ON THE BRAUER GROUP OF ENRIQUES SURFACES

Arnaud Beauville

Abstract. Let $S$ be a complex Enriques surface (quotient of a K3 surface $X$ by a fixed-point-free involution). The Brauer group $\text{Br}(S)$ has a unique nonzero element. We describe its pull-back in $\text{Br}(X)$, and show that the surfaces $S$ for which it is trivial form a countable union of hypersurfaces in the moduli space of Enriques surfaces.

1. Introduction

Let $S$ be a complex Enriques surface, and $\pi : X \to S$ its 2-to-1 cover by a K3 surface. Poincaré duality provides an isomorphism $H^3(S, \mathbb{Z}) \cong H_1(S, \mathbb{Z}) = \mathbb{Z}/2$, so that there is a unique nontrivial element $b_S$ in the Brauer group $\text{Br}(S)$. What is the pull-back of this element in $\text{Br}(X)$? Is it nonzero?\footnote{Received by the editors May 19, 2009.}

The answer to the first question is easy in terms of the canonical isomorphism $\text{Br}(X) \xrightarrow{\sim} \text{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ (see \S 2): $\pi^* b_S$ corresponds to the linear form $\tau \mapsto (\beta \cdot \pi^* \tau)$, where $\beta$ is any element of $H^2(S, \mathbb{Z}/2)$ which does not come from $H^2(S, \mathbb{Z})$. The second question turns out to be more subtle: the answer depends on the surface. We will characterize the surfaces $S$ for which $\pi^* b_S = 0$ (Corollary 5.7), and show that they form a countable union of hypersurfaces in the moduli space of Enriques surfaces (Corollary 6.5).

Part of our results hold over any algebraically closed field, and also in a more general set-up (see Proposition 4.1 below); for the last part, however, we need in a crucial way Horikawa’s description of the moduli space by transcendental methods.

2. The Brauer group of a surface

Let $S$ be a smooth projective variety over a field; we define the Brauer group $\text{Br}(S)$ as the étale cohomology group $H^2_{\text{ét}}(S, \mathbb{G}_m)$. For surfaces this definition coincides with that of Grothendieck [G] by [G], II, Cor. 2.2; this holds in fact in any dimension by a result of Gabber, which we will not need here (see [dJ]).

In this section we assume that $S$ is a complex surface; we recall the description of $\text{Br}(S)$ in that case – this is classical but not so easy to find in the literature. The Kummer exact sequence

$$0 \to \mathbb{Z}/n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

gives rise to an exact sequence

$$0 \to \text{Pic}(S) \otimes \mathbb{Z}/n \to H^2(S, \mathbb{Z}/n) \xrightarrow{\mu} \text{Br}(S)[n] \to 0$$

(2.a)

(we denote by $M[n]$ the kernel of the multiplication by $n$ in a $\mathbb{Z}$-module $M$).

\footnote{The question is mentioned in [H-S], where the authors construct an Enriques surface $S$ over $\mathbb{Q}$ for which $\pi^* b_S \neq 0$ (see Cor. 2.8).}
3. Algebraic topology of Enriques surfaces

3.1. Let S be an Enriques surface (over \( \mathbb{C} \)). We first recall some elementary facts on the topology of \( S \). A general reference is [BHPV], ch. VIII.

The torsion subgroup of \( H^2(S, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}/2 \); its nonzero element is the canonical class \( K_S \). Let \( k_S \) denote the image of \( K_S \) in \( H^2(S, \mathbb{Z}/2) \). The universal coefficient theorem together with Poincaré duality gives an exact sequence

\[
0 \to \mathbb{Z}/2 \to H^2(S, \mathbb{Z}/2) \to \text{Hom}(H^2(S, \mathbb{Z}), \mathbb{Z}/2) \to 0
\]  

(3.a)

where \( v_S \) is deduced from the cup-product.

