STABILITY UNDER DEFORMATIONS OF EXTREMAL
ALMOST-KÄHLER METRICS IN DIMENSION 4.

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Abstract. Given a path of almost-Kähler metrics compatible with a fixed symplectic form on a compact 4-manifold such that at time zero the almost-Kähler metric is an extremal Kähler one, we prove, for a short time and under a certain hypothesis, the existence of a smooth family of extremal almost-Kähler metrics compatible with the same symplectic form, such that at each time the induced almost-complex structure is diffeomorphic to the one induced by the path.

1. Introduction

An almost-Kähler metric on a 2n-dimensional symplectic manifold \((M, \omega)\) is induced by an almost-complex structure \(J\) compatible with \(\omega\) in the sense that the tensor field \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\) is symmetric and positive definite and thus it defines a Riemannian metric on \(M\). The almost-Kähler metric is Kähler if the almost-complex structure \(J\) is integrable. Given an almost-Kähler metric, one can define a canonical hermitian connection \(\nabla\) (see e.g. [16, 24]). The hermitian scalar curvature \(s^\nabla\) is then obtained by taking a trace and contracting the curvature of \(\nabla\) with \(\omega\). In the Kähler case, the hermitian scalar curvature coincides with the Riemannian scalar curvature.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of the infinite dimensional group \(\text{Ham}(M, \omega)\) of hamiltonian symplectomorphisms on the space \(\text{AK}_\omega\) of \(\omega\)-compatible almost-Kähler metrics is hamiltonian with moment map \(\mu : \text{AK}_\omega \to (\text{Lie}(\text{Ham}(M, \omega)))^*\) given by \(\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}\). The critical points of the norm \(\int_M (s^\nabla)^2 \frac{\omega^n}{n!}\) are called extremal almost-Kähler metrics. It turns out that the symplectic gradient of \(s^\nabla\) of such metrics is a holomorphic vector field in the sense that its flow preserves the corresponding almost-complex structure. In particular, extremal Kähler metrics in the sense of Calabi [7] and almost-Kähler metrics with constant hermitian scalar curvature are extremal.

The GIT formal picture in [9] suggests the existence and the uniqueness of an extremal almost-Kähler metric, modulo the action of \(\text{Ham}(M, \omega)\), in each ‘stable complexified’ orbit of the action of \(\text{Ham}(M, \omega)\). However, in this formal infinite dimensional setting, a natural complexification of \(\text{Ham}(M, \omega)\) does not exist. When \(H^1(M, \mathbb{R}) = 0\), an identification of the ‘complexified’ orbit of a Kähler metric \((J, g) \in \text{AK}_\omega\) is given by considering all Kähler metrics \((J, \tilde{g})\) in the Kähler class \([\omega]\) and applying Moser’s Lemma [9]. In this setting, Fujiki–Schumacher [14] and LeBrun–Simanca [21] showed, in the absence of holomorphic vector fields, that the existence of

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an extremal Kähler metric is an open condition on the space of such orbits. Moreover, Apostolov–Calderbank–Gauduchon–Friedman [3] generalized this result by fixing a maximal torus $T$ in the reduced automorphism group of $(M, J)$ and considering $T$-invariant $\omega$-compatible Kähler metrics. In general, for an almost-Kähler metric, a description of these ‘complexified’ orbits is not available, see however [10] for the toric case. Nevertheless, the formal picture suggests that the existence of an extremal Kähler metric should persist for smooth almost-Kähler metrics close to an extremal one.

Thus motivated, we consider in this paper the 4-dimensional case where one can introduce a notion of almost-Kähler potential related to the one defined by Weinkove [27, 28]. In the spirit of [14, 21], we shall apply the Banach Implicit Function Theorem for the hermitian scalar curvature of $\omega$ in the spirit of [14, 21], we shall apply the Banach Implicit Function Theorem. Using a Kodaira–Spencer result [19, 20], one can resolve this problem if we suppose that the dimension of $g_t$-harmonic $J_t$-anti-invariant 2-forms, denoted by $h_{J_t}$ (see [12]), satisfies the condition $h_{J_t} = h_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ along the path $(J_t, g_t) \in AK^T_\omega$ in the space of $T$-invariant $\omega$-compatible almost-Kähler metrics. So, our main theorem claims the following

Theorem 1.1. Let $(M, \omega)$ be a 4-dimensional compact symplectic manifold and $T$ a maximal torus in $Ham(M, \omega)$. Let $(J_t, g_t)$ be any smooth family of almost-Kähler metrics in $AK^T_\omega$ such that $(J_0, g_0)$ is an extremal Kähler metric. Suppose that $h_{J_t} = h_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, there exists a smooth family $(\tilde{J}_t, \tilde{g}_t)$ of extremal almost-Kähler metrics in $AK^T_\omega$, defined for sufficiently small $t$, with $(\tilde{J}_0, \tilde{g}_0) = (J_0, g_0)$ and such that $\tilde{J}_t$ is equivariantly diffeomorphic to $J_t$.

