Re} z) z}.
\]

Thus,
\[
\int \int_{\mathbb{R}^{1+n}} t \partial_t^2 F \cdot \nabla \, dt \, dx = \int \int_{\mathbb{R}^{1+n}} \overline{A} \psi(tA)(f) \cdot \overline{Q} \overline{A} \partial_t \overline{v} \, dt \, dx.
\]
with $Q_t = \Theta_t \mathcal{A}^{-1}$ and $\Theta_t = (tB^*D)(I + (tB^*D)^2)^{-1}$ acting on $\mathbf{v}_t \equiv \mathbf{v}(t, \cdot)$ for each fixed $t$ [The notation $\mathcal{A}$ has nothing to do with complex conjugate and we apologize for any conflict this may cause]. It follows from the quadratic estimates of Proposition 2.3 that
\[
\|\psi(T_A)(\mathbf{f})\| \lesssim \|\mathbf{f}\|_2.
\]

It remains to estimate $\|Q_t \mathbf{v}_t\|$. To do that we follow the principal part approximation of [4] - which is an elaboration of the so-called Coifman-Meyer trick [7] - applied to $Q_t$ instead of $\Theta_t$ there. That is, we write
\[
(4) \quad Q_t \mathbf{v}_t = Q_t \left( \frac{I - P_t}{t(-\Delta)^{1/2}} \right) t(-\Delta)^{-1/2} \mathbf{v}_t + (Q_t P_t - \gamma_t S_t P_t) \mathbf{v}_t + \gamma_t S_t P_t \mathbf{v}_t
\]
where $\Delta$ is the Laplacian on $\mathbb{R}^n$; $P_t$ is a nice scalar approximation to the identity acting componentwise on $L_2(\mathbb{R}^n, C^{(1+n)m})$ and $\gamma_t$ is the element of $L^2_{loc}(\mathbb{R}^n; L(C^{(1+n)m}))$ given by
\[
\gamma_t(x) \mathbf{w} := (Q_t \mathbf{w})(x)
\]
for every $\mathbf{w} \in C^{(1+n)m}$. We view $\mathbf{w}$ on the right-hand side of the above equation as the constant function valued in $C^{(1+n)m}$ defined on $\mathbb{R}^n$ by $\mathbf{w}(x) := \mathbf{w}$. We identify $\gamma_t(x)$ with the (possibly unbounded) multiplication operator $\gamma_t : f(x) \mapsto \gamma_t(x)f(x)$. Finally, the dyadic averaging operator $S_t : L_2(\mathbb{R}^n, C^{(1+n)m}) \to L_2(\mathbb{R}^n, C^{(1+n)m})$ is given by
\[
S_t \mathbf{u}(x) := \frac{1}{|Q|} \int_Q \mathbf{u}(y) \, dy
\]
for every $x \in \mathbb{R}^n$ and $t > 0$, where $Q$ is the unique dyadic cube in $\mathbb{R}^n$ that contains $x$ and has side length $\ell$ with $\ell/2 < \ell \leq \ell$.

With this in hand, we apply the triple bar norm to (4).

Using the uniform $L_2$ boundedness of $Q_t$ and that of $\frac{I - P_t}{t(-\Delta)^{1/2}}$, the first term in the RHS is bounded by $\|t(-\Delta)^{1/2} \mathbf{v}_t\| \leq \|t \nabla \mathbf{v}_t\|$.

Following exactly the computation of Lemma 3.6 in [4], the second term in the RHS is bounded by $C \|\nabla P_t \mathbf{v}_t\| \leq C \|t \nabla \mathbf{v}_t\|$ using the uniform $L_2$ boundedness of $P_t$. This computation makes use of the off-diagonal estimates of $\Theta_t$, hence of $Q_t$, proved in [4] Prop. 3.11.

