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ESTIMATES FOR FUNCTIONS OF THE LAPLACIAN ON MANIFOLDS WITH BOUNDED GEOMETRY

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Abstract. In this paper we consider a complete connected noncompact Riemannian manifold M with Ricci curvature bounded from below and positive injectivity radius. Denote by $\mathcal{L}$ the Laplace–Beltrami operator on M and by $\mathcal{D}$ the operator $\sqrt{\mathcal{L}-b}$, where b denotes the bottom of the spectrum of $\mathcal{L}$. We assume that the kernel associated to the heat semigroup generated by $\mathcal{L}$ satisfies a mild decay condition at infinity. We prove that if m is a bounded, even holomorphic function in a suitable strip of the complex plane, and satisfies Mihlin–Hörmander type conditions of appropriate order at infinity, then the operator $m(\mathcal{D})$ extends to an operator of weak type 1.

This partially extends a celebrated result of J. Cheeger, M. Gromov and M. Taylor, who proved similar results under much stronger curvature assumptions on M, but without any assumption on the decay of the heat kernel.

1. Introduction

The purpose of this paper is to extend a celebrated multiplier result of J. Cheeger, M. Gromov and M. Taylor [8, Thm 10.2], [29, Thm 1.6], by substantially relaxing its geometric assumptions.

Suppose that M is a complete connected noncompact Riemannian manifold. Denote by $-\mathcal{L}$ the Laplace–Beltrami operator on M: $\mathcal{L}$ is a symmetric operator on $C_c^\infty(M)$ (the space of compactly supported smooth complex-valued functions on M). Its closure is a self adjoint operator on $L^2(M)$ which, with a slight abuse of notation, we denote still by $\mathcal{L}$. We denote by b the bottom of the spectrum of $\mathcal{L}$, and by $\{\mathcal{P}_\lambda\}$ the spectral resolution of the identity for which

$$\mathcal{L}f = \int_b^\infty \lambda \,\mathrm{d}\mathcal{P}_\lambda f$$

for every f in the domain of $\mathcal{L}$. For notational convenience, we denote by $\mathcal{D}$ the operator $\sqrt{\mathcal{L}-b}$.

We say that M has C^∞ *bounded geometry* if the injectivity radius of M is positive and the Riemann curvature tensor is bounded in the C^∞ topology. We say that M has *bounded geometry* if the injectivity radius of M is positive and the Ricci curvature is bounded from below. If M has bounded geometry, then there are nonnegative constants α, β and C such that

$$\mu\big(B(x,r)\big) \leq C\, r^\alpha\, \mathrm{e}^{2\beta\, r} \qquad \forall r \in [1,\infty) \quad \forall x \in M, \tag{1.1}$$

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where $\mu\big(B(x,r)\big)$ denotes the Riemannian volume of the geodesic ball with centre x and radius r. The same is *a fortiori* true if M has C^∞ bounded geometry.

For each W in $\mathbb{R}^+$, denote by $\mathbf{S}_W$ the strip $\big\{z \in \mathbb{C} : \operatorname{Im} z \in (-W,W)\big\}$. Taylor [29, Thm 1.6], following up earlier work of Cheeger, Gromov and Taylor [8, Thm 10.2], proved that if M has C^∞ bounded geometry and m is a bounded even holomorphic function in $\mathbf{S}_\beta$ satisfying estimates of the form

$$\big|D^j m(\zeta)\big| \leq C\,\big(1+|\zeta|\big)^{-j} \qquad \forall \zeta \in \mathbf{S}_\beta \quad \forall j \in \{0,1,\ldots,J\}, \tag{1.2}$$

where J is a sufficiently large integer depending on the dimension n of M, then the operator $m(\mathcal{D})$ is bounded on $L^p(M)$ for p in $(1,\infty)$, and of weak type 1.

In fact, the proof of this result requires control only of a finite number of covariant derivatives of the Riemann tensor, but this number is of the same order of magnitude of the dimension of M. Notice that [29, Thm 1.6] extends a previous result of R.J. Stanton and P.A. Tomas [27] in the case where M is a symmetric space of the noncompact type G/K and real rank one (and $\mathcal{L}$ is the Laplace–Beltrami operator associated to the G-invariant metric on M induced by the Killing form of G). See also the pioneering work of J.L. Clerc and E.M. Stein [9] on spherical multipliers on noncompact symmetric spaces associated to complex semisimple Lie groups, and recent related works [1, 20, 21, 25] on general noncompact symmetric spaces, which have been stimulated by [29, Thm 1.6] and [9].

As M. Berger says in his book [3, p. 291] "Up to the end of the 1980s, Ricci curvature was believed to be only useful to control volumes,... ". Since then, various geometric and analytic results on Riemannian manifolds have been established under the hypothesis that the manifold is of bounded geometry. To mention a few, we recall the relationship between isoperimetric inequalities and the behaviour for large time of the heat kernel [11, 7, 30] and local Harnack type estimates for positive solutions of the heat equation [26].

In view of these considerations, it is natural to speculate whether [29, Thm 1.6] may be extended to Riemannian manifolds of bounded geometry. In this paper we assume that M has bounded geometry in the sense specified above, but, for technical reasons, we need also to assume that there exist constants $\rho > 1/2$ and C such that

$$|\!|\!|\mathcal{H}^t|\!|\!|_{1;\infty} \leq C\,\mathrm{e}^{-bt}\,t^{-\rho} \qquad \forall t \in [1,\infty), \tag{1.3}$$

where $\{\mathcal{H}^t\}$ denotes the heat semigroup generated by $\mathcal{L}$, and $|\!|\!|\mathcal{H}^t|\!|\!|_{1;\infty}$ the norm of $\mathcal{H}^t$ *qua* operator from $L^1(M)$ to $L^\infty(M)$. Note that on every manifold M with bounded geometry estimate (1.3) holds, but with $\rho = 0$ (see, for instance, [17, Section 7.5]). Moreover it holds with $\rho > 1$ on nonamenable unimodular Lie groups with a left invariant Riemannian metric [23] and on noncompact Riemannian symmetric spaces [12].

Our main result, Theorem 3.4, states that if M is a Riemannian manifold of bounded geometry satisfying (1.3) with $\rho > 1/2$, then the conclusion of Taylor's result holds with $J > \max([\![n/2+1]\!]+2, [\![n/2+1]\!]+2+\alpha/2-\rho)$.

To prove Theorem 3.4 we decompose, as in [29, Thm 1.6], the Schwartz kernel $k_{m(\mathcal{D})}$ of $m(\mathcal{D})$ as the sum of a kernel with support near the diagonal in $M \times M$, and of a kernel supported off the diagonal. As in [29, 8], we show that the part near the diagonal satisfies a Hörmander type integral condition, and that the part off the diagonal gives rise to a bounded operator on $L^1(M)$. However, the technical details are rather different. In particular, since we do not assume any control on the derivatives of the Riemann tensor, we cannot use either the eikonal equation or the Hadamard parametrix construction to obtain the required estimates of $k_{m(\mathcal{D})}$ near the diagonal. Our approach to these estimates is based on ultracontractive bounds for the heat semigroup and for the restriction of the semigroup generated by the de Rham operator to 1-forms on M and uses an adaptation of L. Hörmander's method [19]. We believe that our approach, though technically elaborate, helps to understand and clarify the relationships between the heat semigroup and singular integral operators on M.

We recall that the idea to use ultracontractive estimates for the heat semigroup in the proof of multiplier results for its generator is not new (see, for instance, [15]), but, to the best of our knowledge, it is indeed new in our setting.

Different endpoint estimates for various classes of multiplier operators on manifolds with bounded geometry will be considered in a forthcoming paper [24]. Those estimates will involve the Hardy space $H^1(M)$ introduced in [5] and some related spaces, which will be defined in [24].

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

2. Notation, background material and preliminary results

Suppose that M is a connected n-dimensional Riemannian manifold of infinite volume with Riemannian measure μ. We assume that M has bounded geometry, i.e., that the injectivity radius of M is positive and that

$$\mathrm{Ric}(M) \geq -\kappa^2 \tag{2.1}$$

for some $\kappa \geq 0$. It is well known that manifolds with bounded geometry satisfy the *uniform ball size condition*, i.e. for every $r \in \mathbb{R}^+$

$$\inf\big\{\mu\big(B(x,r)\big) : x \in M\big\} > 0, \qquad \sup\big\{\mu\big(B(x,r)\big) : x \in M\big\} < \infty \tag{2.2}$$

(see, for instance, [14], where complete references are given). Moreover, by standard comparison theorems [6, Theor. 3.10], the measure μ is *locally doubling*, i.e. for every $R > 0$ there exists a constant C_R such that for every ball $B(x,r)$ such that $r < R$

$$\mu\big(B(x,2r)\big) \leq C_R\, \mu\big(B(x,r)\big).$$

If $\mathcal{T}$ is a bounded linear operator from $L^p(M)$ to $L^q(M)$, we shall denote by $|||\mathcal{T}|||_{p;q}$ the operator norm of $\mathcal{T}$ from $L^p(M)$ to $L^q(M)$. In the case where $p = q$, we shall simply write $|||\mathcal{T}|||_p$ instead of $|||\mathcal{T}|||_{p;p}$.