Remark 1.2. (i) The condition that $h_{\tilde{J}_t} = h_{\tilde{J}_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ is satisfied in the following cases:

1. When $J_t$ are integrable almost-complex structures for each $t$. Then, $h_{J_t} = 2h^{2,0}(M, J_t) = b^+(M) - 1$ by a well-known result of Kodaira [5]. On the other hand, it is unknown whether or not, for an $\omega$-compatible non-integrable almost-complex $J$ on a compact 4-dimensional symplectic manifold $M$ with $b^+(M) \geq 3$, the equality $h_{J_t} = b^+(M) - 1$ is possible (see [12]).

2. When $b^+(M) = 1$, $h_{J_t} = 0$ for each $t$. This condition is satisfied when $(M, \omega)$ admits a non-trivial torus in $Ham(M, \omega)$ [17].

(ii) Theorem 1.1 holds under the weaker assumption that the torus $T \subset Ham(M, \omega)$ is maximal in $Ham(M, \omega) \cap Isom_0(M, g_0)$, where $Isom_0(M, g_0)$ denotes the connected component of the isometry group of the initial metric $g_0$. By a known result of Calabi [8], any extremal Kähler metric is invariant under a maximal connected compact subgroup of $Ham(M, \omega) \cap Aut(M, J_0)$, where $Aut(M, J_0)$ is the reduced automorphism group of $(M, J_0)$. Hence, Theorem 1.1 generalizes [14, 21] in the 4-dimensional case.

(iii) It was kindly pointed out to us by T. Drăghici that using a recent result of Donaldson and Remarks (i) and (ii) above, one can further extend Theorem 1.1 in the case when $b^+(M) = 1$ as follows: Let $(M, \omega_0, J_0, g_0)$ be a compact 4-dimensional extremal Kähler manifold with $b^+(M) = 1$ and $T$ be a maximal torus in $Ham(M, \omega) \cap$
Then, for any smooth family of $T$-invariant almost-complex structures $J(t)$ with $J(t) = J_0$, $J(t)$ is compatible with an extremal almost-Kähler metric $g_t$ for $t \in (-\epsilon, \epsilon)$. Indeed, as $J(t)$ are tamed by $\omega_t$ for $t \in (-\epsilon, \epsilon)$ and $b^+(M) = 1$, one can use the openness result of Donaldson [11, Proposition 1] (see also [12, Sec. 5]) to show that there exists a smooth family of $J(t)$-invariant symplectic forms $\omega_t$ with $[\omega_t] = [\omega_0]$. Averaging $\omega_t$ over the compact group $T$ and using the equivariant Moser Lemma, we obtain a family $J_t$ of $T$-invariant $\omega_0$-compatible almost-complex structures such that $J_t$ is $T$-equivariantly diffeomorphic to $J(t)$. We can then apply Theorem 1.1 to produce compatible extremal metrics.

Kim and Sung [18] showed that, in any dimension, if one starts with a Kähler metric of constant scalar curvature with no holomorphic vector fields, one can construct infinite dimensional families of almost-Kähler metrics of constant hermitian scalar curvature which coincide with the initial metric away from an open set. Similar existence result was presented in [22] when the initial Kähler metric is locally toric.

2. Preliminaries

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$. An almost-complex structure $J$ is compatible with $\omega$ if the tensor field $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$ defines a Riemannian metric on $M$; then, $(J, g)$ is called an ($\omega$-compatible) almost-Kähler metric on $(M, \omega)$. If, additionally, the almost-complex structure $J$ is integrable, then $(J, g)$ is a Kähler metric on $(M, \omega)$.

The almost-complex structure $J$ acts on the cotangent bundle $T^\ast(M)$ by $J_\alpha(X) = -\alpha(JX)$, where $\alpha$ is a 1-form and $X$ a vector field on $M$. Any section $\psi$ of the bundle $\otimes^2 T^\ast(M)$ admits an orthogonal splitting $\psi = \psi^{J,+} + \psi^{J,-}$, where $\psi^{J,+}$ is the $J$-invariant part and $\psi^{J,-}$ is the $J$-anti-invariant part, given by

$$\psi^{J,+}(\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) + \psi(J \cdot, J \cdot)) \quad \text{and} \quad \psi^{J,-}(\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) - \psi(J \cdot, J \cdot)).$$

In particular, the bundle of 2-forms decomposes under the action of $J$

$$\Lambda^2(M) = \mathbb{R} \cdot \omega \oplus \Lambda^2_0(J)(M) \oplus \Lambda^2(-J)(M),$$

where $\Lambda^2_0(J)(M)$ is the subbundle of the primitive $J$-invariant 2-forms (i.e. 2-forms pointwise orthogonal to $\omega$) and $\Lambda^2(-J)(M)$ is the subbundle of $J$-anti-invariant 2-forms.

