INTEGRALLY CLOSED IDEALS ON LOG TERMINAL SURFACES ARE MULTIPLIER IDEALS

KEVIN TUCKER

Abstract. We show that all integrally closed ideals on log terminal surfaces are multiplier ideals by extending an existing proof for smooth surfaces.

1. Introduction

Consider a scheme $X = \text{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over $\mathbb{C}$. Our purpose is to partially address the following question, raised in [6]:

Question. If $X$ has a rational singularity, is every integrally closed ideal which is contained in $\mathcal{J}(X, \mathcal{O}_X)$ a multiplier ideal?

Here, $\mathcal{J}(X, a^\lambda)$ denotes the multiplier ideal corresponding to an ideal $a \subseteq \mathcal{O}_X$ with coefficient $\lambda \in \mathbb{Q}_{>0}$. When $X$ is regular, an affirmative answer was given concurrently by [8] and [3]. Our main result is to generalize their methods to prove the following:

Theorem 1.1. Suppose $X$ has log terminal singularities. Then every integrally closed ideal is a multiplier ideal.

Log terminal singularities satisfy $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$ by definition, and are necessarily rational (see Theorem 5.22 in [4]). Thus, Theorem 1.1 gives a complete answer to the above question in this case.

There are several difficulties in trying to extend the techniques used in [8]. One must show that successful choices can be made in the construction (specifically, the choice of $\epsilon$ and $N$ in Lemma 2.2 of [8]). Here, it is essential that $X$ has log terminal singularities. Further problems arise from the failure of unique factorization to hold for integrally closed ideals. As $X$ is not necessarily factorial, we may no longer reduce to the finite colength case. In addition, the crucial contradiction argument which concludes the proof in [8] does not apply. These nontrivial difficulties are overcome by using a relative numerical decomposition for divisors on a resolution over $X$. Further, appropriately interpreted, the proof of Theorem 1.1 applies over an algebraically closed field of arbitrary characteristic.

Our presentation is self-contained and elementary. Section 2 contains background material covering the relative numerical decomposition, antinef closures, and some computations using generic sequences of blowups. Section 3 is dedicated to the constructions and arguments in the proof of Theorem 1.1.

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2. Background

2.1. Relative Numerical Decomposition. For the remainder, we will consider a scheme $X = \text{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over an algebraically closed field of arbitrary characteristic. Let $x \in X$ be the unique closed point, and suppose $f: Y \to X$ is a projective birational morphism such that $Y$ is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and $\Lambda = \oplus \mathbb{Z} E_i \subset \text{Div}(Y)$ the lattice they generate.

The intersection pairing $\text{Div}(Y) \times \Lambda \to \mathbb{Z}$ induces a negative definite $\mathbb{Q}$-bilinear form on $\Lambda_{\mathbb{Q}}$ (see [4] for an elementary proof). Consequently, there is a dual basis $\hat{E}_1, \ldots, \hat{E}_u$ for $\Lambda_{\mathbb{Q}}$ defined by the property that

$$\hat{E}_i \cdot E_j = -\delta_{ij} = \begin{cases} -1 & i = j \\ 0 & i \neq j \end{cases}.$$ 

Recall that a divisor $D \in \text{Div}_{\mathbb{Q}}(Y)$ is said to be $f$-antinéf if $D \cdot E_i \leq 0$ for all $i = 1, \ldots, u$. In this case, $D$ is effective if and only if $f_* D$ is effective (see Lemma 3.39 in [4]). In particular, $\hat{E}_1, \ldots, \hat{E}_u$ are effective.

If $C \in \text{Div}_{\mathbb{Q}}(X)$, we define the numerical pullback of $C$ to be the unique $\mathbb{Q}$-divisor $f^* C$ on $Y$ such that $f_* f^* C = C$ and $f^* C \cdot E_i = 0$ for all $i = 1, \ldots, u$. Note that, when $C$ is Cartier or even $\mathbb{Q}$-Cartier, this agrees with the standard pullback of $C$. If $D \in \text{Div}_{\mathbb{Q}}(Y)$, we have

$$D = f^* f_* D + \sum_i (-D \cdot E_i) \hat{E}_i. \quad (1)$$

We shall refer to this as a relative numerical decomposition for $D$. Note that, even when $D$ is integral, both $f^* f_* D$ and $\hat{E}_1, \ldots, \hat{E}_u$ are likely non-integral. The fact that $f^* f_* D$ and $\hat{E}_1, \ldots, \hat{E}_u$ are always integral divisors when $X$ is smooth and $D$ is integral is equivalent to the unique factorization of integrally closed ideals. See [7] for further discussion.