Denote by $-\mathcal{L}$ the Laplace–Beltrami operator on M, by b the bottom of the $L^2(M)$ spectrum of $\mathcal{L}$, and set $\beta = \limsup_{r\to\infty} \big[\log \mu\big(B(o,r)\big)\big]/(2r)$. By a result of R. Brooks [4] $b \leq \beta^2$. Denote by $\{\mathcal{H}^t\}$ the semigroup generated by $-\mathcal{L}$ on $L^2(M)$. It is well known that for every p in $[1,\infty)$, the operator $\mathcal{H}^t$ extends to a contraction on $L^p(M)$. Furthermore, $\{\mathcal{H}^t\}$ is ultracontractive, i.e., $\mathcal{H}^t$ maps $L^1(M)$ into $L^\infty(M)$ for every t in $\mathbb{R}^+$. Recall that $\mathcal{H}^t$ satisfies the following estimate [17, Section 7.5]

$$\big|\big|\big|\mathcal{H}^t\big|\big|\big|_{1;2} \leq C\, \mathrm{e}^{-bt}\, t^{-n/4}\, (1+t)^{n/4-\rho/2} \qquad \forall t \in \mathbb{R}^+ \tag{2.3}$$

for some ρ in $[0,\infty)$. Then, by standard subordination techniques of semigroups

$$\big|\!\big|\!\big| \mathrm{e}^{-t\mathcal{D}} \big|\!\big|\!\big|_{1;2} \le C\, t^{-n/2}\,(1+t)^{n/2-\rho} \qquad \forall t \in \mathbb{R}^+. \tag{2.4}$$

The proof of our main result, Theorem 3.4 below, is rather technical and requires some background material and a few preliminary results, which are the content of the following three subsections. Specifically, Subsection 2.1 gives information about the heat semigroup generated by $\mathcal{L}$, and its natural extension to forms, i.e., the semigroup generated by the de Rham operator $\mathbf{L}$, Subsection 2.2 and Subsection 2.3 contain estimates for the kernels of certain functions of the operator $\mathcal{D}$ and some technical lemmata respectively. These results will be directly used in the proof of Theorem 3.4.

2.1. The de Rham operator. Denote by $L^2(\Lambda^k_{\mathbb{C}}M)$ the space of square integrable k-forms on M with complex coefficients, and by $C^\infty_c(\Lambda^k_{\mathbb{C}}M)$ the subspace of smooth forms with compact support. As usual we identify 0-forms with functions on M, and $L^2(\Lambda^0_{\mathbb{C}}M)$ with the space $L^2(M)$. We shall denote by $\langle\cdot,\cdot\rangle_k$ the Hermitian inner product on $L^2(\Lambda^k_{\mathbb{C}}M)$, i.e.

$$\langle \omega,\eta\rangle_k = \int_M \big(\omega(y),\eta(y)\big)_y \,\mathrm{d}\mu(y),$$

where $(\cdot,\cdot)_y$ is the Hermitian inner product induced by the metric of M on the complexification of the space of alternating tensors of order k at the point y.

We denote by d the operator of exterior differentiation, considered as a closed densely defined operator from $L^2(\Lambda^k_{\mathbb{C}}M)$ to $L^2(\Lambda^{k+1}_{\mathbb{C}}M)$ and by δ its adjoint operator, i.e. the closed densely defined operator mapping $L^2(\Lambda^{k+1}_{\mathbb{C}}M)$ to $L^2(\Lambda^k_{\mathbb{C}}M)$ such that

$$\langle \mathrm{d}\omega,\eta\rangle_{k+1} = \langle \omega,\delta\,\eta\rangle_k \qquad \forall \omega \in C^\infty_c(\Lambda^k_{\mathbb{C}}M) \quad \forall \eta \in C^\infty_c(\Lambda^{k+1}_{\mathbb{C}}M).$$

As a consequence, for each nonnegative integer k the de Rham operator $\delta\mathrm{d}+\mathrm{d}\delta$ maps smooth k-forms into smooth k-forms.

We denote by $\mathbf{L}$ the operator on $\bigoplus_{k=0}^n C^\infty_c(\Lambda^k_{\mathbb{C}}M)$, defined by

$$\mathbf{L}\omega = (\delta\mathrm{d}+\mathrm{d}\delta)\omega \qquad \forall \omega \in C^\infty_c(\Lambda^k_{\mathbb{C}}M).$$

With a slight abuse of notation, the closure of $\mathbf{L}$ in $\bigoplus_{k=0}^n L^2(\Lambda^k_{\mathbb{C}}M)$ will be denoted still by $\mathbf{L}$.

It is well known that for each nonnegative integer k, the restriction of $\mathbf{L}$ to $L^2(\Lambda^k_{\mathbb{C}}M)$ is a self adjoint operator [28]. In particular, the restriction of $\mathbf{L}$ to $L^2(\Lambda^0_{\mathbb{C}}M)$ coincides with $\mathcal{L}$. Furthermore, it is known that the restriction of $\mathbf{L}$ to 1-forms is nonnegative. Therefore, the restriction of $-\mathbf{L}$ to $L^2(\Lambda^1_{\mathbb{C}}M)$ generates a strongly continuous one parameter semigroup on $L^2(\Lambda^1_{\mathbb{C}}M)$ that we denote by $\{\mathbf{H}^t\}$.

The next lemma summarises some of the properties of the operator $\mathbf{L}$ on 1-forms that we shall need in the proof of Proposition 2.2.

Lemma 2.1. *Suppose that κ is as in (2.1). Then*

(i) *for every bounded Borel function F on $[0,\infty)$*

$$F(\mathcal{L})\delta\omega = \delta F(\mathbf{L})\omega \qquad \forall \omega \in C^\infty_c(\Lambda^1_{\mathbb{C}}M);$$

(ii) *for every ω in $L^2(\Lambda^1_{\mathbb{C}}M)$*

$$\big|\mathbf{H}^t\omega(x)\big|_x \le \mathrm{e}^{\kappa^2 t}\,\mathcal{H}^t\,|\omega|\,(x) \qquad \forall t \in \mathbb{R}^+ \quad \forall x \in M;$$

(iii) *for every* ω *in* $L^2(\Lambda^1_{\mathbb{C}}M)$

$$\|\mathbf{H}^t\omega\|_2 \leq \mathrm{e}^{(\kappa^2-b)t}\|\omega\|_2 \qquad \forall t \in \mathbb{R}^+.$$

Hence the bottom of the spectrum of $\mathbf{L}$ *on* $L^2(\Lambda^1_{\mathbb{C}}M)$ *is greater than or equal to* $b-\kappa^2$.

Proof. (i) The identity $F(\mathcal{L})\delta = \delta F(\mathbf{L})$ is a straightforward consequence of the identity $\mathcal{L}\delta\,\omega = \delta\mathbf{L}\,\omega$ and of the fact that the operators $\mathcal{L}$ and $\mathbf{L}$ are self-adjoint.

For the proof of (ii) see [2, Prop. 1.7]; now (iii) follows directly from (ii). □

Suppose that ω is a smooth 1-form with compact support. From Lemma 2.1, by using the contractivity of $\mathcal{H}^t$ on $L^p(M)$ for every $p \in [1,\infty]$ and interpolation, one may deduce that if p is in $[1,\infty]$, then

$$\left\|\mathbf{H}^t\omega\right\|_p \leq \mathrm{e}^{\kappa^2|1-2/p|t}\,\|\omega\|_p \qquad \forall t \in \mathbb{R}^+.$$

Thus, $\{\mathbf{H}^t\}$ extends to a semigroup on $L^p(\Lambda^1_{\mathbb{C}}M)$ for all p in $[1,\infty)$, that we denote still by $\{\mathbf{H}^t\}$. From Lemma 2.1 and the ultracontractivity estimates (2.3) for $\mathcal{H}^t$ we also deduce that

$$\begin{aligned}
\left\|\mathbf{H}^t\omega\right\|_2 &= \Big[\int_M \left|\mathbf{H}^t\omega(x)\right|_x^2 \,\mathrm{d}\mu(x)\Big]^{1/2}\\
&\leq \mathrm{e}^{\kappa^2 t}\Big[\int_M \big(\mathcal{H}^t\,|\omega|\,(x)\big)^2\,\mathrm{d}\mu(x)\Big]^{1/2}\\
&\leq C\,\mathrm{e}^{(\kappa^2-b)t}\,t^{-n/4}\,(1+t)^{n/4-\rho/2}\,\|\omega\|_1 \qquad \forall t \in \mathbb{R}^+.
\end{aligned}$$

Thus there exists a constant C such that

$$\left\|\!\left|\mathbf{H}^t\right|\!\right\|_{1;2} \leq C\,\mathrm{e}^{(\kappa^2-b)t}\,t^{-n/4}\,(1+t)^{n/4-\rho/2} \qquad \forall t \in \mathbb{R}^+. \tag{2.5}$$

2.2. Estimates for certain kernels. For notational convenience, we denote by $\mathcal{D}_1$ the operator $\sqrt{\mathcal{L}-b+\kappa^2}$, and by $\mathbf{D}_1$ the operator $\sqrt{\mathbf{L}-b+\kappa^2}$ (the operator $\mathbf{L}-b+\kappa^2$ is nonnegative by Lemma 2.1(iii)).