Hence, the subbundle of primitive 2-forms $\Lambda^2_0(J)(M)$ admits the splitting

$$\Lambda^2_0(J)(M) = \Lambda^2_0(J,+)(M) \oplus \Lambda^2_0(J,-)(M).$$

For an $\omega$-compatible almost-Kähler metric $(J, g)$, the canonical hermitian connection on the complex tangent bundle $(T(M), J, g)$ is defined by

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J(D^g_X J) Y,$$

where $D^g$ is the Levi-Civita connection with respect to $g$ and $X, Y$ are vector fields on $M$. Denote by $R^\nabla$ the curvature of $\nabla$. Then, the hermitian Ricci form $\rho^\nabla$ is the trace of $R^\nabla_{X,Y}$ viewed as an anti-hermitian linear operator of $(T(M), J, g)$, i.e.,

$$\rho^\nabla(X, Y) = -\text{tr}(J \circ R^\nabla_{X,Y}).$$

Hence, the 2-form $\rho^\nabla$ is a closed (real) 2-form and it is a deRham representative of $2\pi c_1(T(M), J)$ in $H^2(M, \mathbb{R})$, where $c_1(T(M), J)$ is the first (real) Chern class. If
the almost-complex structure $J$ is compatible with a symplectic form $\tilde{\omega}$ such that 
\[\tilde{\omega}^n = e^F \omega^n\] for some smooth real-valued function $F$ on $M$, then [26, 27]

\begin{equation}
\tilde{\rho}^\nabla = -\frac{1}{2} d J dF + \rho^\nabla,
\end{equation}

where $\tilde{\rho}^\nabla$ is the hermitian Ricci form of the almost-Kähler metric $(J, \tilde{g})$ (here $\tilde{g}(\cdot, \cdot) = \tilde{\omega}(\cdot, J \cdot)$ is the induced Riemannian metric).

We define the hermitian scalar curvature $s^\nabla$ of an almost-Kähler metric $(J, g)$ as the trace of $\rho^\nabla$ with respect to $\omega$, i.e.

\begin{equation}
\begin{aligned}
s^\nabla \omega^n & = 2n \left( \rho^\nabla \wedge \omega^{n-1} \right).
\end{aligned}
\end{equation}

The (Riemannian) Hodge operator $*_g : \Lambda^p(M) \to \Lambda^{2n-p}(M)$ is defined to be the unique isomorphism such that $\psi_1 \wedge (*_g \psi_2) = g(\psi_1, \psi_2) \frac{\omega^n}{n!}$, for any $p$-forms $\psi_1, \psi_2$. Then, the codifferential $\delta^g$, defined as the formal adjoint of the exterior derivative $d$ with respect to $g$, is related to $d$ by the relation [6, 15]

\[\delta^g = -*_g d*_g.\]

It follows that

\begin{equation}
\begin{aligned}
d &= *_g \delta^g *_g.
\end{aligned}
\end{equation}

In dimension $2n = 4$, the bundle of 2-forms decomposes as

\[\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M),\]

where $\Lambda^\pm(M)$ correspond to the eigenvalue $(\pm 1)$ under the action of the Hodge operator $*_g$. This decomposition is related to the splitting (2.1) as follows

\begin{equation}
\begin{aligned}
\Lambda^+(M) &= \mathbb{R} \cdot \omega \oplus \Lambda^{J,-}(M) \text{ and } \Lambda^-(M) = \Lambda^{J,+}_0(M).
\end{aligned}
\end{equation}

3. Extremal almost-Kähler metrics

Let $(M, \omega)$ be a compact and connected symplectic manifold of dimension $2n$. Any $\omega$-compatible almost-complex structure is identified with the induced Riemannian metric.

Denote by $AK_\omega$ the Fréchet space of $\omega$-compatible almost-complex structures. The space $AK_\omega$ comes naturally equipped with a formal Kähler structure. Let $Ham(M, \omega)$ be the group of hamiltonian symplectomorphisms of $(M^{2n}, \omega)$. The Lie algebra of $Ham(M, \omega)$ is identified with the space of smooth functions on $M$ with zero mean value.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of $Ham(M, \omega)$ on $AK_\omega$ is hamiltonian with momentum given by the hermitian scalar curvature. More precisely, the moment map $\mu : AK_\omega \to (Lie(Ham(M, \omega)))^*$ is

\[\mu_J(f) = \int_M s^\nabla J f \frac{\omega^n}{n!},\]

where $s^\nabla$ is the hermitian scalar curvature of $(J, g)$ and $f$ is a smooth function with zero mean value viewed as an element of $Lie(Ham(M, \omega))$. The square-norm of the
hermitian scalar curvature defines a functional on $AK_{\omega}$
\begin{equation}
J \mapsto \int_M (s^\nabla)^2 \omega^n \overline{n!}.
\end{equation}

**Definition 3.1.** The critical points $(J, g)$ of the functional (3.1) are called *extremal almost-Kähler metrics*.

**Proposition 3.2.** An almost-Kähler metric $(J, g)$ is a critical point of (3.1) if and only if $\text{grad}_g s^\nabla$ is a Killing vector field with respect to $g$.

A proof of Proposition 3.2 is given in [4, 15, 22].

