TILTING INVARIANCE OF THE AUSLANDER-REITEN
CONJECTURE

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Abstract. Let $R$ be an artin algebra and $T$ be a tilting or cotilting $R$-module with $S = \text{End}_RT$. We show that $R$ satisfies the Auslander-Reiten conjecture if and only if so does $S$.

1. Introduction

Throughout this paper, we consider artin algebras and finitely generated left modules over them. Let $R$ be an artin algebra, we denote by $\text{mod}R$ the category of finitely generated left $R$-modules. For an $X \in \text{mod}R$, we denote by $\text{pd}_RX$ (resp., $\text{id}_RX$) the projective (resp., injective) dimension of $X$.

In studying the Nakayama conjecture, Auslander and Reiten [2] proposed the following conjecture, which is also listed as the 10th conjecture in the book [4].

**Auslander-Reiten Conjecture:** Let $R$ be an artin algebra and $X$ be an $R$-module. If $\text{Ext}_R^i(X, X \oplus R) = 0$ for all $i \geq 1$, then $X$ is projective.

Auslander and Reiten [2] proved the conjecture over artin algebras such that every module $M$ has an ultimately closed projective resolution, that is, there is some syzygy $N$ of $M$ such that all indecomposable direct summands of $N$ already appear in earlier syzygies. This includes algebras of finite representation type, algebras with radical square zero and all torsionless-finite algebras. Hoshino [9] proved if $R$ is a self-injective artinian local ring with radical cube zero, then $\text{Ext}_R^1(M, M) = 0$ implies that $M$ is free.

We note that the Auslander-Reiten Conjecture actually makes sense for any ring. In fact, there are already some results in the study of the Auslander-Reiten Conjecture for commutative algebras, see for instance [1, 5, 11, 12] etc.. Recently, Christensen and Holm proved that every left noetherian ring satisfy the Auslander-Reiten Conjecture if it satisfies the Auslander’s condition on vanishing of cohomology [6]. Such rings contain group algebras of finite groups and artin algebras such that every module $M$ has an ultimately closed projective resolution [6, 16].

The Auslander-Reiten Conjecture is related to the Finitistic Dimension Conjecture which reads as follows.

**Finitistic Dimension Conjecture:** Let $R$ be an artin algebra. Then $\text{fdim}R =$
Indeed, if the Finitistic Dimension Conjecture holds for all artin algebras then the Auslander-Reiten Conjecture holds for all artin algebras. However, for an artin algebra satisfying the Finitistic Dimension Conjecture, we don’t know if it also satisfies the Auslander-Reiten Conjecture. For instance, it is still a question if the Auslander-Reiten Conjecture holds for all self-injective artin algebras and in this case, it is just the Tachikawa conjecture [15]. However, there is a counterexample over QF-rings [14].

Our aim in this paper is to show that the Auslander-Reiten conjecture is in fact a tilting invariance. More precisely, we prove the following result.

**Main Theorem** Let $R$ be an artin algebra and $T \in \text{mod } R$ be a tilting module with $S = \text{End}_R T$. Then $R$ satisfies the Auslander-Reiten conjecture if and only if so does $S$.

Similar results for Finitistic Dimension Conjecture had been proved by Happel [7] using the techniques of derived categories. More recently, it was proved that the Finitistic Dimension Conjecture is stable under derived equivalences. It would be interesting to consider whether the Auslander-Reiten conjecture is also stable under derived equivalences.

2. Preliminaries

Let $R$ be an artin algebra and $M \in \text{mod } R$. We denote by $R^\circ$ the opposite algebra and an $R^\circ$-module $M$ means the right $R$-module $M_R$.

Let $C$ be a subcategory of $\text{mod } R$, we denote by $\hat{C}$ the category of all $R$-modules $M$ such that there is an exact sequence $0 \to M \to C_0 \to \cdots \to C_m \to 0$ for some integer $m$ with each $C_i \in C$. Let $M \in C$, we denote by $\text{codim}_C(M)$ the minimal non-negative integer $m$ such that there is an exact sequence $0 \to M \to C_0 \to \cdots \to C_m \to 0$ with each $C_i \in C$. We also denote by $(C)_n$ the category of all $M \in C$ with $\text{codim}_C(M) \leq n$.

Dually, the notion $\hat{C}$ denotes the category of all $R$-modules $M$ such that there is an exact sequence $0 \to C_m \to \cdots \to C_0 \to M \to 0$ for some integer $m$ with each $C_i \in \hat{C}$, and the notion $\text{dim}_C(M)$ denotes the minimal non-negative integer $m$ such that there is an exact sequence $0 \to C_m \to \cdots \to C_0 \to M \to 0$ with each $C_i \in C$. Similarly, the notion $(\hat{C})_n$ is the category of all $M \in \hat{C}$ with $\text{dim}_C(M) \leq n$.