If $\mathcal{T}$ is an operator bounded on $L^2(M)$, then we denote by $k_{\mathcal{T}}$ its Schwartz kernel (with respect to the Riemannian density μ). In this subsection, we prove estimates for $k_{F(t\mathcal{D})}$, $k_{F(t\mathcal{D}_1)}$ and of $\mathrm{d}_2 k_{F(t\mathcal{D}_1)}$, when the function F decays sufficiently fast at infinity; here d_2 denotes the differential with respect to the second variable.

We observe that the only reason to introduce the de Rham operator $\mathbf{L}$ and the auxiliary operator $\mathbf{D}_1$ is that to estimate the kernel of $\mathrm{d}_2 k_{F(t\mathcal{D}_1)}$ we exploit the identity in Lemma 2.1 (i).

Proposition 2.2. *Assume that* ρ *is as in (1.3), that* γ *is in* $(n/2+1,\infty)$, *and that* F *is a bounded function on* $[0,\infty)$ *such that*

$$\sup_{\lambda\in\mathbb{R}^+}\left|\big(1+\lambda\big)^\gamma F(\lambda)\right| < \infty.$$

Then for every t *in* $\mathbb{R}^+$ *the operators* $F(t\mathcal{D})$, $F(t\mathcal{D}_1)$ *and* $\mathrm{d}F(t\mathcal{D}_1)^*$ *are bounded from* $L^1(M)$ *to* $L^2(M)$. *Furthermore, there exists a constant* C *such that for all* t *in* $\mathbb{R}^+$

(i) $\sup_{y\in M}\ \left\|k_{F(t\mathcal{D})}(\cdot,y)\right\|_2 \leq C\,t^{-n/2}\,(1+t)^{n/2-\rho}$;

(ii) $\sup_{y\in M}\ \left\|k_{F(t\mathcal{D}_1)}(\cdot,y)\right\|_2 \leq C\,t^{-n/2}\,(1+t)^{n/2-\rho}$;

(iii) $\sup_{y\in M}\ \left\|\mathrm{d}_2 k_{F(t\mathcal{D}_1)}(\cdot,y)\right\|_2 \leq C\,t^{-n/2-1}\,(1+t)^{n/2+1-\rho}$, *where* d_2 *denotes the exterior differentiation with respect to the second variable.*

Proof. We may assume that the kernels $k_{F(t\mathcal{D})}$ and $k_{F(t\mathcal{D}_1)}$ are smooth. Indeed, it suffices to prove that the desired estimates hold for all functions G with bounded support such that $|G| \leq |F|$, with a constant C that does not depend on the support. Since for such functions the operator $\mathcal{L}^N G(t\mathcal{D})$ is bounded on $L^2(M)$ for every positive integer N, its kernel is a smooth function on M, by elliptic regularity. The general case will follow by approximating F with functions of bounded support.

First we prove (i). Suppose that $\sigma > n/4$. Then, by (2.3),

$$\begin{aligned}
\left|\!\left|\!\left| (1+t^2\mathcal{D}^2)^{-\sigma} \right|\!\right|\!\right|_{1;2} &\leq \frac{1}{\Gamma(\sigma)} \left|\!\left|\!\left| \int_0^\infty s^\sigma \,\mathrm{e}^{-s}\, \mathrm{e}^{-st^2\mathcal{D}^2} \frac{\mathrm{d}s}{s} \right|\!\right|\!\right|_{1;2} \\
&\leq C \int_0^{1/t^2} s^\sigma \,\mathrm{e}^{-s}\, (t^2 s)^{-n/4} \frac{\mathrm{d}s}{s} \\
&\quad + C \int_{1/t^2}^\infty s^\sigma \,\mathrm{e}^{-s}\, (t^2 s)^{-\rho/2} \frac{\mathrm{d}s}{s} \\
&\leq C\, t^{-n/2}\, (1+t)^{n/2-\rho} \qquad \forall t \in \mathbb{R}^+. \qquad (2.6)
\end{aligned}$$

By the spectral theorem and the assumption on F

$$\sup_{t>0} \left|\!\left|\!\left| (1+t^2\mathcal{D}^2)^{\gamma/2}\, F(t\mathcal{D}) \right|\!\right|\!\right|_2 = \sup_{\lambda>0} \left| (1+\lambda^2)^{\gamma/2}\, F(\lambda) \right| < \infty.$$

Thus, by applying (2.6) with $\sigma = \gamma/2$, we get

$$\begin{aligned}
\left|\!\left|\!\left| F(t\mathcal{D}) \right|\!\right|\!\right|_{1;2} &= \left|\!\left|\!\left| (1+t^2\mathcal{D}^2)^{-\gamma/2}\, (1+t^2\mathcal{D}^2)^{\gamma/2}\, F(t\mathcal{D}) \right|\!\right|\!\right|_{1;2} \\
&\leq \left|\!\left|\!\left| (1+t^2\mathcal{D}^2)^{\gamma/2}\, F(t\mathcal{D}) \right|\!\right|\!\right|_2 \left|\!\left|\!\left| (1+t^2\mathcal{D}^2)^{-\gamma/2} \right|\!\right|\!\right|_{1;2} \\
&\leq C\, t^{-n/2}\, (1+t)^{n/2-\rho} \qquad \forall t \in \mathbb{R}^+. \qquad (2.7)
\end{aligned}$$

Then the adjoint operator $F(t\mathcal{D})^*$ maps $L^2(M)$ boundedly into $L^\infty(M)$ and

$$\left|\!\left|\!\left| F(t\mathcal{D})^* \right|\!\right|\!\right|_{2;\infty} = \left|\!\left|\!\left| F(t\mathcal{D}) \right|\!\right|\!\right|_{1;2}.$$

Thus, by Dunford-Pettis' Theorem [16, Thm 6, p. 503], the kernel $k_{F(t\mathcal{D})^*}$ of $F(t\mathcal{D})^*$ satisfies the estimate

$$\begin{aligned}
\sup_{x\in M} \|k_{F(t\mathcal{D})^*}(x,\cdot)\|_2 &= \left|\!\left|\!\left| F(t\mathcal{D}) \right|\!\right|\!\right|_{1;2} \\
&\leq C\, t^{-n/2}\, (1+t)^{n/2-\rho} \qquad \forall t \in \mathbb{R}^+.
\end{aligned}$$

Estimate (i) follows from this and the fact that $k_{F(t\mathcal{D})}(x,y) = \overline{k_{F(t\mathcal{D})^*}(y,x)}$.

The proof of (ii) is, *mutatis mutandis*, the same as the proof of (i). We simply replace $\mathcal{D}^2$ by $\mathcal{D}_1^2$ in the proof of (i), and use the obvious ultracontractive bounds for $\mathrm{e}^{-st^2\mathcal{D}_1^2}$, instead of those for $\mathrm{e}^{-st^2\mathcal{D}^2}$.

Finally we prove (iii). By arguing as in the proof of (i) (with $\mathbf{D}_1$ in place of $\mathcal{D}$, and using the ultracontractivity estimates (2.5) in place of (2.3)), we may show that there exists a constant C such that

$$\left|\!\left|\!\left| F(t\mathbf{D}_1) \right|\!\right|\!\right|_{1;2} \leq C\, t^{-n/2}\, (1+t)^{n/2-\rho} \qquad \forall t \in \mathbb{R}^+. \qquad (2.8)$$

We claim that there exists a constant C such that

$$\left|\!\left|\!\left| \delta\, F(t\mathbf{D}_1) \right|\!\right|\!\right|_{1;2} \leq C\, t^{-n/2-1}\, (1+t)^{n/2+1-\rho} \qquad \forall t \in \mathbb{R}^+. \qquad (2.9)$$

To prove (2.9), observe that for every ω in $C_c^\infty(\Lambda^1_{\mathbb{C}}M)$

$$\begin{aligned}\|\delta\, F(t\mathbf{D}_1)\,\omega\|_2^2 &= \langle \delta\, F(t\mathbf{D}_1)\,\omega, \delta\, F(t\mathbf{D}_1)\,\omega\rangle_0 \\ &\leq \langle \delta\, F(t\mathbf{D}_1)\,\omega, \delta\, F(t\mathbf{D}_1)\,\omega\rangle_0 + \langle \mathrm{d}\, F(t\mathbf{D}_1)\,\omega, \mathrm{d}\, F(t\mathbf{D}_1)\,\omega\rangle_2 \\ &= \langle \mathbf{L}\, F(t\mathbf{D}_1)\,\omega, F(t\mathbf{D}_1)\,\omega\rangle_1 \\ &= \|\mathbf{D}_1\, F(t\mathbf{D}_1)\omega\|_2^2 + (b-\kappa^2)\,\|F(t\mathbf{D}_1)\omega\|_2^2.\end{aligned}$$

This and (2.8) imply that

$$\begin{aligned}\left|\!\left|\!\left|\delta\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} &\leq \left|\!\left|\!\left|\mathbf{D}_1\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} + b\,\left|\!\left|\!\left|F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} \\ &\leq \left|\!\left|\!\left|\mathbf{D}_1\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} + C\,t^{-n/2}\,(1+t)^{n/2-\rho} \qquad \forall t\in\mathbb{R}^+.\end{aligned} \tag{2.10}$$

To conclude the proof of the claim it suffices to observe that

$$\begin{aligned}\left|\!\left|\!\left|t\mathbf{D}_1\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} &= \left|\!\left|\!\left|(1+t^2\mathbf{D}_1^2)^{-(\gamma-1)/2}\,(1+t^2\mathbf{D}_1^2)^{(\gamma-1)/2}\,t\mathbf{D}_1\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_{1;2} \\ &\leq \left|\!\left|\!\left|(1+t^2\mathbf{D}_1^2)^{(\gamma-1)/2}\,t\mathbf{D}_1\, F(t\mathbf{D}_1)\right|\!\right|\!\right|_2\,\left|\!\left|\!\left|(1+t^2\mathbf{D}_1^2)^{-(\gamma-1)/2}\right|\!\right|\!\right|_{1;2} \\ &\leq C\,t^{-n/2}\,(1+t)^{n/2-\rho} \qquad \forall t\in\mathbb{R}^+,\end{aligned}$$

where we have used (2.6) with $\sigma=(\gamma-1)/2$.