3.2. The linear form \( \alpha \mapsto (k_S \cdot \alpha) \) on \( H^2(S, \mathbb{Z}/2) \) vanishes on the image of \( H^2(S, \mathbb{Z}) \), hence coincides with the map \( H^2(S, \mathbb{Z}/2) \to H^3(S, \mathbb{Z}) = \mathbb{Z}/2 \) from the exact sequence (2.b). Note that \( k_S \) is the second Stiefel-Whitney class \( w_2(S) \); in particular, we have \( (k_S \cdot \alpha) = \alpha^2 \) for all \( \alpha \in H^2(S, \mathbb{Z}/2) \) (Wu formula, see [M-S]).

3.3. The map \( c_1 : \text{Pic}(S) \to H^2(S, \mathbb{Z}) \) is an isomorphism, hence (2.e) provides an isomorphism \( \text{Br}(S) \cong \text{Tors} H^3(S, \mathbb{Z}) \cong \mathbb{Z}/2 \). We will denote by \( b_S \) the nonzero element of \( \text{Br}(S) \).

Let \( \pi : X \to S \) be the 2-to-1 cover of \( S \) by a K3 surface. The aim of this note is to study the pull-back \( \pi^* b_S \) in \( \text{Br}(X) \).

**Proposition 3.4.** The class \( \pi^* b_S \) is represented, through the isomorphism \( \text{Br}(X) \cong \text{Hom}(T_X, \mathbb{Q}/\mathbb{Z}) \), by the linear form \( \tau \mapsto (\beta \cdot \pi_* \bar{\tau}) \), where \( \tau \) is the image of \( \tau \) in \( H^2(X, \mathbb{Z}/2) \) and \( \beta \) any element of \( H^2(S, \mathbb{Z}/2) \) which does not come from \( H^2(S, \mathbb{Z}) \).
Proof. Let \( \beta \) be an element of \( H^2(S, \mathbb{Z}/2) \) which does not come from \( H^2(S, \mathbb{Z}) \), so that \( p(\beta) = b_S \) \((2.a)\). The pull-back \( \pi^* b_S \in \text{Br}(X) \) is represented by \( \pi^* \beta \in H^2(X, \mathbb{Z}/2) \cong H^2(X, \mathbb{Z}) \otimes \mathbb{Z}/2 \); its image in \( \text{Hom}(T_X, \mathbb{Z}/2) \) is the linear form \( \tau \mapsto (\pi^* \beta \cdot \tau) \). Since \( (\pi^* \beta \cdot \tau) = (\beta \cdot \pi_* \tau) \), the Proposition follows.

Part \((i)\) of the following Proposition shows that the class \( \pi^* \beta \in H^2(X, \mathbb{Z}/2) \) which appears above is nonzero. This does not say that \( \pi^* b_S \) is nonzero, as \( \pi^* \beta \) could come from a class in \( \text{Pic}(X) \) – see \S6.

**Proposition 3.5.**

\((i)\) The kernel of \( \pi^* : H^2(S, \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2) \) is \( \{0, k_S\} \).

\((ii)\) The Gysin map \( \pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \) is surjective.

**Proof.** To prove \((i)\) we use the Hochschild-Serre spectral sequence:

\[
E_2^{p,q} = H^p(\mathbb{Z}/2, H^q(X, \mathbb{Z}/2)) \Rightarrow H^{p+q}(S, \mathbb{Z}/2).
\]

We have \( E_2^{1,1} = 0 \), and \( E_2^{2,0} = E_2^{3,0} = H^2(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \). Thus the kernel of \( \pi^* : H^2(S, \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2) \) is isomorphic to \( \mathbb{Z}/2 \). Since it contains \( k_S \), it is equal to \( \{0, k_S\} \).