Let $M \in \text{mod } R$. We denote by $\text{add}_R M$ the category of modules isomorphic to direct summands of finite direct sums of $M$. The notion $M^\perp$ denotes the category of all modules $N \in \text{mod } R$ such that $\text{Ext}_R^{i}(M, N) = 0$. Dually, the notion $^\perp M$ denotes the category of all modules $N$ such that $\text{Ext}_R^{i}(N, M) = 0$.

We denote by $D$ the usual duality functor between $\text{mod } R$ and $\text{mod } R^\circ$. For an $M \in \text{mod } R$ and a positive integer $t$, the notion $\Omega^t_R M$ denote the $t$-th syzygy of $M$.

We recall now some necessary basic tilting theory. The readers are suggested to refer to [3, 8, 13] for more details.

Let $R$ be an artin algebra and $n$ a non-negative integer. Recall that $T \in \text{mod } R$ is called a tilting module of projective dimension at most $n$ if it satisfies the following three conditions:

(T1) $\text{pd} T \leq n$, i.e., there is an exact sequence $0 \to R_n \to \cdots \to R_0 \to T \to 0$ with each $R_i$ projective.

(T2) $\text{Ext}^i_R(T, T) = 0$ for all $i > 0$, and
(T3) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some $n$, where each $T_i \in \text{add} T$.

The notion of tilting modules is left-right symmetric in the sense that if $R T$ is a tilting module of projective dimension at most $n$, then $T S$, where $S = \text{End}_R T$, is also a tilting module of projective dimension at most $n$.

Dually, $C \in \text{mod} R$ is a cotilting module of injective dimension at most $n$ if it satisfies

(C1) $\text{id} T \leq n$,
(C2) $\text{Ext}^i_R(T, T) = 0$ for all $i > 0$, and
(C3) there is an exact sequence $0 \to C_n \to \cdots \to C_0 \to D(R_R) \to 0$ with each $C_i \in \text{add} R C$.

Note that $T \in \text{mod} R$ is a tilting module of projective dimension at most $n$ if and only if $D(R T) \in \text{mod} R^o$ is a cotilting module of injective dimension at most $n$.

Lemma 2.1. Let $R$ be an artin algebra and $T \in \text{mod} R$ be a tilting module of projective dimension at most $n$. Then for any $M \in \text{mod} R$, there is an exact sequence $0 \to M \to U_M \to V_M \to 0$ with $U_M \in \text{add}_R T$ and $V_M \in (\text{add}_R T)_{n-1}$. In particular, $V_M$ has the projective dimension at most $n$.

Proof. The claimed exact sequence exists by for instance [3, Section 5]. □

The following is the well-known Brenner-Butler Theorem in the tilting theory, see for instance [8, 13].

Lemma 2.2. Let $R$ be an artin algebra and $T \in \text{mod} R$ be a tilting module with $S = \text{End}_R T$. Denote $C = D(T S)$. Then there is an equivalence between $T^⊥$ and $C^⊥$, given by the functor $\text{Hom}_R(T, -)$. Moreover, for any $U, W \in T^⊥$ and any $i \geq 0$, we have that $\text{Ext}^i_R(U, W) \simeq \text{Ext}^i_S(\text{Hom}_R(T, U), \text{Hom}_R(T, W))$ canonically.

3. The proof of Main Theorem

Throughout this section, we fix an artin algebra $R$ and a tilting $R$-module $T$ with $S = \text{End}_R T$. We set $n = \text{pd}_R T$.

To prove the Main Theorem, we need some lemmas.

Lemma 3.1. Assume that $M \in \text{mod} R$ satisfies that $M \in T^⊥$. Then $\Omega^n_R M \in T^⊥$.