Recall that $F(t\mathcal{D}_1)\,\delta = \delta\, F(t\mathbf{D}_1)$ by Lemma 2.1 (i). Thus, by (2.9), the operator $F(t\mathcal{D}_1)\,\delta$ maps $L^1(\Lambda^1_{\mathbb{C}}M)$ to $L^2(M)$, and

$$\left|\!\left|\!\left|F(t\mathcal{D}_1)\,\delta\right|\!\right|\!\right|_{1;2} \leq C\,t^{-n/2-1}\,(1+t)^{n/2+1-\rho} \qquad \forall t\in\mathbb{R}^+.$$

Hence its adjoint $\mathrm{d}\, F(t\mathcal{D}_1)^*$ maps $L^2(M)$ to $L^\infty(\Lambda^1_{\mathbb{C}}M)$ and

$$\left|\!\left|\!\left|\mathrm{d}\, F(t\mathcal{D}_1)^*\right|\!\right|\!\right|_{2;\infty} \leq C\,t^{-n/2-1}\,(1+t)^{n/2+1-\rho}.$$

Thus, by Dunford-Pettis' Theorem, the kernel $k_{\mathrm{d}F(t\mathcal{D}_1)^*}$ of the operator $\mathrm{d}\, F(t\mathcal{D}_1)^*$ satisfies the estimate

$$\sup_{y\in M}\,\left\|k_{\mathrm{d}F(t\mathcal{D}_1)^*}(y,\cdot)\right\|_2 \leq C\,t^{-n/2-1}\,(1+t)^{n/2+1-\rho} \qquad \forall t\in\mathbb{R}^+.$$

The desired conclusion follows because $\mathrm{d}_2\, k_{F(t\mathcal{D}_1)}(x,y) = \overline{k_{\mathrm{d}F(t\mathcal{D}_1)^*}(y,x)}$. □

2.3. Some technical lemmata. To motivate the technical result contained in this subsection, we briefly recall the main features of Taylor's method to prove spectral multiplier theorems for the Laplace–Beltrami operator on a Riemannian manifold M of bounded geometry. Consider an operator of the form $m(\mathcal{D})$, where m is an even, bounded, holomorphic function in the strip $\mathbf{S}_\beta$ and satisfies Mihlin-type conditions at infinity (see (1.2)). One of the main ingredients of Taylor's method is the functional calculus formula

$$m(\mathcal{D}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{m}(t)\,\cos(t\mathcal{D})\,\mathrm{d}t, \tag{2.11}$$

based on the Fourier inversion formula and the spectral theorem. The analysis of $m(\mathcal{D})$ ultimately relies on the finite propagation speed property for the wave equation and uniform Sobolev estimates on M, proved in [8] under rather strong bounded curvature assumptions on the manifold M. Since, in this paper, we want to relax the latter assumption by requiring only a lower bound on the Ricci curvature of M, we need to modify Taylor's proof. The aim of this section is to provide some of the required technical ingredients.

The first step consists in replacing the cosine in the right hand side of (2.11) by a modified Bessel function (see Lemma 2.4 (ii)). For each $\nu \geq -1/2$, denote by $\mathcal{J}_\nu : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ the modified Bessel function of order ν, defined by

$$\mathcal{J}_\nu(t) = \frac{J_\nu(t)}{t^\nu},$$

where J_ν denotes the standard Bessel function of the first kind and order ν (see, for instance, [22, formula (5.10.2), p. 114]). We recall that, if $\operatorname{Re}\nu > -1/2$,

$$J_\nu(t) = \frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\; t^\nu \int_0^1 (1-s^2)^{\nu-1/2} \cos(ts)\,\mathrm{d}s \tag{2.12}$$

and that

$$\mathcal{J}_{-1/2}(t) = \sqrt{\frac{2}{\pi}}\,\cos t \qquad \text{and} \qquad \mathcal{J}_{1/2}(t) = \sqrt{\frac{2}{\pi}}\,\frac{\sin t}{t}.$$

We recall the definition of the generalised Riesz means, introduced in [CM, Section 1], and summarise some of their properties.

Suppose that d and z are complex numbers such that $\operatorname{Re} d > 0$ and that $\operatorname{Re} z > 0$. For every f in the Schwartz class $\mathcal{S}(\mathbb{R})$, the *generalised Riesz mean of order* (d,z) of f is the function $R_{d,z}f$, defined by

$$R_{d,z}f(t) = \frac{2}{\Gamma(z)} \int_0^1 s^{d-1}\,(1-s^2)^{z-1}\,f(st)\,\mathrm{d}s \qquad \forall t \in \mathbb{R}.$$

For fixed d and t, the function $z \mapsto R_{d,z}f(t)$ has analytic continuation to an entire function.

For every f in $L^1(\mathbb{R})$ define its Fourier transform $\widehat{f}$ by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(s)\,e^{-ist}\,\mathrm{d}s \qquad \forall t \in \mathbb{R}.$$

Sometimes we write $\mathcal{F}f$ instead of $\widehat{f}$, and denote by $\mathcal{F}^{-1}f$ the inverse Fourier transform of f.

Suppose that f is a function on $\mathbb{R}$, and that λ is in $\mathbb{R}^+$. We denote by f^λ and f_λ the λ-dilates of f, defined by

$$f^\lambda(x) = f(\lambda x) \qquad \text{and} \qquad f_\lambda(x) = \lambda^{-1}\,f(x/\lambda) \qquad \forall x \in \mathbb{R}. \tag{2.13}$$

For each positive integer h, we denote by $\mathcal{O}^h$ the differential operator $t^h\,D^h$ on the real line.

Lemma 2.3 ([CM])**.** *Suppose that k is a positive integer, d and w are complex numbers, and $\operatorname{Re} d > 0$. The following hold:*

(i) *if $\operatorname{Re}(d-2w) > 0$, then $R_{d-2w,w}\,R_{d,z} = R_{d-2w,w+z}$;*
(ii) *$R_{d,0}$ is the identity operator;*
(iii) *there exist constants $C_{j,d,k}$ such that $R_{d,-k} = \sum_{j=0}^k C_{j,d,k}\,\mathcal{O}^j$.*

Proof. The proofs of (i), (ii), and (iii) may be found in [13, Section 1]. ∎

We shall make repeated use of the operator $R_{1+2k,-k}$. For notational convenience, in the rest of this paper we shall write R_k instead of $R_{1+2k,-k}$, and we shall denote the formal adjoint of R_k by R_k^*. Thus

$$\int_{-\infty}^{\infty} f(t)\,R_k g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} R_k^* f(t)\,g(t)\,\mathrm{d}t \qquad \forall f,g \in \mathcal{S}(\mathbb{R}).$$

Lemma 2.4. *Suppose that k is a positive integer. The following hold:*

(i) *if g is a bounded smooth function with bounded derivatives up to the order k, and f is rapidly decreasing together with its derivatives up to the order k, then there exist constants $C^*_{h,k}$ such that*
$$\langle f, R_k g\rangle = \sum_{h=0}^{k} C^*_{h,k} \int_{-\infty}^{\infty} \mathcal{O}^h f(t)\, g(t)\, \mathrm{d}t.$$
Thus $R_k^* = \sum_{h=0}^k C^*_{h,k}\mathcal{O}^h$.