### 3.1. The extremal vector field.

We fix a maximal torus $T$ in $\text{Ham}(M, \omega)$ and denote by $t_{\omega}$ the finite dimensional space of real-valued smooth functions on $M$ which are hamiltonians with zero mean value of elements of $\mathfrak{t} = \text{Lie}(T)$. Denote by $\Pi_{\omega}^T$ the $L^2$-orthogonal projection of $T$-invariant smooth functions on $t_{\omega}$ with respect to the volume form $\frac{\omega^n}{n!}$. Let $AK_{\omega}^T$ be the space of $\omega$-compatible $T$-invariant almost-complex structures. Given any $J \in AK_{\omega}^T$, we define $z^T_{\omega} := \Pi_{\omega}^T s^\nabla$, where $s^\nabla$ is the hermitian scalar curvature of $(J, g)$. Then, we have the following (for more details see [3, 15, 22])

**Proposition 3.3.** The potential $z^T_{\omega}$ is independant of $(J, g)$. Furthermore, a $\omega$-compatible $T$-invariant almost-Kähler metric $(J, g)$ is extremal if and only if

$$s^\nabla = z^T_{\omega},$$

where $s^\nabla$ is the integral zero part of the hermitian scalar curvature $s^\nabla$ of $(J, g)$.

**Definition 3.4.** The vector field $Z^T_{\omega} := \text{grad}_\omega z^T_{\omega}$ is called the *extremal vector field* relative to $T$.

**Proposition 3.5.** The vector field $Z^T_{\omega}$ is invariant under $T$-invariant isotopy of $\omega$.

**Remark 3.6.** The assumption that $T \subset \text{Ham}(M, \omega)$ is a maximal torus is used only in the second part of Proposition 3.3. Indeed, the arguments in [22] show that $z^T_{\omega} = \Pi_{\omega}^T s^\nabla$ is independent of $(J, g)$ for any torus $T \subset \text{Ham}(M, \omega)$ and Proposition 3.5 still holds true for the corresponding vector field $Z^T_{\omega} = \text{grad}_\omega z^T_{\omega}$.

### 4. Almost-Kähler potentials in dimension 4

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n = 4$ and $(J, g)$ a $\omega$-compatible almost-Kähler metric. In order to define the almost-Kähler potentials, we consider the following second order linear differential operator [23] on the smooth sections $\Omega^{J,-}(M)$ of the bundle of $J$-anti-invariant 2-forms.

$$P : \Omega^{J,-}(M) \xrightarrow{\psi} \Omega^{J,-}(M) \xrightarrow{(d\delta^g \psi)^{J,-}},$$

where $\delta^g$ is the codifferential with respect to the metric $g$.

**Lemma 4.1.** $P$ is a self-adjoint strongly elliptic linear operator with kernel the $g$-harmonic $J$-anti-invariant 2-forms.
Proof. The principal symbol of $P$ is given by the linear map 
\[ \sigma(P)_\xi(\psi) = -\frac{1}{2}|\xi|^2\psi, \quad \forall \xi \in T^*_x(M), \psi \in \Omega^{-1}_{\infty}(M). \]
So, $P$ is a self-adjoint elliptic linear operator with respect to the global inner product $\langle \cdot, \cdot \rangle = \int_M g(\cdot, \cdot) \frac{\omega^n}{n!}$. Now, let $\psi \in \Omega^1(M)$ and suppose that $P(\psi) = 0$. Then, $0 = \langle (d\delta g)^1, \psi \rangle = \langle \delta^g, \psi \rangle$ which means that $\delta^g = 0$. It follows from (2.5) and since $\psi$ is $J$-anti-invariant that $*_{\omega}\psi = \psi$. Using the relation (2.4), we obtain $d\psi = *_{\omega}\delta^g *_{\omega}\psi = *_{\omega}\delta^g \psi = 0$. Hence, $d\psi = \delta^g \psi = 0$ and thus $\psi$ is a $g$-harmonic $J$-anti-invariant 2-form.

\[ \square \]

**Corollary 4.2.** For $f \in C_c^\infty(M, \mathbb{R})$, there exist a unique $\psi_f \in \Omega^1(M)$ orthogonal to the kernel of $P$ such that $(d\delta^g \psi_f)_J = (dJdf)^1$.

**Proof.** For a smooth real-valued function $f \in C_c^\infty(M, \mathbb{R})$ and any $\alpha$ in the kernel of $P$, we have $\langle (dJdf)^1, \alpha \rangle = \langle Jdf, \alpha \rangle = (dJdf, \delta^g \alpha) = 0$. By a standard result of elliptic theory [6, 29] and since $P$ is self-adjoint, there exist a smooth section $\psi_f \in \Omega^1(M)$ such that $P(\psi_f) = (dJdf)^1$. Moreover, $\psi_f$ is unique if one requires $\psi_f$ be orthogonal to the kernel of $P$.

\[ \square \]