Proof. Consider the exact sequence $0 \to \Omega^n M \to R_{n-1} \to \cdots \to R_0 \to M \to 0$ with each $R_i$ projective. By applying functors $\text{Hom}_R(-, R)$, $\text{Hom}_R(-, \Omega^n M)$ and $\text{Hom}_R(M, -)$ in turn, we obtain for all $i \geq 1$ that first

$$\text{Ext}^i_R(\Omega^n M, R) \simeq \text{Ext}^{i+n}_R(M, R) = 0,$$

secondly

$$\text{Ext}^i_R(\Omega^n M, \Omega^n M) \simeq \text{Ext}^{i+n}_R(M, \Omega^n M),$$

and lastly

$$\text{Ext}^{i+n}_R(M, \Omega^n M) \simeq \text{Ext}^i_R(M, M) = 0,$$

by the dimension shift and the assumption. Hence the conclusion follows. □

Lemma 3.2. Assume that $M \in T^⊥$. Then $\Omega^n_R M \in T^⊥$.
Consider the exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ in the definition of tilting modules. Applying the functor $\text{Hom}_R(M, -)$, we obtain that $\text{Ext}^i_R(M, R) \simeq \text{Ext}^i_R(M, T_i)$ for all $i \geq 1$ by the dimension shift, since $M \in \perp T$. It follows that $\text{Ext}^i_R(\Omega^n_R M, R) \simeq \text{Ext}^{i+n}_R(M, R) = 0$ for all $i \geq 1$, i.e., $\Omega^n_R M \in \perp R$. 

The following lemma is important for the proof.

**Lemma 3.3.** For any $M \in \text{mod} R$, there is an exact sequence

$$0 \to M \to V \to U \to 0$$

such that $U \in \text{add}_R T$ and $V$ satisfies an exact sequence

$$0 \to \Omega^n_R M \to T_{n-1} \to \cdots \to T_0 \to V \to 0$$

with each $T_i \in \text{add}_R T$.

**Proof.** Clearly we have the exact sequence $0 \to \Omega^n_R M \to R_{n-1} \to \cdots \to R_0 \to M \to 0$ with each $R_i$ projective. Since $R \in (\text{add}_R T)_n$, the conclusion then follows from [17, Lemma 2.3].

**Lemma 3.4.** Assume that $M \in \perp (T \oplus M) \cap T^\perp$. Then $\Omega^n_R M \in \perp \Omega^n_R M$.

**Proof.** We consider the two exact sequences in Lemma 3.3. Since $\text{pd}_R T \leq n$ and $\text{Ext}^i_R(T, T) = 0$ for all $i \geq 1$, we easily obtain that $V \in T^\perp$ from the exact sequence $(\ddagger)$. It follows that $U \in T^\perp$ from the exact sequence $(\ddagger)$, since $M \in T^\perp$ too. Hence we obtain that $U \in T^\perp \cap \text{add}_R T = \text{add}_R T$. It turns out that the sequence $(\ddagger)$ splits, and consequently $V \simeq M \oplus U \in \text{add}_R (T \oplus M)$. Since $M \in \perp (T \oplus M)$ by assumption, $M \in \perp V$ too. Hence, applying the functor $\text{Hom}_R(M, -)$ to the exact sequence $(\ddagger)$, we obtain that $\text{Ext}^{i+n}_R(M, \Omega^n_R M) \simeq \text{Ext}^i_R(M, V) = 0$ for all $i \geq 1$ by the dimension shift. It follows that $\text{Ext}^i_R(\Omega^n_R M, \Omega^n_R M) \simeq \text{Ext}^{i+n}_R(M, \Omega^n_R M) = 0$ for all $i \geq 1$, i.e., $\Omega^n_R M \in \perp \Omega^n_R M$.

**Lemma 3.5.** Assume that $M \in \perp (T \oplus M) \cap T^\perp$. If $R$ satisfies the Auslander-Reiten conjecture, then $M \in \text{add}_R T$.

**Proof.** Since $M \in \perp (T \oplus M) \cap T^\perp$, we obtain that $\Omega^n_R M \in \perp (\Omega^n_R M \oplus R)$ by Lemmas 3.2 and 3.4. If $R$ satisfies the Auslander-Reiten conjecture, then $\Omega^n_R M$ must be projective. It follows that $\text{pd}_R M < \infty$. Combining with the assumption $M \in T^\perp$, we have that $M \in \text{add}_R T$. Combining with the assumption $M \in \perp T$, we easily obtain that $M \in \text{add}_R T$.

We can now prove the one-part of the Main Theorem.