(ii) *if f is a tempered distribution such that $\mathcal{O}^h f$ is in $L^1(\mathbb{R})\cap C_0(\mathbb{R})$ for all h in $\{0,1,\ldots,k\}$, then*
$$\int_{-\infty}^{\infty} f(t)\, \cos(vt)\, \mathrm{d}t = \sqrt{\pi}\, 2^{k-1/2} \int_{-\infty}^{\infty} R_k^* f(t)\, \mathcal{J}_{k-1/2}(tv)\, \mathrm{d}t.$$

Proof. First we prove (i). By using Lemma 2.3 (iii), and then integrating by parts
$$\begin{aligned}\langle f, R_k g\rangle &= \sum_{j=0}^{k} C_{j,1+2k,k} \int_{-\infty}^{\infty} f(t)\, \mathcal{O}^j g(t)\, \mathrm{d}t\\ &= \sum_{j=0}^{k} C_{j,1+2k,k} (-1)^j \int_{-\infty}^{\infty} D^j\big(t^j\, f\big)(t)\; g(t)\, \mathrm{d}t.\end{aligned}$$

Define $a_{j,\ell} = \binom{j}{\ell}\dfrac{j!}{\ell!}$. By Leibniz's rûle $D^j\big(t^j\, f\big)(t) = \sum_{\ell=0}^{j} a_{j,\ell}\, \mathcal{O}^{j-\ell} f(t)$. Then
$$\begin{aligned}\langle f, R_k g\rangle &= \sum_{j=0}^{k} C_{j,1+2k,k} (-1)^j \sum_{\ell=0}^{j} a_{j,\ell} \int_{-\infty}^{\infty} \mathcal{O}^{j-\ell} f(t)\, g(t)\, \mathrm{d}t\\ &= \sum_{j=0}^{k} C_{j,1+2k,k} (-1)^j \sum_{h=0}^{j} a_{j,j-h} \int_{-\infty}^{\infty} \mathcal{O}^{h} f(t)\, g(t)\, \mathrm{d}t\\ &= \sum_{h=0}^{k} C^*_{h,k} \int_{-\infty}^{\infty} \mathcal{O}^{h} f(t)\; g(t)\, \mathrm{d}t,\end{aligned}$$
where $C^*_{h,k} = \sum_{j=h}^{k} (-1)^j\, C_{j,1+2k,k}\, a_{j,j-h}$, as required.

Next we prove (ii). For every v in $\mathbb{R}^+$, denote by C^v the function $C^v(t) = \cos(tv)$. The required formula follows from (i), once we prove that
$$C^v = \sqrt{\pi}\, 2^{k-1/2}\, R_k\big(\mathcal{J}^v_{k-1/2}\big).$$
To prove this formula, observe that for every positive integer k
$$\begin{aligned}\big(R_{1,k} C^v\big)(t) &= \frac{2}{\Gamma(k)} \int_0^1 (1-s^2)^{k-1}\, \cos(stv)\, \mathrm{d}s\\ &= \sqrt{\pi}\, 2^{k-1/2}\, \mathcal{J}_{k-1/2}(tv)\end{aligned}$$
by (2.12). Then we use Lemma 2.3 (i) and (ii), and write
$$\sqrt{\pi}\, 2^{k-1/2}\, R_k\big(\mathcal{J}^v_{k-1/2}\big) = R_k\big(R_{1,k} C^v\big) = R_{1+2k,0} C^v = C^v,$$
as required. □

In the rest of this section we shall provide various estimates of functions of the form $R_k^*\widehat{g}$ that, in combination with Lemma 2.4 (i), will be needed in the proof of Theorem 3.4.

Suppose that J is a positive number. Denote by $\varphi : \mathbb{R} \to [0,1]$ a smooth even function, supported in $[-4,-1/4] \cup [1/4,4]$, equal to one on $[-2,-1/2] \cup [1/2,2]$, and such that $\sum_{j\in\mathbb{Z}} \varphi^{2^{-j}} = 1$ on $\mathbb{R}\setminus\{0\}$. We denote by $H^J(\mathbb{R})$ the standard Sobolev space on $\mathbb{R}$, modelled over $L^2(\mathbb{R})$.

Definition 2.5. We say that a function $g : \mathbb{R} \to \mathbb{C}$ satisfies a *Hörmander condition* [19] of order J on the real line if

$$\sup_{\lambda>0} \|\varphi\, g^\lambda\|_{H^J(\mathbb{R})} < \infty. \tag{2.14}$$

We set $\|g\|_{\mathrm{Horm}(J)} := \sup_{\lambda>0} \|\varphi\, g^\lambda\|_{H^J(\mathbb{R})}$.

Note that (2.14) implies that $\|g\|_\infty \leq 2\,\|g\|_{\mathrm{Horm}(J)}$ if $J > 1/2$. We need a technical lemma, which is a version of Hörmander's method [19].

Lemma 2.6. *Suppose that s is in $(1/2,\infty)$, that k is a positive integer and that $g : \mathbb{R} \to \mathbb{C}$ is a bounded even function that extends to an entire function of exponential type 1. For each integer j define the functions g_j by*

$$g_j = g\,\varphi^{2^{-j}}.$$

Then $\widehat{g_j}$ is an entire function of exponential type and $\widehat{g} = \sum_j \widehat{g_j}$ in the sense of distributions. Furthermore, for every ε in $[0, s-1/2)$ there exists a constant C such that for all r in $\mathbb{R}^+$

(i) $\displaystyle\int_{|t|>r} |R_k^*\widehat{g_j}(t)|\,\mathrm{d}t \leq C\,(2^j\,r)^{-\varepsilon}\,\|g\|_{\mathrm{Horm}(s+k)}$;

(ii) $\displaystyle r\int_{|t|>r} \frac{|R_k^*\widehat{g_j}(t)|}{|t|}\,\mathrm{d}t \leq C\,(2^j\,r)^{1/2}\,\|g\|_{\mathrm{Horm}(k)}$;

(iii) $\displaystyle\sup_{j\in\mathbb{Z}} \|\widehat{g_j}\|_1 \leq C\,\|g\|_{\mathrm{Horm}(s)}$.

Proof. For every integer ℓ in $\{0,\ldots,k\}$ define the tempered distribution G^ℓ and the functions G_j^ℓ by

$$G^\ell = \mathcal{O}^\ell\widehat{g}, \qquad \text{and} \qquad G_j^\ell = \mathcal{O}^\ell\widehat{g_j}.$$

By Lemma 2.4 (i), $R_k^*\widehat{g_j} = \sum_{\ell=0}^k C_{\ell,k}^*\,\mathcal{O}^\ell\widehat{g_j}$. Thus, to prove (i) and (ii) it suffices to prove similar estimates with G_j^ℓ in place of $R_k^*\widehat{g_j}$ for all ℓ in $\{0,\ldots,k\}$.

Note that both $\widehat{g_j}$ and G_j^ℓ are entire functions of exponential type 2^{j+2}. Observe that

$$\widehat{g_j} = \mathcal{F}\big[(g^{2^j}\varphi)^{2^{-j}}\big] = \big[\mathcal{F}(g^{2^j}\varphi)\big]_{2^{-j}}.$$

Hence

$$G_j^\ell = \mathcal{O}^\ell\big\{\big[\mathcal{F}(g^{2^j}\varphi)\big]_{2^{-j}}\big\} = \big[\mathcal{O}^\ell\mathcal{F}(g^{2^j}\varphi)\big]_{2^{-j}}. \tag{2.15}$$

By elementary Fourier analysis $\mathcal{F}^{-1}\big[\mathcal{O}^\ell\,\mathcal{F}(g^{2^j}\,\varphi)\big](\xi) = (-1)^\ell\,D^\ell\big[\xi^\ell\,g^{2^j}\,\varphi\big](\xi)$.

Now we prove (i). Note that

$$
\begin{aligned}
\int_{|t|>r} \left|G_j^\ell(t)\right| \,\mathrm{d}t &= \int_{|t|>r} \left|\left[\mathcal{O}^\ell \mathcal{F}(g^{2^j}\varphi)\right]_{2^{-j}}(t)\right| \mathrm{d}t \\
&= \int_{|t|>2^j\, r} \left|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\varphi)(t)\right| \mathrm{d}t \\
&\le (2^j\, r)^{-\varepsilon} \int_{\mathbb{R}} |t|^{\varepsilon} \left|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\varphi)(t)\right| \mathrm{d}t \\
&\le C\,(2^j\, r)^{-\varepsilon} \left\|\mathcal{F}^{-1}\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)\right\|_{H^s(\mathbb{R})}
\end{aligned}
\tag{2.16}
$$

by the classical Bernstein's Theorem, where ε is in $[0, s-1/2)$. Note that C depends on ε but is independent of j. Now, by Plancherel's Theorem,

$$
\begin{aligned}
\|\mathcal{F}^{-1}\left[\mathcal{O}^\ell\, \mathcal{F}(g^{2^j}\,\varphi)\right]\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} \left|\mathcal{O}^\ell \mathcal{F}\left[g^{2^j}\,\varphi\right](t)\right|^2 \left(1+|t|^2\right)^s \mathrm{d}t \\
&\le \int_{\mathbb{R}} \left|D^\ell \mathcal{F}\left[g^{2^j}\,\varphi\right](t)\right|^2 \left(1+|t|^2\right)^{s+\ell} \mathrm{d}t \\
&= \int_{\mathbb{R}} \left|\mathcal{F}\left[\xi^\ell\, g^{2^j}\,\varphi\right](t)\right|^2 \left(1+|t|^2\right)^{s+\ell} \mathrm{d}t.
\end{aligned}
$$

The square root of the last integral is a constant times $\|\xi^\ell\, g^{2^j}\,\varphi\|_{H^{s+\ell}(\mathbb{R})}$, which is clearly dominated by $\|g^{2^j}\,\varphi\|_{H^{s+\ell}(\mathbb{R})}$. Then (2.16) implies that

$$
\int_{|t|>r} \left|G_j^\ell(t)\right| \,\mathrm{d}t \le C\,(2^j\, r)^{-\varepsilon}\, \|g\|_{\mathrm{Horm}(s+\ell)},
\tag{2.17}
$$

as required to conclude the proof of (i).