Let us prove \((ii)\). Because of the formula \( \pi_* \pi^* \alpha = 2\alpha \), the cokernel of \( \pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \) is a \( (\mathbb{Z}/2) \)-vector space; therefore it suffices to prove that the transpose map

\[
^t \pi_* : \text{Hom}(H^2(S, \mathbb{Z}), \mathbb{Z}/2) \to \text{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z}/2)
\]

is injective. This is implied by the commutative diagram

\[
\begin{array}{ccc}
H^2(S, \mathbb{Z}/2) & \xrightarrow{v_S} & \text{Hom}(H^2(S, \mathbb{Z}), \mathbb{Z}/2) \\
\downarrow^* & & \downarrow^t \pi_* \\
H^2(X, \mathbb{Z}/2) & \xrightarrow{v_X} & \text{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z}/2)
\end{array}
\]

plus the fact that \( \text{Ker} \pi^* = \text{Ker} v_S = \{0, k_S\} \) by \((i)\) and \((3.a)\). \( \square \)

4. Brauer groups and cyclic coverings

**Proposition 4.1.** Let \( \pi : X \to S \) be an étale, cyclic covering of smooth projective varieties over an algebraically closed field \( k \). Let \( \sigma \) be a generator of the Galois group \( G \) of \( \pi \), and let \( \text{Nm} : \text{Pic}(X) \to \text{Pic}(S) \) be the norm homomorphism. The kernel of \( \pi^* : \text{Br}(S) \to \text{Br}(X) \) is canonically isomorphic to \( \text{Ker} \text{Nm} / (1 - \sigma^*)(\text{Pic}(X)) \).

**Proof.** We consider the Hochschild-Serre spectral sequence

\[
E_2^{p,q} = H^p(G, H^q(X, \mathbb{G}_m)) \Rightarrow H^{p+q}(S, \mathbb{G}_m).
\]

Since \( E_2^{2,0} = H^2(G, k^*) = 0 \), the kernel of \( \pi^* : \text{Br}(S) \to \text{Br}(X) \) is identified with \( E_\infty^{1,1} = \text{Ker}(d_2 : E_2^{1,1} \to E_2^{3,0}) \). We have \( E_2^{3,0} = H^3(G, k^*) \); by periodicity of the cohomology of \( G \), this group is canonically isomorphic to \( H^1(G, k^*) = \text{Hom}(G, k^*) \), the character group of \( G \), which we denote by \( G \). So we view \( d_2 \) as a map from \( H^1(G, \text{Pic}(X)) \) to \( \hat{G} \).
Let $S$ be the endomorphism $L \mapsto \bigotimes_{g \in G} g^* L$ of $\text{Pic}(X)$; recall that $H^1(G, \text{Pic}(X))$ is isomorphic to $\text{Ker} S / \text{Im}(1 - \sigma^*)$. We have $\pi^* \text{Nm}(L) = S(L)$ for $L \in \text{Pic}(X)$, hence $\text{Nm}$ maps $\text{Ker} S$ into $\text{Ker} \pi^* \subset \text{Pic}(S)$. Now recall that $\text{Ker} \pi^*$ is canonically isomorphic to $\hat{G}$: to $\chi \in \hat{G}$ corresponds the subsheaf $L_\chi$ of $\pi_* O_X$ where $G$ acts through the character $\chi$. Since $\text{Nm} \circ (1 - \sigma^*) = 0$, the norm induces a homomorphism $H^1(G, \text{Pic}(X)) \to \text{Ker} \pi^* \cong \hat{G}$. The Proposition will follow from:

**Lemma 4.2.** The map $d_2 : H^1(G, \text{Pic}(X)) \to \hat{G}$ coincides with the homomorphism induced by the norm.

*Proof.* We apply the formalism of [S], Proposition 1.1, where a very close situation is considered. This Proposition, together with property (1) which follows it, tells us that $d_2$ is given by cup-product with the extension class in $\text{Ext}_G^2(\text{Pic}(X), k^*)$ of the exact sequence of $G$-modules

$$1 \to k^* \to R_X^* \to \text{Div}(X) \to \text{Pic}(X) \to 0,$$

where $R_X$ is the field of rational functions on $X$. This means that $d_2$ is the composition

