SMASH-NILPOTENT CYCLES ON ABELIAN 3-FOLDS

BRUNO KAHN AND RONNIE SEBASTIAN

Abstract. We show that homologically trivial algebraic cycles on a 3-dimensional abelian variety are smash-nilpotent.

Introduction

Let $X$ be a smooth projective variety over a field $k$. An algebraic cycle $Z$ on $X$ (with rational coefficients) is smash-nilpotent if there exists $n > 0$ such that $Z^n$ is rationally equivalent to 0 on $X^n$. Smash-nilpotent cycles have the following properties:

1. The sum of two smash-nilpotent cycles is smash-nilpotent.
2. The subgroup of smash-nilpotent cycles forms an ideal under the intersection product as $(x \cdot y) \times (x \cdot y) \cdots \times (x \cdot y) = (x \times x \times \cdots \times x) \cdot (y \times y \times \cdots \times y)$.
3. On an abelian variety, the subgroup of smash-nilpotent cycles forms an ideal under the Pontryagin product as $(x \ast y) \times (x \ast y) \cdots \times (x \ast y) = (x \times x \times \cdots \times x) \ast (y \times y \times \cdots \times y)$ where $\ast$ denotes the Pontryagin product.

Voevodsky [11, Cor. 3.3] and Voisin [12, Lemma 2.3] proved that any cycle algebraically equivalent to 0 is smash-nilpotent. On the other hand, because of cohomology, any smash-nilpotent cycle is numerically equivalent to 0; Voevodsky conjectured that the converse is true [11, Conj. 4.2].

This conjecture is open in general. The main result of this note is:

Theorem 1. Let $A$ be an abelian variety of dimension $\leq 3$. Any homologically trivial cycle on $A$ is smash-nilpotent.

In characteristic 0 we can improve “homologically trivial” to “numerically trivial”, thanks to Lieberman’s theorem [7].

Nori’s results in [8] give an example of a group of smash-nilpotent cycles which is not finitely generated modulo algebraic equivalence. The proof of Theorem 1 actually gives the uniform bound 21 for the degree of smash-nilpotence on this group, see Remark 2. See Proposition 2 for partial results in dimension 4.

1. Beauville’s decomposition, motivically

For any smooth projective variety $X$ and any integer $n \geq 0$, we write as in [1] $CH^n_\mathbb{Q}(X) = CH^n(X) \otimes \mathbb{Q}$, where $CH^n(X)$ is the Chow group of cycles of codimension $n$ on $X$ modulo rational equivalence.

Let $A$ be an abelian variety of dimension $g$. For $m \in \mathbb{Z}$, we denote by $\langle m \rangle$ the endomorphism of multiplication by $m$ on $A$, viewed as an algebraic correspondence.
In [1], Beauville introduces an eigenspace decomposition of the rational Chow groups of $A$ for the actions of the operators $\langle m \rangle$, using the Fourier transform. Here is an equivalent definition: in the category of Chow motives with rational coefficients, the endomorphism $1_A \in \text{End}(h(A)) = CH^2_Q(A \times A)$ is given by the class of the diagonal $\Delta_A$. We have the canonical Chow-Künneth decomposition of Deninger-Murre

$$1_A = \sum_{i=0}^{2g} \pi_i$$

[4, Th. 3.1], where the $\pi_i$ are orthogonal idempotents and $\pi_i$ is characterised by $\pi_i\langle m \rangle^* = m^i\pi_i$ for any $m \in \mathbb{Z}$. This yields a canonical Chow-Künneth decomposition of the Chow motive $h(A)$ of $A$:

$$h(A) = \bigoplus_{i=0}^{2g} h^i(A), \quad h^i(A) = (A, \pi_i)$$

(see [10, Th. 5.2]). Then, under the isomorphism

$$CH^n_Q(A) = \text{Hom}(L^n, h(A))$$

(where $L$ is the Lefschetz motive) we have

$$CH^n(A)_{[\tau]} := \{ x \in CH^n_Q(A) | \langle m \rangle^* x = m^\tau x \ \forall m \in \mathbb{Z} \} = \text{Hom}(L^n, h^\tau(A)),$$

Remark 1. In Beauville’s notation, we have

$$CH^n(A)_{[\tau]} = CH^n_{2n-\tau}(A).$$

We shall use his notation in §3.