2.2. Antinef Closures and Global Sections. Suppose now that $D' = \sum_E a_E' E$ and $D'' = \sum_E a_E'' E$ are $f$-antinef divisors, where the sums range over the prime divisors $E$ on $Y$. It is easy to check that $D' \wedge D'' = \sum_E \min\{a_E', a_E''\} E$ is also $f$-antinef. Further, any integral $D \in \text{Div}(Y)$ is dominated by some integral $f$-antinef divisor (e.g. $(f^{-1})_* D + M(\hat{E}_1 + \cdots + \hat{E}_u)$ for sufficiently large and divisible $M$). In particular, there is a unique smallest integral $f$-antinef divisor $D^\sim$, called the $f$-antinef closure of $D$, such that $D^\sim \geq D$. One can verify that $f_* D = f_* D^\sim$, and in addition the following important lemma holds (see Lemma 1.2 of [8]). The proof also gives an effective algorithm for computing $f$-antinef closures.

Lemma 2.1. For any $D \in \text{Div}(Y)$, we have $f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(-D^\sim)$.

Proof. Let $s_D \in \mathbb{N}$ be the sum of the coefficients of $D^\sim - D$ when written in terms of $E_1, \ldots, E_u$. If $s_D = 0$, then $D = D^\sim$ is $f$-antinef and the statement follows trivially. Else, there is an index $i$ such that $D \cdot E_i > 0$. As $E_i \cdot E_j \geq 0$ for $j \neq i$, we must have $D \leq D + E_i \leq (D + E_i)^\sim$. 


Thus, $s_{D+E_i} = s_D - 1$. By induction, we may assume

$$f_*O_Y(-(D+E_i)) = f_*O_Y(-(D+E_i)^\sim) = f_*O_Y(-D^\sim)$$

and it is enough to show $f_*O_Y(-D) = f_*O_Y(-(D+E_i))$. Consider the exact sequence

$$0 \rightarrow O_Y(-(D+E_i)) \rightarrow O_Y(-D) \rightarrow O_{E_i}(-D) \rightarrow 0.$$ 

Since $\deg(O_{E_i}(-D)) = -D \cdot E_i < 0$, we have $f_*O_{E_i}(-D) = 0$; applying $f_*$ yields the desired result.

2.3. Generic Sequences of Blowups. In the proof of Theorem 1.1, we will make use of the following auxiliary construction. Suppose $x^{(i)}$ is a closed point of $E_i$ with $x^{(i)} \notin E_j$ for $j \neq i$. A generic sequence of $n$-blowups over $x^{(i)}$ is:

$$Y = Y_0 \leftarrow \sigma_1 Y_1 \leftarrow \sigma_2 \cdots \leftarrow \sigma_{n-1} Y_{n-1} \leftarrow \sigma_n Y_n$$

where $\sigma_1: Y_1 \rightarrow Y_0$ is the blowup of $Y_0 = Y$ at $x_1 := x^{(i)}$, and $\sigma_k: Y_k \rightarrow Y_{k-1}$ is the blowup of $Y_{k-1}$ at a generic closed point $x_k$ of $(\sigma_{k-1})^{-1}(x_{k-1})$ for $k = 2, \ldots, n$. Let $\sigma: Y_n \rightarrow Y$ be the composition $\sigma_n \circ \cdots \circ \sigma_1$. We will denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$ on $Y_n$. Also, let $E(i, x^{(i)}, k)$, $k = 1, \ldots, n$, be the strict transforms of the $n$ new $\sigma$-exceptional divisors created by the blowups $\sigma_1, \ldots, \sigma_n$, respectively.

Lemma 2.2. (a) Let $\sigma: Y_n \rightarrow Y$ be a generic sequence of blowups over $x^{(i)} \in E_i$. Then one has

$$\bar{E}(i) \leq \bar{E}(i, x^{(i)}, 1) \leq \cdots \leq \bar{E}(i, x^{(i)}, n).$$

(b) Suppose $D \in \text{Div}(Y_n)$ is an integral $(f \circ \sigma)$-anti-ef divisor such that $E_i$ is the unique component of $\sigma_* D$ containing $x^{(i)}$. If $\text{ord}_{E(i)} D = a_0$ and $\text{ord}_{E(i, x^{(i)}, k)} D = a_k$ for $k = 1, \ldots, n$, then

$$a_0 \leq a_1 \leq \cdots \leq a_n.$$

Further, $a_0 < a_n$ if and only if

$$\left(\sum_{k=1}^{n} (-D \cdot E(i, x^{(i)}, k))\bar{E}(i, x^{(i)}, k) \right) \geq \bar{E}(i).$$