$$H^1(G, \text{Pic}(X)) \xrightarrow{\partial} H^2(G, R_X^*/k^*) \xrightarrow{\partial'} H^3(G, k^*)$$

where $\partial$ and $\partial'$ are the coboundary maps associated to the short exact sequences

$$0 \to R_X^*/k^* \to \text{Div}(X) \to \text{Pic}(X) \to 0$$

and

$$0 \to k^* \to R_X^* \to R_X^*/k^* \to 0.$$

Let $\lambda \in H^1(G, \text{Pic}(X))$, represented by $L \in \text{Pic}(X)$ with $\bigoplus_{g \in G} g^* L \cong O_X$. Let $D \in \text{Div}(X)$ such that $L = O_X(D)$. Then $\sum_g g^* D$ is the divisor of a rational function $\psi \in R_X^*$, whose class in $R_X^*/k^*$ is well-defined. This class is invariant under $G$, and defines the element $\partial(\lambda) \in H^2(G, R_X^*/k^*)$. Since $\text{div} \psi$ is invariant under $G$, there exists a character $\chi \in \hat{G}$ such that $g^* \psi = \chi(g) \psi$ for each $g \in G$. Then $d_2^1(\lambda) = \chi$ viewed as an element of $H^3(G, k^*) = \hat{G}$.

It remains to prove that $O_S(\pi_* D) = L_\chi$. Since $\text{div} (\psi) = \pi^* \pi_* D$, multiplication by $\psi$ induces a global isomorphism $u : \pi^* O_S(\pi_* D) \xrightarrow{\sim} O_X$. Let $\varphi \in R_X$ be a generator of $O_X(D)$ on an open $G$-invariant subset $U$ of $X$. Then $\text{Nm}(\varphi)$ is a generator of $O_S(\pi_* D)$ on $\pi(U)$, and $\pi^* \text{Nm}(\varphi)$ is a generator of $\pi^* O_S(\pi_* D)$ on $U$; the function $h := \psi \pi^* \text{Nm}(\varphi)$ on $U$ satisfies $g^* h = \chi(g) h$ for all $g \in G$. This proves that the homomorphism $u^* : O_S(\pi_* D) \to \pi_* O_X$ deduced from $u$ maps $O_S(\pi_* D)$ onto the subsheaf $L_\chi$ of $\pi_* O_X$, hence our assertion. \hfill $\Box$

We will need a complement of the Proposition in the complex case:

**Corollary 4.3.** Assume $k = \mathbb{C}$, and $H^1(X, O_X) = H^2(S, O_S) = 0$. The following conditions are equivalent:

(i) The map $\pi^* : \text{Br}(S) \to \text{Br}(X)$ is not injective;

(ii) there exists $L \in \text{Pic}(X)$ whose class $\lambda = c_1(L)$ in $H^2(X, \mathbb{Z})$ satisfies $\pi_* \lambda = 0$ and $\lambda \notin (1 - \sigma^*)(H^2(X, \mathbb{Z}))$.

Observe that the hypotheses of the Corollary are satisfied when $S$ is a complex Enriques surface and $\pi : X \to S$ its universal cover.
Proof. By Proposition 4.1 (i) is equivalent to the existence of a line bundle \( L \) on \( X \) with \( \text{Nm}(L) = \mathcal{O}_S \) and \([L] \neq 0 \) in \( H^1(G, \text{Pic}(X)) \), while (ii) means that there exists such \( L \) with \([c_1(L)] \neq 0 \) in \( H^1(G, H^2(X, \mathbb{Z})) \). Therefore it suffices to prove that the map

\[
H^1(c_1) : H^1(G, \text{Pic}(X)) \to H^1(G, H^2(X, \mathbb{Z}))
\]

is injective.