For the third term in the RHS, we observe that $\gamma_t(x) \mathbf{w} = \Theta_t(\mathcal{A}^{-1} \mathbf{w})(x)$. Hence, the square-function estimate on $\Theta_t$, proved in [4] Theorem 1.1], the off-diagonal estimates of $\Theta_t$ and the fact that $\mathcal{A}^{-1}$ is bounded imply that $|\gamma_t(x)|^2 \frac{dtdx}{t}$ is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes $\|N_x(S_t P_t \mathbf{w})\|_2$, which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening.

**4. The domain of the Dirichlet semi-group**

Assume (Dir-A) in the sense of Definition 2.1 is well-posed. If we set
\[
\mathcal{P}_t u_0 = (e^{-tT_A} \mathbf{f})_0, \quad \mathbf{f} = S^{-1} u_0 \in \overline{R(\chi(T_A))}
\]
for all $t > 0$, then Lemma 2.6 implies that $(\mathcal{P}_t)_{t>0}$ is a bounded $C_0$-semigroup on $L_2(\mathbb{R}^n, C^m)$ [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].
Furthermore, with our definition of well-posedness of the Dirichlet problem, the domain of the infinitesimal generator $A$ of this semi-group is contained in the Sobolev space $W^{1,2}(\mathbb{R}^n, C^m)$ and $\|\nabla_x u_0\|_2 \lesssim \|Au_0\|_2$. Indeed, from Lemma 2.6 we have for all $t > 0$, $\partial_t e^{-tT}A f = \nabla_t x U(t, \cdot)$. Also $\partial_t e^{-tT}A f \in R(\chi_+(T_A))$ and the invertibility of $S$ tells that $\nabla_t x U(t, \cdot) = S^{-1}(\partial_t U(t, \cdot))$. Therefore $\|\nabla_x U(t, \cdot)\|_2 \lesssim \|\partial_t U(t, \cdot)\|_2$.

By definition of $A$, $\partial_t U(t, \cdot) = AU(t, \cdot)$, thus we have for all $t > 0$ $\|\nabla_x U(t, \cdot)\|_2 \lesssim \|AU(t, \cdot)\|_2$.

The conclusion for the domain follows easily.

The question of whether this domain coincides with $W^{1,2}(\mathbb{R}^n, C^m)$ is answered by the following theorem:

**Theorem 4.1.** Assume that (Dir-$A$) and (Dir-$A^*$) are well-posed. Then the domain of the infinitesimal generator $A$ of $(P_t)_{t>0}$ coincides with the Sobolev space $W^{1,2}(\mathbb{R}^n, C^m)$ and $\|\nabla_x u_0\|_2 \sim \|Au_0\|_2$.

This theorem applies to the three situations listed in Theorem 2.4.

**Proof.** Combining [4, Lemma 4.2] (which says that (Dir-$A^*$) is equivalent to an auxiliary Neumann problem for $A^*$), [3, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for $A$: this is non trivial) with the proof of Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for $A$), we have that (Dir-$A^*$) is well-posed if and only if

$$R : R(\chi_+(T_A)) \rightarrow \{g \in L_2(\mathbb{R}^n; C^{nm}) ; \text{curl}_x(g) = 0\}, f \mapsto f_1$$

is invertible. This implies that for $f \in R(\chi_+(T_A))$, we have that $\|f\|_2 \sim \|f_1\|_2$.

Therefore, the conjunction of well-posedness for (Dir-$A$) and (Dir-$A^*$) gives $\|f_0\|_2 \sim \|f_1\|_2$, $f \in R(\chi_+(T_A))$.

From this, it is easy to identify the domain of $A$ by an argument as before. □

We have seen that invertibility of $S$ reduces to that of $R$ (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that $\|f_1\|_2 \sim \|(Af)_0\|_2$ for all $f \in R(\chi_+(T_A))$. This method was first used in the context of Laplace equation on Lipschitz domains by Verchota [20]. This depends strongly on $A$. Various relations between Dirichlet, regularity and Neumann problems for $L^p$ data in the sense of non tangential approach for second order real symmetric equations are studied in [12, 13] and more recently in [14, 18].
References


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