Next we prove (ii). Observe that

$$
\begin{aligned}
r \int_{|t|>r} \frac{\left|G_j^\ell(t)\right|}{|t|} \,\mathrm{d}t &= r \int_{|t|>r} \frac{\left|\left[\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)\right]_{2^{-j}}(t)\right|}{|t|} \,\mathrm{d}t \\
&= 2^j\, r \int_{|t|>2^j\, r} \frac{\left|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)(t)\right|}{|t|} \,\mathrm{d}t \\
&\le 2^j\, r \left(\int_{|t|>2^j\, r} |t|^{-2} \,\mathrm{d}t\right)^{1/2} \left\|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)\right\|_2 \\
&\le C\left(2^j\, r\right)^{1/2} \|g^{2^j}\,\varphi\|_{H^\ell(\mathbb{R})} \\
&\le C\,(2^j\, r)^{1/2}\, \|g\|_{\mathrm{Horm}(\ell)},
\end{aligned}
\tag{2.18}
$$

as required.

The inequality (iii) follows from (i) by taking $k=\varepsilon=0$. □

Remark 2.7. Notice the following variant of Lemma 2.6 (ii) that will be used in the proof of Theorem 3.4 (i) below. For every η in $(1/2,1]$ and for every R in $\mathbb{R}^+$ there exists a constant C such that

$$
r \int_{|t|>r} \frac{|R_k^*\widehat{g}_j(t)|}{|t|^\eta} \,\mathrm{d}t \le C\,(2^j\, r)^{1/2}\, \|g\|_{\mathrm{Horm}(k)} \qquad \forall r \in (0,R].
\tag{2.19}
$$

The proof is much the same as the proof of Lemma 2.6 (ii). As before, it suffices to prove (2.19) with G_j^ℓ in place of $R_k^*\widehat{g}_j$ for all ℓ in $\{0,\dots,k\}$.

Observe that

$$
\begin{aligned}
r \int_{|t|>r} \frac{\big|G_j^\ell(t)\big|}{|t|^\eta}\,\mathrm{d}t &= r \int_{|t|>r} \frac{\big|\big[\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)\big]_{2^{-j}}(t)\big|}{|t|^\eta}\,\mathrm{d}t \\
&= 2^{\eta j}\, r \int_{|t|>2^j\, r} \frac{\big|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)(t)\big|}{|t|^\eta}\,\mathrm{d}t \\
&\leq 2^{\eta j}\, r \Big(\int_{|t|>2^j\, r} |t|^{-2\eta}\,\mathrm{d}t\Big)^{1/2} \,\big\|\mathcal{O}^\ell \mathcal{F}(g^{2^j}\,\varphi)\big\|_2 \\
&\leq C\, 2^{j/2}\, r^{3/2-\eta}\, \|g^{2^j}\,\varphi\|_{H^\ell(\mathbb{R})} \\
&\leq C\,(2^j\, r)^{1/2}\, \|g\|_{\mathrm{Horm}(\ell)},
\end{aligned}
\tag{2.20}
$$

as required. Note that in the last inequality we have used the fact that r varies in a bounded set.

3. Spectral multipliers on Riemannian manifolds

In this section we prove our main result, Theorem 3.4. To treat the part of the kernel $k_{m(\mathcal{D})}$ near the diagonal of $M \times M$, we shall need the following result, which is the analogue on manifolds with bounded geometry of a well known result in the setting of spaces of homogeneous type in the sense of Coifman and Weiss [10]. For the reader's convenience we sketch its proof, but omit the details of the part which is very similar to the proof of [10, Théorème 2.4].

We denote by $\mathcal{B}_s$ the family of all balls with radius at most s. Given a ball B, we denote by c_B its centre, by r_B its radius and by $2B$ the ball with the same centre as B and radius $2r_B$.

Proposition 3.1. *Suppose that $\mathcal{T}$ is a bounded operator on $L^2(M)$ and that*

(i) *its Schwartz kernel $k_\mathcal{T}$ is locally integrable in $(M \times M) \setminus \{(x,x) : x \in M\}$, and is supported in $\{(x,y) \in M \times M : d(x,y) \leq 1\}$;*

(ii) *the following* Hörmander integral condition at scale 1 *holds*

$$
A := \sup_{B \in \mathcal{B}_1} \sup_{y \in B} \int_{B(c_B,2)\setminus(2B)} |k_\mathcal{T}(x,y) - k_\mathcal{T}(x,c_B)|\,\mathrm{d}\mu(x) < \infty.
$$

Then $\mathcal{T}$ extends to an operator of weak type 1 and there exists a constant C such that

$$
\|\mathcal{T}\|_{L^1(M);L^{1,\infty}(M)} \leq CA.
$$

Proof. Denote by $\mathfrak{M}$ a 1-discretisation of M, i.e., a subset of M that is maximal with respect to the following property:

$$
d(z_1,z_2) \geq 1 \qquad \forall z_1, z_2 \in \mathfrak{M} \qquad \text{and} \qquad d(x,\mathfrak{M}) \leq 1 \quad \forall x \in M.
$$

We denote by $\{z_j : j \in \mathbb{N}\}$ the points of $\mathfrak{M}$. Since the measure μ is locally doubling, the family $\{B(z_j,1) : z_j \in \mathfrak{M}\}$ is a covering of M such that $\{B(z_j,2) : z_j \in \mathfrak{M}\}$ has the *bounded overlap property*, i.e., there exists a positive integer N_2 such that

$$
1 \leq \sum_{j\in\mathbb{N}} \mathbf{1}_{B(z_j,1)} \leq \sum_{j\in\mathbb{N}} \mathbf{1}_{B(z_j,2)} \leq N_2,
$$

where $\mathbf{1}_E$ denotes the indicator function of the set E. Given f in $L^1(M)$ and a nonnegative integer j, we define f_j by $f_j = f\,\mathbf{1}_{B(z_j,1)}/\sum_\ell \mathbf{1}_{B(z_\ell,1)}$. Then $f = \sum_{j\in\mathbb{N}} f_j$, and

$$\mathcal{T}f = \sum_{j\in\mathbb{N}} \mathcal{T}f_j.$$

Note that this sum is locally uniformly finite, because the function $\mathcal{T}f_j$ is supported in the ball $B(z_j,2)$, by (i) above, and the family $\{B(z_j,2) : z_j \in \mathfrak{M}\}$ has the bounded overlap property. Then there exists a constant C such that

$$\mu\big(\{x\in M : |\mathcal{T}f(x)| > s\}\big) \leq C \sum_{j\in\mathbb{N}} \mu\big(\{x\in M : |\mathcal{T}f_j(x)| > s/N_2\}\big) \qquad \forall s\in\mathbb{R}^+.$$

Thus, to conclude the proof it suffices to show that there exists a constant C such that

$$s\;\mu\big(\{x\in M : |\mathcal{T}f_j(x)| > s\}\big) \leq C\,A\,\|f_j\|_1 \qquad \forall s\in\mathbb{R}^+ \quad \forall j\in\mathbb{N}, \tag{3.1}$$

for then we may conclude that

$$\begin{aligned} s\;\mu\big(\{x\in M : |\mathcal{T}f(x)| > s\}\big) &\leq C\,A \sum_{j\in\mathbb{N}} \|f_j\|_1 \\ &\leq C\,A\,\|f\|_1 \qquad \forall s\in\mathbb{R}^+. \end{aligned}$$

by the bounded overlap property, as required.

To prove (3.1), we may follow the proof of the original result of R.R. Coifman and G. Weiss on spaces of homogeneous type [10, Théorème 2.4]. Define the *local doubling constant* D_2 by

$$D_2 = \sup_{B\in\mathcal{B}_2} \frac{\mu(2B)}{\mu(B)}.$$

Then, given s in $\mathbb{R}^+$, consider a Calderón–Zygmund decomposition of f_j at height s. Note that, though M need not be a space of homogeneous type, each f_j is supported in a ball of radius 1, and all these balls are spaces of homogeneous type with doubling constant dominated by D_2. Thus, the constants appearing in the Calderón–Zygmund decompositions of the functions f_j depend on D_2, but not on j. Then the proof of (3.1) is exactly as in the setting of spaces of homogeneous type, and the constant C in (3.1) depends on D_2, but not on s or j. We omit the details. □

Now we define an appropriate function space of holomorphic functions which will be needed in the statement of Theorem 3.4. Then, for the reader's convenience, we recall one of its properties, which will be key in the proof of our main result.

Definition 3.2. Suppose that J is a positive integer and that W is in $\mathbb{R}^+$. We recall that $\mathbf{S}_W$ denotes the strip $\{\zeta\in\mathbb{C} : \mathrm{Im}(\zeta)\in(-W,W)\}$ and we denote by $H^\infty(\mathbf{S}_W;J)$ the vector space of all bounded *even* holomorphic functions f in $\mathbf{S}_W$ for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C\,(1+|\zeta|)^{-j} \qquad \forall \zeta\in\mathbf{S}_W \quad \forall j\in\{0,1,\dots,J\}. \tag{3.2}$$

The infimum of all constants C for which (3.2) holds will be denoted by $\|f\|_{\mathbf{S}_W;J}$.