Since \( H^1(X, \mathcal{O}_X) = 0 \) we have an exact sequence

\[
0 \to \text{Pic}(X) \overset{\cdot c_1}{\to} H^2(X, \mathbb{Z}) \to Q \to 0 \quad \text{with } Q \subset H^2(X, \mathcal{O}_X).
\]

Since \( H^2(S, \mathcal{O}_S) = 0 \), there is no nonzero invariant vector in \( H^2(X, \mathcal{O}_X) \), hence in \( Q \). Then the associated long exact sequence implies that \( H^1(c_1) \) is injective. \( \square \)

5. More algebraic topology

5.1. As in §3, we denote by \( S \) a complex Enriques surface, by \( \pi : X \to S \) its universal cover and by \( \sigma \) the corresponding involution of \( X \). We will need some more precise results on the topology of the surfaces \( X \) and \( S \). We refer again to [BHPV], ch. VIII.

Let \( E \) be the lattice \((-E_8) \oplus H\), where \( H \) is the rank 2 hyperbolic lattice. Let \( H^2(S, \mathbb{Z})_{\text{ff}} \) be the quotient of \( H^2(S, \mathbb{Z}) \) by its torsion subgroup \{0, \( K_S \)\}. We have isomorphisms

\[
H^2(S, \mathbb{Z})_{\text{ff}} \cong E \quad H^2(X, \mathbb{Z}) \cong E \oplus E \oplus H
\]

such that \( \pi^* : H^2(S, \mathbb{Z})_{\text{ff}} \to H^2(X, \mathbb{Z}) \) is identified with the diagonal embedding \( \delta : E \hookrightarrow E \oplus E \), and \( \sigma^* \) is identified with the involution

\[
\rho : (\alpha, \alpha', \beta) \mapsto (\alpha', \alpha, -\beta) \quad \text{of } E \oplus E \oplus H.
\]

5.2. We consider now the cohomology with values in \( \mathbb{Z}/2 \). For a lattice \( M \), we will write \( M_2 := M/2M \). The scalar product of \( M \) induces a product \( M_2 \otimes M_2 \to \mathbb{Z}/2 \); if moreover \( M \) is even, there is a natural quadratic form \( q : M_2 \to \mathbb{Z}/2 \) associated with that product, defined by \( q(m) = \frac{1}{2} \tilde{m}^2 \), where \( \tilde{m} \in M \) is any lift of \( m \in M_2 \). In particular, \( H_2 \) contains a unique element \( \varepsilon \) with \( q(\varepsilon) = 1 \): it is the class of \( e + f \) where \( (e, f) \) is a hyperbolic basis of \( H \).

Using the previous isomorphism we identify \( H^2(X, \mathbb{Z}/2) \) with \( E_2 \oplus E_2 \oplus H_2 \).

Proposition 5.3. The image of \( \pi^* : H^2(S, \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2) \) is \( \delta(E_2) \oplus (\mathbb{Z}/2)\varepsilon \).

Proof. This image is invariant under \( \sigma^* \), hence is contained in \( \delta(E_2) \oplus H_2 \); by Proposition 3.6 (i) it is 11-dimensional, hence a hyperplane in \( \delta(E_2) \oplus H_2 \), containing \( \delta(E_2) \) (which is spanned by the classes coming from \( H^2(S, \mathbb{Z}) \)). So \( \pi^*H^2(S, \mathbb{Z}/2) \) is spanned by \( \delta(E_2) \) and a nonzero element of \( H_2 \); it suffices to prove that this element is \( \varepsilon \). Since the elements of \( H^2(S, \mathbb{Z}/2) \) which do not come from \( H^2(S, \mathbb{Z}) \) have square 1 (3.2), this is a consequence of the following lemma. \( \square \)