**Proposition 3.6.** If $R$ satisfies the Auslander-Reiten conjecture, then so does $S$.

**Proof.** Take any $N \in \text{mod} S$ such that $N \in \perp (N \oplus S)$. Then $\Omega^n_S N \in \perp (\Omega^n_S N \oplus S)$ by Lemma 3.1. Note that $\Omega^n_S N \in \mathbb{D}(T_S)$, so $\Omega^n_S N = \text{Hom}_R(T, M)$ for some $M \in T^\perp$ by the tilting equivalence in Lemma 2.2. Since $S = \text{Hom}_R(T, T)$ and $M \oplus T \in T^\perp$, we obtain that $\text{Ext}^i_R(M, M \oplus T) \simeq \text{Ext}^i_S(\text{Hom}_R(T, M), \text{Hom}_R(T, M) \oplus \text{Hom}_R(T, T)) \simeq \text{Ext}^i_S(\Omega^n_S N, \Omega^n_S N \oplus S) = 0$ for all $i \geq 1$ by assumption and Lemma 2.2 again. Hence we have that $M \in \perp (T \oplus M) \cap T^\perp$. Since $R$ satisfies the Auslander-Reiten conjecture, we obtain from Lemma 3.5 that $M \in \text{add}_R T$. It follows that $\Omega^n_S N(= \text{Hom}_R(T, M))$ is projective. Consequently, $\text{pd}_S N < \infty$. Since $N \in \perp S$ too, it is easy to see that $N$ is projective. It follows that $S$ satisfies the Auslander-Reiten conjecture.
Proof of the Main Theorem:

By the previous proposition, we need only to show that if $S$ satisfies the Auslander-Reiten conjecture, then so does $R$. To this end, let us take any $M \in \mod R$ such that $M \in M T$. Then we need to show that $M$ is projective.

By Lemma 2.1, there is exact sequence

$$0 \rightarrow M \rightarrow U_M \rightarrow V_M \rightarrow 0$$

with $U_M \in T^\perp$ and $V_M \in (\add R T)$, $N$. Note that for any $N \in \mod R$ with $\pd R N < \infty$, we have that $\Ext_R^i(M, N) = 0$ for all $i \geq 1$ since $M \in M T$. It follows that $M \in M T$ and $M \in M V_M$. Since clearly $V_M \in M T$ too, we obtain that $U_M \in M T$ from the sequence $(\sharp)$. By assumption, $\Ext_R^i(M, M) = 0$ for all $i \geq 1$. It follows that $\Ext_R^i(M, U_M) = 0$ for all $i \geq 1$ by applying the functor $\Hom_R(M, -)$ to the sequence $(\sharp)$. Note also that $\Ext_R^i(V_M, U_M) = 0$ for all $i \geq 1$ since $U_M \in M T$ and $V_M \in \add R T$, so applying the functor $\Hom_R(-, U_M)$ to the exact sequence $(\sharp)$, we further obtain that $\Ext_R^i(U_M, U_M) = 0$ for all $i \geq 1$. It amounts to that $U_M \in M (T \oplus U_M) \cap T^\perp$.

Denote $N = \Hom_R(T, U_M)$. Then by Lemma 2.2 and the above arguments, we obtain that $\Ext_S^i(N, N + S) = \Ext_S^i(\Hom_R(T, U_M), \Hom_R(T, U_M) \oplus \Hom_R(T, T)) \simeq \Ext_R^i(U_M, U_M \oplus T) = 0$ for all $i \geq 1$. Hence if $S$ satisfies the Auslander-Reiten conjecture, then $N$ is projective. Consequently $U_M \in \add R T$ by the tilting equivalence in Lemma 2.2. Thus from the exact sequence $(\sharp)$ we obtain that $M \in \add R T$. It follows that $\pd R M < \infty$, and hence $M$ is projective. Thus $R$ satisfies the Auslander-Reiten conjecture.

Corollary 3.7. Let $R$ be an artin algebra and $T \in \mod R$ with $S = \End R T$.

(1) If $T$ is a tilting module, then $T^o$ satisfies the Auslander-Reiten conjecture if and only if so does $S^o$.

(2) If $T$ is a cotilting module, then $R$ (resp., $R^o$) satisfies the Auslander-Reiten conjecture if and only if so does $S$ (resp., $S^o$).

Proof. (1) Note that $T$ is also a tilting $S^o$-module with $T^o \simeq \End S^o T$, so the conclusion follows from the Main Theorem.

(2) Denote $C = D(T_S)$. Then $C$ is a tilting $S$-module. Now the conclusion follows from the Main Theorem and the first part. \qed

Let $A, B$ be two artin algebras. We say that $A$ is tilting-cotilting equivalent to $B$, provided that there are some artin algebras $A_i$, $0 \leq i \leq n$, and some tilting or cotilting $A_i$-modules $T_i$ such that $A = A_0$, $B = A_n$, and $A_{i+1} \simeq \End A_i T_i$ for $0 \leq i \leq n - 1$. Clearly the tilting-cotilting equivalence is a kind of derived equivalences. Results in this paper show that if two artin algebras $A$ and $B$ are tilting-cotilting equivalent, then $A$ satisfies the Auslander-Reiten conjecture if and only if so does $B$.

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