Lemma 3.3 ([18, Lemma 5.4]). *Suppose that J is an integer ≥ 2, and that W is in $\mathbb{R}^+$. Then there exists a positive constant C such that for every function f in $H^\infty\big(\mathbf{S}_W; J\big)$, and for every positive integer $h \leq J-2$*

$$\big|\mathcal{O}^h \widehat{f}(t)\big| \leq C\,\|f\|_{\mathbf{S}_W;J}\ |t|^{h-J}\ \mathrm{e}^{-W|t|} \qquad \forall t \in \mathbb{R}\setminus\{0\}.$$

Theorem 3.4. *Suppose that M is a Riemannian manifold with bounded geometry, and suppose that (1.3) holds for some $\rho > 1/2$. Assume that α and β are as in (1.1), and denote by N the integer $[\![n/2+1]\!]+1$. Suppose that J is an integer $>$ $\max(N+1, N+1+\alpha/2-\rho)$. Then there exists a constant C such that*

$$|\!|\!| m(\mathcal{D}) |\!|\!|_{L^1(M);L^{1,\infty}(M)} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty(\mathbf{S}_\beta; J).$$

Proof. For notational convenience in this proof we shall write $\mathcal{J}$ instead of $\mathcal{J}_{N-1/2}$.

Denote by ω an even function in $C_c^\infty(\mathbb{R})$ which is supported in $[-1,1]$, is equal to 1 in $[-1/4,1/4]$, and satisfies

$$\sum_{j\in\mathbb{Z}} \omega(t-j) = 1 \qquad \forall t \in \mathbb{R}.$$

Clearly $\widehat{\omega}*m$ and $m-\widehat{\omega}*m$ are bounded functions. We follow the strategy of Taylor (see [29, Thm 1]), and define the operators $\mathcal{A}$ and $\mathcal{B}$ spectrally by

$$\mathcal{A} = (\widehat{\omega}*m)(\mathcal{D}) \qquad \text{and} \qquad \mathcal{B} = (m-\widehat{\omega}*m)(\mathcal{D}).$$

Then $m(\mathcal{D}) = \mathcal{A}+\mathcal{B}$. We shall prove that there exists a constant C such that

$$|\!|\!|\mathcal{A}|\!|\!|_{L^1(M);L^{1,\infty}(M)} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \tag{3.3}$$

and

$$|\!|\!|\mathcal{B}|\!|\!|_{L^1(M)} \leq C\,\|m\|_{\mathbf{S}_\beta;J}. \tag{3.4}$$

These estimates clearly imply the desired conclusion.

First we analyse the operator $\mathcal{A}$. Since $\widehat{\omega}*m$ is an even entire function of exponential type 1, the function A, defined by

$$A(\zeta) = (\widehat{\omega}*m)\big(\sqrt{\zeta^2-\kappa^2}\big) \qquad \forall \zeta \in \mathbb{C},$$

is entire of exponential type 1. The reason for introducing the new function A is to write $\mathcal{A}$ as a function of the operator $\mathcal{D}_1$ (defined at the beginning of Subsection 2.2) rather than of the operator $\mathcal{D}$. Observe that

$$\mathcal{A} = A(\mathcal{D}_1),$$

and that the support of $k_{\mathcal{A}}$ is contained in $\{(x,y) \in M\times M : d(x,y) \leq 1\}$. It is straightforward to check that

$$\big\|A\big\|_{\mathrm{Horm}(J)} \leq C\ \big\|\widehat{\omega}*m\big\|_{\mathrm{Horm}(J)}\,, \tag{3.5}$$

where the constant C does not depend on m. By arguing much as in the proof of [18, Proposition 5.3], we may show that the function $\widehat{\omega}*m$ satisfies a Mihlin–Hörmander condition of order J, with $\|\widehat{\omega}*m\|_{\mathrm{Horm}(J)}$ bounded by a constant times $\|m\|_{\mathrm{Horm}(J)}$. Furthermore, it is clear that

$$\|m\|_{\mathrm{Horm}(J)} \leq C\,\|m\|_{\mathbf{S}_\beta;J},$$

with C independent of m. In view of this observation and of Proposition 3.1, to prove that $\mathcal{A}$ is of weak type 1, with the required norm estimate, it suffices prove that its integral kernel $k_{\mathcal{A}}$ satisfies the following

$$\sup_{y\in B\in\mathcal{B}_1}\int_{B(c_B,2)\setminus(2B)}|k_{\mathcal{A}}(x,y)-k_{\mathcal{A}}(x,c_B)|\,\mathrm{d}\mu(x)\le C\,\|A\|_{\mathrm{Horm}(J)}. \tag{3.6}$$

To prove (3.6) we further decompose the function A, and then decompose the operator $\mathcal{A}$ accordingly. For all j in $\mathbb{Z}$ define the functions A_j by

$$A_j = A\,\varphi^{2^{-j}},$$

where φ is defined just above Lemma 2.6. Then $\widehat{A}_j$ is an entire function of exponential type and $\widehat{A}=\sum_j \widehat{A}_j$ in the sense of distributions. Furthermore, Lemma 2.6 (with A in place of g) and Remark 2.7 imply that for every η in $(1/2,1]$ there exists a constant C such that for every j in $\mathbb{Z}$ and for every ℓ in $\{0,1,\dots,N\}$

$$\int_{|t|>r}\big|R_N^*\widehat{A}_j(t)\big|\,\mathrm{d}t\le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j\,r)^{-\varepsilon}\qquad\forall r\in\mathbb{R}^+\ , \tag{3.7}$$

$$r\int_{|t|>r}\frac{\big|R_N^*\widehat{A}_j(t)\big|}{|t|^{\eta}}\,\mathrm{d}t\le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j\,r)^{1/2}\qquad\forall r\in(0,2]. \tag{3.8}$$

Here we have used the fact that $J>N+1/2$.

For each ball B in $\mathcal{B}_1$ and for each integer j, define $I_j(B)$ by

$$I_j(B)=\sup_{y\in B}\int_{E_B}\big|k_{A_j(\mathcal{D}_1)}(x,y)-k_{A_j(\mathcal{D}_1)}(x,c_B)\big|\,\mathrm{d}\mu(x),$$

where, for notational convenience, we write E_B instead of $B(c_B,2)\setminus(2B)$. To prove (3.6), it suffices to show that there exists a constant C such that for

$$I_j(B)\le C\,\|A\|_{\mathrm{Horm}(J)}\,\min\big((2^j\,r_B)^{-\varepsilon},(2^j\,r_B)^{1/2}\big)\qquad\forall B\in\mathcal{B}_1\quad\forall j\in\mathbb{Z}.$$

To prove this, we shall prove separately that

$$I_j(B)\le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j\,r_B)^{-\varepsilon}, \tag{3.9}$$

and that

$$I_j(B)\le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j\,r_B)^{1/2}. \tag{3.10}$$

The key formula here is

$$A_j(\lambda)=\frac{2^{N-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}R_N^*\widehat{A}_j(t)\ \mathcal{J}(t\lambda)\,\mathrm{d}t,$$

which follows from the Fourier inversion formula and Lemma 2.4 (ii), and its consequence

$$k_{A_j(\mathcal{D}_1)}=\frac{2^{N-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}R_N^*\widehat{A}_j(t)\ k_{\mathcal{J}(t\mathcal{D}_1)}\,\mathrm{d}t.$$

Note that the modified Bessel function $\lambda\mapsto\mathcal{J}(t\lambda)$ is an even entire function of exponential type $|t|$. Thus, by the finite propagation speed property for $\mathcal{L}$, the kernel $k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)$ vanishes outside the ball $B(y,|t|)$.

To prove (3.9), note that

$$\begin{aligned} I_j(B) &\leq 2 \sup_{y\in B} \big\|k_{A_j(\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(E_B)} \\ &\leq C \sup_{y\in B} \int_{|t|\geq r_B} \big|R_N^* \widehat{A}_j(t)\big| \; \big\|k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(E_B)} \,\mathrm{d}t. \end{aligned}$$

Now we split each of these integrals as the sum of the corresponding integrals over the sets $\{t\in\mathbb{R} : r_B \leq |t| \leq 1\}$ and $\{t\in\mathbb{R} : |t| > 1\}$. We denote these two integrals by Υ_1 and Υ_2 respectively. They depend on y in B and j.

By the asymptotics of Bessel functions [22, formula (5.11.6), p.122]

$$\sup_{s>0} \big|(1+s)^N \, \mathcal{J}(s)\big| < \infty,$$

so that $\mathcal{J}$ satisfies the assumptions of Proposition 2.2 (with N in place of γ). Hence by Schwarz's inequality, (2.2) and Proposition 2.2 (ii)

$$\begin{aligned} \big\|k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(B(c_B,2))} &\leq \mu\big(B(y,|t|)\big)^{1/2} \; \big\|k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^2(B(y,|t|))} \\ &\leq C \qquad \forall t\in[-1,1]\setminus\{0\}. \end{aligned}$$

As a consequence

$$\begin{aligned} \Upsilon_1 &\leq C \int_{r_B\leq|t|\leq 1} \big|R_N^*\widehat{A}_j(t)\big| \,\mathrm{d}t \\ &\leq C\, \|A\|_{\mathrm{Horm}(J)} \,(2^j\, r_B)^{-\varepsilon}. \end{aligned}$$

Note that we have used (3.7) above in the last inequality.