Lemma 5.4. For every \( \alpha \in H^2(S, \mathbb{Z}/2) \), \( q(\pi^*\alpha) = \alpha^2 \).
Proof. This proof has been shown to me by J. Lannes. The key ingredient is the Pontryagin square, a cohomological operation
\[ \mathcal{P} : H^{2m}(M, \mathbb{Z}/2) \longrightarrow H^{4m}(M, \mathbb{Z}/4) \]
defined for any reasonable topological space \( M \) and satisfying a number of interesting properties (see [M-T], ch. 2, exerc. 1). We will state only those we need in the case of interest for us, namely \( m = 2 \) and \( M \) is a compact oriented 4-manifold. We identify \( H^4(M, \mathbb{Z}/4) \) with \( \mathbb{Z}/4 \); then \( \mathcal{P} : H^2(M, \mathbb{Z}) \to \mathbb{Z}/4 \) satisfies:

a) For \( \alpha \in H^2(M, \mathbb{Z}/2) \), the class of \( \mathcal{P}(\alpha) \) in \( \mathbb{Z}/2 \) is \( \alpha^2 \);

b) If \( \alpha \in H^2(M, \mathbb{Z}/2) \) comes from \( \tilde{\alpha} \in H^2(M, \mathbb{Z}) \), then \( \mathcal{P}(\alpha) = \tilde{\alpha}^2 \) (mod. 4). In particular, if \( M \) is a K3 surface, we have \( \mathcal{P}(\alpha) = 2q(\alpha) \) in \( \mathbb{Z}/4 \).

Coming back to our situation, let \( \alpha \in H^2(S, \mathbb{Z}/2) \). We have in \( \mathbb{Z}/4 \):

\[
\mathcal{P}(\pi^*\alpha) = 2\mathcal{P}(\alpha) \quad \text{by functoriality}
\]

\[
= 2\alpha^2 \quad \text{by a), and}
\]

\[
\mathcal{P}(\pi^*\alpha) = 2q(\pi^*\alpha) \quad \text{by b).}
\]

Comparing the two last lines gives the lemma. \( \square \)

**Corollary 5.5.** The kernel of \( \pi_* : H_2 \to \{0, k_S\} \) is \( \{0, \varepsilon\} \).

*Proof.* By Proposition 5.3 \( \varepsilon \) belongs to \( \text{Im} \pi^* \), hence \( \pi_*\varepsilon = 0 \). It remains to check that \( \pi_* \) is nonzero on \( H^1(\mathbb{Z}/2, H^2(X, \mathbb{Z})) \cong H_2 \). We know that there is an element \( \alpha \in H^2(X, \mathbb{Z}) \) with \( \pi_*\alpha = K_S \) (Prop. 3.6 (ii)); it belongs to \( \text{Ker}(1 + \pi^*) \), hence defines an element \( \bar{\alpha} \) of \( H^1(\mathbb{Z}/2, H^2(X, \mathbb{Z})) \) with \( \pi_*\bar{\alpha} \neq 0 \). \( \square \)

**Corollary 5.6.** Let \( \lambda \in H^2(X, \mathbb{Z}) \). The following conditions are equivalent:

(i) \( \pi_*\lambda = 0 \) and \( \lambda \notin (1 - \sigma^*)(H^2(X, \mathbb{Z})) \);

(ii) \( \sigma^*\lambda = -\lambda \) and \( \lambda^2 \equiv 2 \) (mod. 4).

*Proof.* Write \( \lambda = (\alpha, \alpha', \beta) \in E \oplus E \oplus H \); let \( \bar{\beta} \) be the class of \( \beta \) in \( H_2 \). Both conditions imply \( \sigma^*\lambda = -\lambda \), hence \( \alpha' = -\alpha \). Since \( (\alpha, -\alpha) = (1 - \sigma^*)(\alpha, 0) \) and \( 2\beta = (1 - \sigma^*)(\beta) \), the conditions of (i) are equivalent to \( \pi_*\bar{\beta} = 0 \) and \( \bar{\beta} \neq 0 \), that is, \( \beta = \varepsilon \) (Corollary 5.5). On the other hand we have \( \lambda^2 = 2\alpha^2 + \beta^2 \equiv 2q(\beta) \) (mod. 4), hence (ii) is also equivalent to \( \bar{\beta} = \varepsilon \). \( \square \)