2. Skew cycles on abelian varieties

Let $\beta \in CH^n_Q(A)$. Assume that $\langle -1 \rangle^* \beta = -\beta$: we say that $\beta$ is skew. This implies that $\beta$ is homologically equivalent to 0.

For $g \leq 2$, the Griffiths group of $A$ is 0 and there is nothing to prove. For $g = 3$, the Griffiths group of $A$ is a quotient of $CH^2(A)[3]$ [1, Prop. 6]; thus Theorem 1 follows from the more general

\textbf{Proposition 1.} Any skew cycle on an abelian variety is smash-nilpotent.

This applies notably to the Ceresa cycle [3], for any genus.

\textbf{Proof.} We may assume $\beta$ homogeneous, say, $\beta \in CH^n_Q(A)$. View $\beta$ as a morphism $L^n \to h(A)$ in the category of Chow motives. Thus, for all $i$:

$$-\pi_i\beta = \pi_i(-1)^*\beta = (-1)^i\pi_i\beta$$

hence $\pi_i\beta = 0$ for $i$ even.

This shows that $\beta$ factors through a morphism

$$\tilde{\beta} : L^n \to h^{\text{odd}}(A)$$

with $h^{\text{odd}}(A) = \bigoplus_{i \text{ odd}} h^i(A)$.

But $L^n$ is evenly finite-dimensional and $h^{\text{odd}}(A)$ is oddly finite-dimensional in the sense of S.-I. Kimura. (Indeed, $S^{2g+1}(h^1(A)) = h^{2g+1}(A) = 0$ by [9, Theorem], and a direct summand of an odd tensor power of an oddly finite-dimensional motive is
oddly finite dimensional by [6, Prop. 5.10 p. 186].) Hence the conclusion follows from [6, prop. 6.1 p. 188]. □

Remark 2. Kimura’s proposition 6.1 shows in fact that all $z \in \text{Hom}(\mathbb{L}^n, h^{\text{odd}}(A))$ verify $z^\otimes N+1 = 0$ for a fixed $N$, namely, the sum of the odd Betti numbers of $A$. If $z \in \text{Hom}(\mathbb{L}^n, h^i(A))$ for some odd $i$, then we may take for $N$ the $i$-th Betti number of $A$. Thus, for $i = 3$ and if $A$ is a 3-fold, we find that all $z \in \text{Hom}(\mathbb{L}, h^3(A))$ verify $z^{\otimes 21} = 0$.

3. The 4-dimensional case

Proposition 2. If $g = 4$, homologically trivial cycles on $A$, except perhaps those which occur in parts $CH^p_0(A)$ or $CH^3_2(A)$ of the Beauville decomposition, are smash-nilpotent.

Proof. Let $A$ be an abelian variety and let $\hat{A}$ denote its dual abelian variety. We know, from [1], the following:

1. $CH^p_0(A) = 0$ for $p \in \{0, 1, g-2, g-1, g\}$ and $s < 0$. [1, Prop. 3a].
2. $CH^p_0(A)$ and $CH^3_2(A)$ consist of cycles algebraically equivalent to 0 for all values of $p$ and all values of $s > 0$. [1, Prop. 4].

For $g = 4$, using these results and Proposition 1 one can conclude smash nilpotence for homologically trivial cycles which are not in $CH^3_0(A)$ or $CH^3_2(A)$. Note that, with the notation of §3,

$$CH^3_0(A) = \text{Hom}(\mathbb{L}^3, h^4(A)), \quad CH^3_2(A) = \text{Hom}(\mathbb{L}^2, h^4(A)).$$

In the case of $CH^3_0(A)$, the problem is whether there are any homologically trivial cycles: in view of the above expression, this is conjecturally not the case, cf. [5, Prop. 5.8]. □

Remark 3. Some of these arguments also follow from a paper of Bloch and Srinivas [2].

Acknowledgements

The second author would like to thank Prof. J.P. Murre for his course on Motives at TIFR in January 2008, from where he learnt about the conjecture. Both authors would like to thank V. Srinivas for discussions leading to this result. The second author is being funded by CSIR.

References