From Corollary 4.2, it follows that, for $f \in C_c^\infty(M, \mathbb{R})$, the symplectic form $\omega_f = \omega + d(Jdf - \delta^g \psi_f)$ is a $J$-invariant closed 2-form. Then, the function $f$ is called an **almost-Kähler potential** if the induced symmetric tensor $g_f(\cdot, \cdot) := \omega_f(\cdot, J \cdot)$ is a Riemannian metric. This notion of almost-Kähler potential is closely related but different (in general) from the one defined by Weinkove in [28]. More precisely, if the almost-complex structure $J$ is compatible with a symplectic form $\omega$ which is cohomologous to $\omega$ i.e. $\tilde{\omega} - \omega = d\alpha$ (for some 1-form $\alpha$), then the almost-Kähler potential defined by Weinkove is given by the function $f$ which is uniquely determined (up to the addition of constant) by the Hodge decomposition of $\alpha$ with respect to the (self-adjoint elliptic) twisted Laplace operator $\tilde{\Delta}^c = J\Delta^g J^{-1}$, where $\Delta^g$ is the (Riemannian) Laplace operator with respect to the induced metric $\tilde{g}(\cdot, \cdot) = \tilde{\omega}(\cdot, J \cdot)$. In other words, we have the decomposition $\alpha = \alpha_H + \Delta^c \mathcal{G} \alpha$, where $\mathcal{G}$ is the Green operator associated to $\tilde{\Delta}^c$ and $\alpha_H$ is the harmonic part of $\alpha$ with respect to $\Delta^c$.

Thus, $f = -\delta^g J \tilde{g} \alpha$, where $\delta^g$ is the codifferential with respect to the metric $\tilde{g}$.

Note that $(dJdf)^{-1} = D^g_{(df)\tilde{g}^t} \omega$ (see e.g. [15]), where $\tilde{g}^t$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by $g^{-1}$. Hence, in the Kähler case, $(dJdf)^{-1} = 0$ which implies that $\psi_f = 0$ and thus this almost-Kähler potential coincides with the usual Kähler one.

### 5. Main Theorem

Let $(M, \omega)$ be a compact and connected symplectic manifold of dimension $2n = 4$ and $J_t \in AK_\omega$ be a smooth path of $\omega$-compatible almost-complex structures. We define the following family of differential operators associated to $J_t$:


g_t : \Omega^2_\omega(M) \rightarrow \Omega^2_\omega(M), \quad g_t(\Delta^g \psi, \omega),

where $\Omega^2_\omega(M)$ is the space of smooth sections of the bundle $\Lambda^2(M)$ of primitive 2-forms (pointwise orthogonal to $\omega$) and $\Delta^g$ is the (Riemannian) Laplacian with respect to the metric $g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$ (here we use the convention $g_t(\omega, \omega) = 2$).
One can easily check that $P_t$ preserves the decomposition

$$\Omega^2_0(M) = \Omega^J_0(M) \oplus \Omega^{\bar J}(M).$$

Furthermore,

$$P_t|_{\Omega^J_0(M)}(\psi) = (d\delta^g \omega)^J_t \psi \quad \text{and} \quad P_t|_{\Omega^{\bar J}(M)}(\psi) = \frac{1}{2} \Delta^g \psi .$$

It follows that the kernel of $P_t$ consists of primitive harmonic 2-forms which splits as anti-selfdual and $J_t$-anti-invariant ones so we have

$$\dim \ker(P_t) = b^-(M) + h^-_{J_t},$$

where $h^-_{J_t}$ is introduced by Drăghici–Li–Zhang in [12].

Moreover, $P_t - \frac{1}{2} \Delta^g$ is a linear differential operator of order 1. Indeed, a direct computation shows that

$$\left( P_t - \frac{1}{2} \Delta^g \right) (\psi) = \frac{1}{2} \left[ \frac{1}{2} \delta^g (D^g \omega(\psi)) - \frac{1}{2} g_t(D^g \psi, D^g \omega) \right] + \frac{s_{g_t}}{6} g_t(\omega, \psi) - W^g(\omega, \psi),$$

where $W^g$ stands for the Weyl tensor (see e.g. [6]), $D^g$ (resp. $\delta^g$) for the Levi-Civita connection (resp. the codifferential) with respect to the metric $g_t$ and $s_{g_t}$ for the Riemannian scalar curvature defined as the trace of the (Riemannian) tensor.

The operator $P_t$ is a self-adjoint strongly elliptic linear operator of order 2. We obtain then a family of Green operators $G_t$ associated to $P_t$. If $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$, then $G_t$ is $C^\infty$ differentiable in $t \in (-\epsilon, \epsilon)$ [19, 20], meaning that $G_t(\psi_t)$ is a smooth family of sections of $\Lambda^2_0(M)$ for any smooth sections $\psi_t$.

To show Theorem 1.1, we consider the extension of $G_t$ to the Sobolev spaces $W^{k,p}(M, \Lambda^2_0(M))$ involving derivatives up to $k$.