Proof. If $n = 1$, we have

$$\bar{E}(i, x^{(i)}, 1) = \left(\sigma^* \bar{E}_i + E(i, x^{(i)}, 1)\right) \geq \sigma^* \bar{E}_i = \bar{E}(i)$$

$$D = \sigma^* \sigma_* D + (-D \cdot E(i, x^{(i)}, 1))\bar{E}(i, x^{(i)}, 1).$$

The general case of both statements follows easily by induction.
3. Main Theorem

3.1. Log Terminal Singularities and Multiplier Ideals. Once more, suppose $x \in X$ is the unique closed point and $f: Y \to X$ is a projective birational morphism such that $Y$ is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and let $K_Y$ be a canonical divisor on $Y$. Then $K_X := f_*K_Y$ is a canonical divisor on $X$. If we write the relative canonical divisor as

$$K_f := K_Y - f^*K_X = \sum_i b_i E_i$$

then $X$ has (numerically) log terminal singularities if and only if $b_i > -1$ for all $i = 1, \ldots, u$. In this case, when working over $\mathbb{C}$, $X$ is automatically $\mathbb{Q}$-factorial (see Proposition 4.11 in [4], as well as [2] for recent developments).

If $a \subseteq O$ is an ideal, recall that $f: Y \to X$ as above is said to be a log resolution of $a$ if $aO_Y = O_Y(-G)$ for an effective divisor $G$ such that $\text{Ex}(f) \cup \text{Supp}(G)$ has simple normal crossings. In this case, we can define the multiplier ideal of $(X, a)$ with coefficient $\lambda \in \mathbb{Q}_{>0}$ as

$$\mathcal{J}(X, a^\lambda) = f_*O_Y([K_f - \lambda G]).$$

See [9] for an introduction in a similar setting, or [5] for a more comprehensive overview. Also recall that $a$ is integrally closed if and only if

$$a = f_*O_Y(-G).$$

3.2. Choosing $a$ and $\lambda$. We now begin the proof of Theorem 1.1. For the remainder, assume $X$ is log terminal, and let $I \subseteq O_X$ be an integrally closed ideal. In this section, we construct another ideal $a \subseteq O_X$ along with a coefficient $\lambda \in \mathbb{Q}_{>0}$; and in the following section it will be shown that $\mathcal{J}(X, a^\lambda) = I$. Let $f: Y \to X$ a log resolution of $I$ with exceptional divisors $E_1, \ldots, E_u$. Suppose $I_O = O_Y(-F^0)$, and write

$$K_f = \sum_{i=1}^u b_i E_i$$

$$F^0 = (f^{-1})_*(F^0) + \sum_{i=1}^u a_i E_i.$$  

Choose $0 < \epsilon < 1/2$ such that $|\epsilon(f^{-1})_*F^0| = 0$ and

$$\epsilon(a_i + 1) < 1 + b_i$$

for $i = 1, \ldots, u$. Note that, since $X$ is log terminal, $1 + b_i > 0$ and any sufficiently small $\epsilon > 0$ will do. Let $n_i := \lfloor 1 + b_i \epsilon - (a_i + 1) \rfloor \geq 0$, and $e_i := (F^0 \cdot E_i)$. Choose $e_i$ distinct closed points $x_1^{(i)}, \ldots, x_{e_i}^{(i)}$ on $E_i$ such that $x_j^{(i)} \not\in \text{Supp}((f^{-1})_*F^0)$ and $x_j^{(i)} \not\in E_l$ for $l \neq i$. Denote by $g: Z \to Y$ the composition of $n_i$ generic blowups at each of the points $x_j^{(i)}$ for $j = 1, \ldots, e_i$ and $i = 1, \ldots, u$. As in Section 2.3 denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$, and $E(i, x_j^{(i)}, 1), \ldots, E(i, x_j^{(i)}, n_i)$ the strict transforms of the $n_i$ exceptional divisors over $x_j^{(i)}$. 


Let \( h := f \circ g, F = g^*(F') \), and choose an effective \( h \)-exceptional integral divisor \( A \) on \( Z \) such that \( -A \) is \( h \)-ample. It is easy to see that 
\[
K_g = \sum_{i=1}^n \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} k E(i, x_j^{(i)}, k)
\]
and one checks 
\[
K_g \cdot E(i) = e_i \quad K_g \cdot E(i, x_j^{(i)}, k) = \begin{cases} 0 & k \neq n_i \\ 1 & k = n_i \end{cases}.
\]

It follows immediately that \( F + K_g \) is \( h \)-antinef. Choose \( \mu > 0 \) sufficiently small that 
\[
[(1 + \epsilon)(F + K_g + \mu A) - K_h] = [(1 + \epsilon)(F + K_g) - K_h].
\]

As \(-(F + K_g + \mu A)\) is \( h \)-ample, there exists \( N \gg 0 \) such that \( G := N(F + K_g + \mu A) \) is integral and \(-G\) is relatively globally generated. In other words, \( a := h_\ast \mathcal{O}_Z(-G) \) is an integrally closed ideal such that \( a \mathcal{O}_Z = \mathcal{O}_Z(-G) \). Set \( \lambda = \frac{1 + \epsilon}{N} \).