This allows us to rephrase Corollary 4.3 in a simpler way:

**Corollary 5.7.** We have \( \pi^*b_S = 0 \) if and only if there exists a line bundle \( L \) on \( X \) with \( \sigma^*L = L^{-1} \) and \( c_1(L)^2 \equiv 2 \) (mod. 4). \( \square \)

**Remark.**—My original proof of (5.3-5) was less direct and less general, but still perhaps of some interest. The key point is to show that on \( H_2 q \) takes the value 1 exactly on the nonzero element of \( \text{Ker} \pi_* \), or equivalently that an element \( \alpha \in H_2 \) with \( \pi_*\alpha = k_S \) satisfies \( q(\alpha) = 0 \). Using deformation theory (see (6.1) below), one can assume that \( \alpha \) comes from a class in \( \text{Pic}(X) \). To conclude I applied the following lemma:

**Lemma 5.8.** Let \( L \) be a line bundle on \( X \) with \( \text{Nm}(L) = K_S \). Then \( c_1(L)^2 \) is divisible by 4.
6.1. We briefly recall the theory of the period map for Enriques surfaces, due to Horikawa (see [BHPV], ch. VIII, or [N]). We keep the notations of (5.1). We denote $\phi$ by $\sigma$, and we let $L$ be a primitive element of $L$.

Let $H^1(S, E) \otimes H^1(S, E) \to H^2(S, K_S) \cong \mathbb{C}$ which is skew-symmetric and non-degenerate. Thus $h^1(E)$ is even; since $h^0(E) = h^2(E)$ by Serre duality, $\chi(E)$ is even, and so is $\chi(L) = \chi(E)$. By Riemann-Roch this implies that $\frac{1}{2}c_1(L)^2$ is even. \hfill \Box

6. The vanishing of $\pi^*b_S$ on the moduli space

6.1. We briefly recall the theory of the period map for Enriques surfaces, due to Horikawa (see [BHPV], ch. VIII, or [N]). We keep the notations of (5.1). We denote by $L$ the lattice $E \oplus E \oplus H$, and by $L^-$ the $(-1)$-eigenspace of the involution $\rho : (\alpha, \alpha', \beta) \mapsto (\alpha', -\alpha, -\beta)$, that is, the submodule of elements $(\alpha, -\alpha, \beta)$.

A marking of the Enriques surface $S$ is an isometry $\varphi : H^2(X, \mathbb{Z}) \to L$ which conjugates $\sigma^*$ to $\rho$. The line $H^{2,0}$ is anti-invariant under $\sigma^*$, so its image by $\varphi_C : H^2(X, \mathbb{C}) \to L_C$ lies in $L^-$. The corresponding point $[\omega]$ of $\mathbb{P}(L^-_C)$ is the period $\varphi(S, \varphi)$. It belongs to the domain $\Omega \subset \mathbb{P}(L^-_C)$ defined by the equations

$$(\omega \cdot \omega) = 0 \quad (\omega \cdot \bar{\omega}) > 0 \quad (\omega \cdot \lambda) \neq 0 \quad \text{for all } \lambda \in L^- \text{ with } \lambda^2 = -2.$$ 

This is an analytic manifold, which is the moduli space for marked Enriques surfaces. To each class $\lambda \in L^-$ we associate the hypersurface $H_\lambda$ of $\Omega$ defined by $(\lambda \cdot \omega) = 0$.

Proposition 6.2. We have $\pi^*b_S = 0$ if and only if $\varphi(S, \varphi)$ belongs to one of the hypersurfaces $H_\lambda$ for some vector $\lambda \in L^-$ with $\lambda^2 \equiv 2 \pmod{4}$.