To estimate Υ_2 we argue similarly, using Proposition 2.2 (i) and the fact that $\mu(E_B) \leq \mu\big(B(y,3)\big) \leq C$ by (2.2). Thus we obtain

$$\begin{aligned} \Upsilon_2 &\leq C \int_{|t|>1} \big|R_N^*\widehat{A}_j(t)\big|\, |t|^{-\rho} \,\mathrm{d}t \\ &\leq C \int_{|t|\geq r_B} \big|R_N^*\widehat{A}_j(t)\big| \,\mathrm{d}t \\ &\leq C\, \|A\|_{\mathrm{Horm}(J)} \,(2^j\, r_B)^{-\varepsilon}. \end{aligned}$$

Then (3.9) follows.

To prove (3.10), observe that

$$\begin{aligned} I_j(B) &\leq C\, r_B \sup_{y\in B} \int_{E_B} \big|\mathrm{d}_2 k_{A_j(\mathcal{D}_1)}(x,y)\big| \,\mathrm{d}\mu(x) \\ &\leq C\, r_B \sup_{y\in B} \int_{|t|\geq r_B} \big|R_N^*\widehat{A}_j(t)\big| \; \big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(E_B)} \,\mathrm{d}t. \end{aligned} \tag{3.11}$$

Much as before, we split each of these integrals as the sum of the integrals over the sets $\{t\in\mathbb{R} : r_B\leq|t|<1\}$ and $\{t\in\mathbb{R} : |t|\geq 1\}$, and denote them by $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$.

By the finite propagation speed for $\mathcal{L}$, the kernel $\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)$ vanishes outside the ball $B(y,|t|)$. Hence by Schwarz's inequality and Proposition 2.2 (iii)

$$\begin{aligned} \big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(B(c_B,2))} &\leq \mu\big(B(y,|t|)\big)^{1/2} \; \big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^2(B(y,|t|))} \\ &\leq C\; |t|^{n/2}\; |t|^{-n/2-1} \qquad \forall t\in[-1,1]\setminus\{0\} \end{aligned}$$

and

$$\begin{aligned}\big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_{L^1(B(c_B,2))} &\le \mu\big(B(c_B,2)\big)^{1/2}\ \big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,y)\big\|_2 \\ &\le C\ |t|^{-\rho} \qquad \forall t\in\mathbb{R}\setminus[-1,1].\end{aligned}$$

Then, by (3.8),

$$\widetilde{\Upsilon}_1 \le C \int_{r_B\le|t|\le 1} \frac{\big|R_N^*\widehat{A}_j(t)\big|}{|t|}\,\mathrm{d}t \le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j/r_B)^{1/2},$$

and

$$\widetilde{\Upsilon}_2 \le C \int_{|t|>1} \big|R_N^*\widehat{A}_j(t)\big|\,|t|^{-\rho}\,\mathrm{d}t \le C\,\|A\|_{\mathrm{Horm}(J)}\,(2^j/r_B)^{1/2}.$$

The required estimate (3.10) follows from this and (3.11). This concludes the proof of (3.6), hence of (3.3).

Next we estimate $\|\mathcal{B}\|_{L^1(M)}$. For each j in $\{2,3,\dots,\}$, define ω_j by the formula

$$\omega_j(t) = \omega(t-j+1)+\omega(t+j-1) \qquad \forall t\in\mathbb{R}.$$

Observe that $\mathcal{F}(m-\widehat{\omega}*m) = \sum_{j=2}^\infty \omega_j\,\widehat{m}$. Since m is in $H^\infty\big(\mathbf{S}_\beta;J\big)$ and $J\ge N+2$, the function $\widehat{m}$ and its derivatives up to the order N are rapidly decreasing at infinity by Lemma 3.3, so that $\mathcal{O}^h\big((1-\omega)\,\widehat{m}\big)$ is in $L^1(\mathbb{R})\cap C_0(\mathbb{R}^+)$ for all h in $\{0,\dots,N\}$. Hence we may use Lemma 2.4 (ii) and write

$$\begin{aligned}(m-\widehat{\omega}*m)(\lambda) &= \frac{1}{2\pi}\sum_{j=2}^\infty \int_{-\infty}^\infty \omega_j(t)\,\widehat{m}(t)\,\cos(t\lambda)\,\mathrm{d}t \\ &= \frac{2^{N-1}}{\sqrt{2\pi}}\sum_{j=2}^\infty \int_{-\infty}^\infty R_N^*\big(\omega_j\,\widehat{m}\big)(t)\,\mathcal{J}(t\lambda)\,\mathrm{d}t,\end{aligned}$$

for all λ in $\mathbb{R}$. Define the kernel $k_{\mathcal{B}}^j$ by

$$k_{\mathcal{B}}^j = \frac{2^{N-1}}{\sqrt{2\pi}} \int_{-\infty}^\infty R_N^*\big(\omega_j\,\widehat{m}\big)(t)\,k_{\mathcal{J}(t\mathcal{D})}\,\mathrm{d}t.$$

Then, at least formally, $k_{\mathcal{B}} = \sum_{j=2}^\infty k_{\mathcal{B}}^j$. Note that $k_{\mathcal{B}}^j$ is supported in $\{(x,y)\in M\times M : d(x,y)\le j\}$ by finite propagation speed. For all positive integer ℓ and for each p in M, denote by $A(p,\ell)$ the annulus with centre p and radii $\ell-1$ and ℓ. Fix y in M. Then

$$\begin{aligned}\big\|k_{\mathcal{B}}(\cdot,y)\big\|_1 &= \sum_{\ell=1}^\infty \int_{A(y,\ell)} \big|k_{\mathcal{B}}(x,y)\big|\,\mathrm{d}\mu(x) \\ &\le \sum_{\ell=1}^\infty \mu\big(B(y,\ell)\big)^{1/2}\Big[\int_{A(y,\ell)} \big|k_{\mathcal{B}}(x,y)\big|^2\,\mathrm{d}\mu(x)\Big]^{1/2}\end{aligned}$$

Note that if $j\le\ell-1$, then the restriction of $k_{\mathcal{B}}^j$ to $A(y,\ell)$ vanishes, because $k_{\mathcal{B}}^j(\cdot,y)$ is supported in the ball $B(y,j)$. Thus, by Schwarz's inequality

$$\begin{aligned}\big\|k_{\mathcal{B}}(\cdot,y)\big\|_1 &\le \sum_{\ell=1}^\infty \mu\big(B(y,\ell)\big)^{1/2} \sum_{j=\ell}^\infty \big\|k_{\mathcal{B}}^j(\cdot,y)\big\|_2 \\ &\le C\sum_{\ell=1}^\infty \ell^{\alpha/2}\,\mathrm{e}^{\beta\ell} \sum_{j=\ell}^\infty \int_{-\infty}^\infty \big|R_N^*\big(\omega_j\,\widehat{m}\big)(t)\big|\ \big\|k_{\mathcal{J}(t\mathcal{D})}(\cdot,y)\big\|_2\ \mathrm{d}t,\end{aligned}$$

where we have used (1.1) and the formula above for $k_{\mathcal{B}}^j$. Recall that $N - 1/2 > (n+1)/2$. Then, by Proposition 2.2 (i) there exists a constant C such that

$$\sup_{y\in M} \big\|k_{\mathcal{J}(t\mathcal{D})}(\cdot,y)\big\|_2 \leq C\, |t|^{-n/2}\, \big(1+|t|\big)^{n/2-\rho} \qquad \forall t\in\mathbb{R}^+.$$

Furthermore, by Lemma 3.3, there exists a constant C such that for h in $\{0,\ldots,N\}$

$$\big|\mathcal{O}^h(\omega_j\,\widehat{m})(t)\big| \leq C\, \|m\|_{\mathbf{S}_\beta;J}\, \mathrm{e}^{-\beta|t|}\, |t|^{h-J} \qquad \forall t\in\mathbb{R}\setminus\{0\};$$

here we have used the fact that $J \geq N+2$. Since $\mathcal{O}^h(\omega_j\widehat{m})$ vanishes in $[2-j, j-2]$,

$$\begin{aligned}\sup_{y\in M}\big\|k_{\mathcal{B}}(\cdot,y)\big\|_1 &\leq C\, \|m\|_{\mathbf{S}_\beta;J} \sum_{\ell=1}^\infty \ell^{\alpha/2}\, \mathrm{e}^{\beta\ell} \sum_{j=\ell}^\infty \int_{|t|\geq j-2} \mathrm{e}^{-\beta|t|}\, |t|^{N-J-\rho}\, \mathrm{d}t \\ &\leq C\, \|m\|_{\mathbf{S}_\beta;J} \sum_{\ell=1}^\infty \ell^{\alpha/2+N-J-\rho}.\end{aligned}$$

The series above is convergent, because $\alpha/2 + N - J - \rho < -1$ by assumption, so that

$$\sup_{y\in M}\big\|k_{\mathcal{B}}(\cdot,y)\big\|_1 \leq C\, \|m\|_{\mathbf{S}_\beta;J}.$$

Since $k_{\mathcal{B}}(x,y) = \overline{k_{\mathcal{B}}(y,x)}$,

$$\sup_{x\in M}\big\|k_{\mathcal{B}}(x,\cdot)\big\|_1 \leq C\, \|m\|_{\mathbf{S}_\beta;J}.$$

Hence $\mathcal{B}$ maps $L^1(M)$ into $L^1(M)$, with operator norm bounded by $C\,\|m\|_{\mathbf{S}_\beta;J}$, as required to to prove (3.4).

The proof of the theorem is complete. □

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