3.3. Conclusion of Proof. Here, we will show \( \mathcal{J}(X, a^h) = I = h_\ast \mathcal{O}_Z(-F) \). Since 
\[
\mathcal{J}(X, a^h) = h_\ast \mathcal{O}_Z([-K_h - \lambda G]) = h_\ast \mathcal{O}_Z([-\lambda G - K_h]),
\]
by Lemma 3.1 it suffices to show \( F' := [\lambda G - K_h] \sim F \). In particular, we have reduced to showing a purely numerical statement.

**Lemma 3.1.** We have \( F' \leq F \) and \( h_\ast F' = h_\ast F \). In addition, for \( i = 1, \ldots, u \) and \( j = 1, \ldots, e_i \), 
\[
\text{ord}_{E(i, x_j^{(i)}, n_i)}(F') = \text{ord}_{E(i, x_j^{(i)}, n_i)}(F) = \text{ord}_{E(i)}(F).
\]

**Proof.** Since \( F' = [\lambda G - K_h] \sim F \) is \( h \)-antinef \((-F\) is relatively globally generated), it suffices to show these statements with \([\lambda G - K_h]\) in place of \( F'\). By 3.2, we have 
\[
[\lambda G - K_h] = [(1 + \epsilon)(F + K_g) - K_h] = F + [\epsilon(F + K_g) - g^*K_f].
\]

Since \([\epsilon(f^{-1})f_*F] = 0\), it follows immediately that \( h_\ast [\lambda G - K_h] = h_\ast F \). For the remaining two statements, consider the coefficients of \( \epsilon(F + K_g) - g^*K_f \). Along \( E(i) \), we have \( e_i - b_i \), which is less than one by choice of \( \epsilon \). Along \( E(i, x_j^{(i)}, k) \), we have \( \epsilon(a_i + k) - b_i \). This expression is greatest when \( k = n_i \), where our choice of \( n_i \) guarantees 
\[
0 \leq \epsilon(a_i + n_i) - b_i < 1.
\]

It follows that \([\lambda G - K_h] \leq F\), with equality along \( E(i, x_j^{(i)}, n_i) \).

**Lemma 3.2.** For each \( i = 1, \ldots, u \), 
\[
(-F' \cdot E(i)) \tilde{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \tilde{E}(i, x_j^{(i)}, k) \geq (-F \cdot E(i)) \tilde{E}(i).
\]

\(^1\)Over \( \mathbb{C} \), as \( X \) is log terminal, it also has rational singularities and by Theorem 12.1 of [2], it follows that \(-F + K_g\) is already globally generated without the addition of \(-A\). However, the above approach seems more elementary, and avoids unnecessary reference to these nontrivial results.
Proof. If \( \text{ord}_{E(i)} F' = \text{ord}_{E(i)} F \), as \( F' \leq F \) we have \( F' \cdot E(i) \leq F \cdot E(i) \) and the conclusion follows as \( \hat{E}(i) \) and \( \hat{E}(i, x_j^{(i)}, k) \) are effective and \( F' \) is \( h \)-antinef. Otherwise, if \( \text{ord}_{E(i)} F' < \text{ord}_{E(i)} F = \text{ord}_{E(i, x_j^{(i)}, n_i)} F' \), then for each \( j = 1, \ldots, e_i \) we saw in Lemma 2.2 that
\[
\sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \hat{E}(i, x_j^{(i)}, k) \geq \hat{E}(i).
\]
Summing over all \( j \) gives the desired conclusion. \( \square \)

We now finish the proof by showing that \( F' \geq F \). Using the relative numerical decomposition (1) and the previous two Lemmas, we compute
\[
F' = h^* h_* F' + \sum_{i=1}^u (-F' \cdot E(i)) \hat{E}(i) + \sum_{i=1}^u \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \hat{E}(i, x_j^{(i)}, k)
\]
\[
= h^* (h_* F) + \sum_{i=1}^u \left( (-F' \cdot E(i)) \hat{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \hat{E}(i, x_j^{(i)}, k) \right)
\]
\[
\geq h^* (h_* F) + \sum_{i=1}^u (-F \cdot E(i)) \hat{E}(i) = F.
\]
This concludes the proof of Theorem 1.1.

References


Department of Mathematics, University of Michigan, 2704 East Hall, 525 East University Avenue, Ann Arbor, MI 48109-1109

E-mail address: kevtuck@umich.edu