Proof. The period point $\varphi(S, \varphi)$ belongs to $H_\lambda$ if and only if $\lambda$ belongs to $c_1(\text{Pic}(X))$; by Corollary 5.7, this is equivalent to $\pi^*b_S = 0$. \hfill \Box

To get a complete picture we want to know which of the $H_\lambda$ are really needed:

Lemma 6.3. Let $\lambda$ be a primitive element of $L^-$. Then $H_\lambda$ is non-empty if and only if $\lambda^2 < -2$. If $\mu$ is another primitive element of $L^-$ with $H_\mu = H_\lambda \neq \emptyset$, then $\mu = \pm \lambda$.

Proof. Let $W$ be the subset of $L^-_C$ defined by the conditions $\omega^2 = 0$, $\omega \cdot \bar{\omega} > 0$. If we write $\omega = \alpha + i \beta$ with $\alpha, \beta \in L^-_R$, these conditions translate as $\alpha^2 = \beta^2 > 0$, $\alpha \cdot \beta = 0$. Thus $W \cap \lambda^\perp \neq \emptyset$ is equivalent to the existence of a positive 2-plane in $L^-_R$ orthogonal to $\lambda$. Since $L^-$ has signature $(2, 10)$, this is also equivalent to $\lambda^2 < 0$.

If $W \cap \lambda^\perp$ is non-empty, $\lambda^\perp$ is the only hyperplane containing it, and $C \lambda$ is the orthogonal of $\lambda^\perp$ in $L^-$. Then $\lambda$ and $-\lambda$ are the only primitive vectors of $L^-$ contained in $C \lambda$. In particular $\lambda$ is determined up to sign by $H_\lambda$, which proves (ii).

Let us prove (i). We have seen that $H_\lambda$ is empty for $\lambda^2 \geq 0$, and also for $\lambda^2 = -2$ by definition of $\Omega$. Assume $\lambda^2 < -2$ and $H_\lambda = \emptyset$; then $H_\lambda$ must be contained in one of the hyperplanes $H_\mu$ with $\mu^2 = -2$; by (ii) this implies $\lambda = \pm \mu$, a contradiction. \hfill \Box
Let \( \Gamma \) be the group of isometries of \( L^- \). The group \( \Gamma \) acts properly discontinuously on \( \Omega \), and the quotient \( \mathcal{M} = \Omega / \Gamma \) is a quasi-projective variety. The image in \( \mathcal{M} \) of the period \( \wp(S, \varphi) \) does not depend on the choice of \( \varphi \); let us denote it by \( \wp(S) \). The map \( S \mapsto \wp(S) \) induces a bijection between isomorphism classes of Enriques surfaces and \( \mathcal{M} \); the variety \( \mathcal{M} \) is a (coarse) moduli space for Enriques surfaces.

**Corollary 6.5.** The surfaces \( S \) for which \( \pi^* b_S = 0 \) form an infinite, countable union of (non-empty) hypersurfaces in the moduli space \( \mathcal{M} \).

**Proof.** Let \( \Lambda \) be the set of primitive elements \( \lambda \) in \( L^- \) with \( \lambda^2 < -2 \) and \( \lambda^2 \equiv 2 \, (\text{mod } 4) \). For \( \lambda \in \Lambda \), let \( H_\lambda \) be the image of \( H_\lambda \) in \( \mathcal{M} \); the argument of [BHPV], ch. VIII, Cor. 20.7 shows that \( H_\lambda \) is an algebraic hypersurface in \( \mathcal{M} \). By Proposition 6.2 and Lemma 6.3 the surfaces \( S \) with \( \pi^*(b_S) = 0 \) form the subset \( \bigcup_{\lambda \in \Lambda} H_\lambda \). By Lemma 6.3 (ii) we have \( H_\lambda = H_\mu \) if and only if \( \mu = \pm g\lambda \) for some element \( g \) of \( \Gamma \). This implies \( \lambda^2 = \mu^2 \); but \( \lambda^2 \) can be any number of the form \(-2k\) with \( k \) odd \( > 1 \) (take for instance \( \lambda = e - kf \), where \((e, f)\) is a hyperbolic basis of \( H \)), so there are infinitely many distinct hypersurfaces among the \( H_\lambda \). \( \square \)

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**References**