**Lemma 5.1.** Let $G_t : \Omega^2_0(M) \to \Omega^2_0(M)$ the family of the above Green operators associated to $P_t$ and suppose that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, the extension of $G_t$ to Sobolev spaces, still denoted by $G_t$, defines a $C^1$ map $G : (\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2_0(M)) \to W^{k+2,p}(M, \Lambda^2_0(M))$.

**Proof.** Denote by $\Pi_0$ the $L^2$-orthogonal projection to the kernel of $P_0$ with respect to $\langle \cdot, \cdot \rangle_{L^2_0} = \int_M g_0(\cdot, \cdot) \omega_0^2$. We claim that $G_0 \circ \Pi_0$ and $\Pi_0 \circ G : (\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2_0(M)) \to W^{k+2,p}(M, \Lambda^2_0(M))$ are $C^1$ maps. Indeed, let $\{\psi_0^i\}$ be an orthonormal basis of the kernel of $P_0$ with respect to $\langle \cdot, \cdot \rangle_{L^2_0}$. Note that $\psi_0^i$ are smooth since $P_0$ is elliptic.
Then, we have
\[
(G_t \circ \Pi_0)(\psi) = \sum_i \langle \psi, \psi_0^i \rangle_{L^2_{\Pi_0}} G_t(\psi_0^i),
\]
\[
(\Pi_0 \circ G_t)(\psi) = \sum_i \langle G_t(\psi), (\psi_0^i)^{J_{0^+}} + (\psi_0^i)^{J_{0^-}} \rangle_{L^2_{\Pi_0}} \psi_0^i
\]
\[
= \sum_i \left( \int_M -G_t(\psi) \wedge (\psi_0^i)^{J_{0^+}} + G_t(\psi) \wedge (\psi_0^i)^{J_{0^-}} \right) \psi_0^i
\]
\[
= \sum_i \left( \int_M -G_t(\psi) \wedge ((\psi_0^i)^{J_{0^+}})^{J_{1^+}} - G_t(\psi) \wedge ((\psi_0^i)^{J_{0^-}})^{J_{1^-}} \right) \psi_0^i
+ \ G_t(\psi) \wedge ((\psi_0^i)^{J_{0^-}})^{J_{1^+}} + G_t(\psi) \wedge ((\psi_0^i)^{J_{0^+}})^{J_{1^-}} \psi_0^i
\]
\[
= \sum_i \left[ \langle \psi, G_t ((\psi_0^i)^{J_{0^+}})^{J_{1^+}} \rangle_{L^2_{\Pi_1}} - \langle \psi, G_t ((\psi_0^i)^{J_{0^-}})^{J_{1^-}} \rangle_{L^2_{\Pi_1}} \right] \psi_i^0
- \langle \psi, G_t ((\psi_0^i)^{J_{0^-}})^{J_{1^+}} \rangle_{L^2_{\Pi_1}} + \langle \psi, G_t ((\psi_0^i)^{J_{0^+}})^{J_{1^-}} \rangle_{L^2_{\Pi_1}} \psi_i^0
\]

(in the latter equality, we used the fact that \(G_t\) is self-adjoint with respect to \(L^2_{\Pi_1}\)). The claim follows from the result of Kodaira–Spencer [19, 20].

Denote by \(W^{k,p}(M, \Lambda^2_0(M))^\perp\) the space of 2-forms in \(W^{k,p}(M, \Lambda^2_0(M))\) which are orthogonal to the kernel of \(P_0\) with respect to \(L^2_{\Pi_0}\) and consider the map

\[
\Phi: (-\epsilon, \epsilon) \times W^{k+2,p}(M, \Lambda^2_0(M))^\perp \longrightarrow (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2_0(M))^\perp
\]
\[
(t, \psi) \mapsto (t, (Id - \Pi_0)P_t(\psi)),
\]

Clearly, the map \(\Phi\) is of class \(C^1\) and its differential at \((0, \psi)\) is an isomorphism so by the inverse function theorem for Banach spaces there exist a neighborhood \(V\) of \((0, \psi)\) such that \(\Phi|_V\) admits an inverse of class \(C^1\). By the The Kodaira–Spencer result [19, 20], the map \(\Pi: (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2_0(M)) \rightarrow W^{k,p}(M, \Lambda^2_0(M))\) is \(C^1\) and thus the map \(P_t(Id - \Pi_0)G_t(Id - \Pi_0) = (Id - \Pi_t)(Id - \Pi_0) - P_t(\Pi_0G_t)(Id - \Pi_0)\) is clearly \(C^1\) since it is a composition of such operators. Then, the map

\[
\Phi|_V^{-1} (t, (Id - \Pi_0)P_t(Id - \Pi_0)G_t(Id - \Pi_0)) = (t, (Id - \Pi_0)G_t(Id - \Pi_0))
\]
\[
= (t, G_t - \Pi_0G_t - G_t\Pi_0 + \Pi_0G_t\Pi_0)
\]
is \(C^1\) and hence \(G_t\) is \(C^1\).

\(\square\)

**Proof of Theorem 1.1** Let \((M, \omega)\) be a 4-dimensional compact and connected symplectic manifold and \(T\) a maximal torus in \(Ham(M, \omega)\). Let \((J_t, g_t)\) a smooth family of \(\omega\)-compatible almost-Kähler metrics in \(AK^T_\omega\) such that \((J_0, g_0)\) is an extremal Kähler metric.

Following [21], we consider the almost-Kähler deformations

\[
\omega_{t,f} = \omega + d(J_t df - \delta^g_t \psi_f^t),
\]

where \(f\) belongs to the Fréchet space \(\tilde{C}^\infty_T(M, \mathbb{R})\) of \(T\)-invariant smooth functions (with zero integral), which are \(L^2\)-orthogonal, with respect to \(\frac{\omega}{2}\), to \(\mathfrak{t}_\omega\) and where the 2-form \(\psi_f^t\) is given by Corollary 4.2.
Let $\mathcal{U}$ be an open set in $\mathbb{R} \times \bar{C}_T^\infty(M, \mathbb{R})$ containing $(0, 0)$ such that the symmetric tensor $g_{t,f}(\cdot, \cdot) := \omega_{t,f}(\cdot, J_{t,f})$ is a Riemannian metric.

By possibly replacing $\mathcal{U}$ with a smaller open set, we may assume as in [21] that the kernel of the operator $(Id - \Pi_T^f) \circ (Id - \Pi_{\omega_{t,f}}^f)$ is equal to the kernel of $(Id - \Pi_{\omega_{t,f}}^f)$. Indeed, let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{t} = \text{Lie}(T)$. Then, the corresponding Hamiltonians with zero mean value $\{\xi_{\omega_{t,f}}^1, \cdots, \xi_{\omega_{t,f}}^n\}$ resp. $\{\xi_{\omega_{t,f}}^1, \cdots, \xi_{\omega_{t,f}}^n\}$, with respect to $\omega$ resp. $\omega_{t,f}$, form a basis of $\mathfrak{t}_\omega$ resp. $\mathfrak{t}_{\omega_{t,f}}$. Let $\{\xi_{\omega_{t,f}}^1, \cdots, \xi_{\omega_{t,f}}^n\}$ resp. $\{\xi_{\omega_{t,f}}^1, \cdots, \xi_{\omega_{t,f}}^n\}$, the corresponding orthonormal basis obtained by the Gram–Schmidt procedure. Since 

$$\det \left[ \begin{bmatrix} \xi_{\omega_{t,f}}^1, \xi_{\omega_{t,f}}^1 \\ \xi_{\omega_{t,f}}^2, \xi_{\omega_{t,f}}^2 \\ \vdots \\ \xi_{\omega_{t,f}}^n, \xi_{\omega_{t,f}}^n \end{bmatrix} \right] \neq 0$$

on an eventually smaller open set than $\mathcal{U}$, then we may suppose that 

$$\det \left[ \begin{bmatrix} \xi_{\omega_{t,f}}^1, \xi_{\omega_{t,f}}^1 \\ \xi_{\omega_{t,f}}^2, \xi_{\omega_{t,f}}^2 \\ \vdots \\ \xi_{\omega_{t,f}}^n, \xi_{\omega_{t,f}}^n \end{bmatrix} \right] \neq 0$$

implies that $v \equiv 0$ and then $ker \left( (Id - \Pi_T^f) \circ (Id - \Pi_{\omega_{t,f}}^f) \right) = ker(Id - \Pi_{\omega_{t,f}}^f).

We then consider the map:

$$\Psi : \mathcal{U} \rightarrow \mathbb{R} \times \bar{C}_T^\infty(M, \mathbb{R})$$

$$\begin{array}{c}
(t, f) \mapsto (t, (Id - \Pi_T^f) \circ (Id - \Pi_{\omega_{t,f}}^f)(s^{\nabla_{t,f}}))
\end{array}$$

where $s^{\nabla_{t,f}}$ is the zero integral part of the hermitian scalar curvature $s^{\nabla_{t,f}}$ of $(J_t, g_{t,f})$.

It follows from Proposition 3.3 that $\Psi(t, f) = (t, 0)$ if and only if $(J_t, g_{t,f})$ is an extremal almost-Kähler metric. In particular, $\Psi(0, 0) = (0, 0)$.

Let $\alpha_{t,f} = J_tdf - \delta^t \psi^f = J_tdf - \delta^t \mathcal{G}_t \left( (dJ_t df)^{\delta t} \right) = J_tdf - \delta^t \mathcal{G}_t (D_{\delta^t} df, \omega)$, where $\mathcal{G}_t$ is the Green operator associated to the elliptic operator $P_t : \Omega^{j_t}(M) \rightarrow \Omega^{j_t}(M)$. In order to extend the map $\Psi$ to Sobolev spaces, we give an explicit expression of $(Id - \Pi_{\omega_{t,f}}^f)(s^{\nabla_{t,f}})$. A direct computation using (2.2) shows that

$$(5.1) s^{\nabla_{t,f}} = \Delta^t \mathcal{G}_t f_{t,f} + g_{t,f}(\rho^{\nabla_{t,f}}, \omega_{t,f})$$

where $F_{t,f} = \log \left( \frac{1}{2} \left( (1 + g_{t,f}(\delta^t df, df))^2 + 1 - g_{t,f}(\delta^t df, df) \right) \right)$ satisfying the relation $\omega_{t,f}^2 = e^{F_{t,f}} \omega^2$. Then

$$(5.2) (Id - \Pi_{\omega_{t,f}}^f)(s^{\nabla_{t,f}}) = \Delta^t \mathcal{G}_t F_{t,f} + g_{t,f}(\rho^{\nabla_{t,f}}, \omega_{t,f}) - \sum_j \left( s^{\nabla_{t,f}}, \xi_{\omega_{t,f}}^j \right) \xi_{\omega_{t,f}}^j.$$
Clearly \( \Psi^{(p,k)} \) is a \( C^1 \) map (in a small enough open around \((0,0)\)). Indeed, it is obtained by a composition of \( C^1 \) maps by Lemma 5.1 and (5.2).

As in [21] and using Proposition 3.3, the differential of \( \Psi^{(p,k)} \) at \((0,0)\) is given by
\[
\left( T_{(0,0)} \Psi^{(p,k)} \right) (t, f) = (t, t \delta^{\rho_0} \delta^{\rho_0} h - 2 \delta^{\rho_0} \delta^{\rho_0} (D^{\rho_0} df)^J)_{0-},
\]
where \( h = \frac{d}{dt} |_{t=0} g_t \).

The operator \( L := \frac{\partial \Psi}{\partial t} |_{(0,0)} \) given by \( L(f) = -2 \delta^{\rho_0} \delta^{\rho_0} (D^{\rho_0} df)^J_{0-} \) is called the Lichnerowicz operator. It is a 4-th order self-adjoint \( T \)-invariant elliptic linear operator leaving invariant \((t_\omega)^+\) since \( L(f) = 0 \) for any \( f \in t_\omega \). By a known result of the elliptic theory [6, 29], we obtain the \( L^2 \)-orthogonal splitting \( C_\infty^\infty (M, \mathbb{R}) = \operatorname{ker}(L) \oplus \operatorname{Im}(L) \).

Following [11, 28], the linearisation of the equation (5.4) \( \tilde{\Psi}^{(p,k)} (\Psi^{(p,k)})^{-1} (t, 0) = (t, 0) \). By Sobolev embedding, we can choose a \( k \) large enough, such that \( \tilde{W}_T^{p,k+4} \subset C_\infty^\infty (M, \mathbb{R}) \). Thus, for \( |t| < \mu \), \((J_t, g_{\Psi^{(p,k)}})^{-1} (t, 0)\) is an extremal almost-Kähler metric of regularity at least \( C^4 \) (so we ensure, in this case, that \( \operatorname{grad}_\omega s^{\nabla_{\omega}} \) is of regularity \( C^1 \)).

By Proposition 3.5, the extremal vector field \( Z_{\omega_{\psi}, f}^T = \tilde{Z}_{\omega}^T \) is smooth for any almost-Kähler metric \((J_t, g_{t,f})\). In particular, for an extremal almost-Kähler metric \((J_t, g_{t,f})\) of regularity \( C^4 \), the dual \( ds^{\nabla_{\omega}} \) of \( Z_{\omega}^T \) with respect to \( \omega_{t,f} \) is of regularity \( C^4 \), then the hermitian scalar curvature \( s^{\nabla_{\omega}} \) of \((J_t, g_{t,f})\) is of regularity \( C^5 \). From (5.1), it follows that the hermitian scalar curvature is given by the pair of equations
\[
(5.3) \quad s^{\nabla_{\omega}} = g_{t,f} (\rho^{\nabla_{\omega}}, \omega_{t,f}) = \Delta g_{t,f} (u),
\]
\[
(5.4) \quad e^u = \frac{\omega_{t,f}^2}{\omega^2}.
\]

From (5.3), using the ellipticity [6] of the (Riemannian) Laplacian \( \Delta^{\rho_0} \) and since the l.h.s of (5.3) is of Hölder class \( C^{3,\beta} \) for any \( \beta \in (0, 1) \), it follows that \( u \) is of class \( C^{5,\beta} \). Following [11, 28], the linearisation of the equation (5.4) \((\omega + d\alpha) \wedge d\alpha = 0 \) together with the constraints \( \delta^{\rho_0} \alpha = 0 \) and \( (d\alpha)^{J_{t,-}} = 0 \) form a linear elliptic system in \( \alpha \). Elliptic theory [2, 6] ensures that the almost-Kähler metric \( g_{t,f} \) is of class \( C^{5,\beta} \) as the volume form and we can prove that any extremal almost-Kähler metric of regularity \( C^4 \) is smooth by a bootstrapping argument (in the Kähler case see [21]).

We obtain then a smooth family of \( T \)-invariant extremal almost-Kähler structures \((J_t, \omega_t = \omega + d\alpha_t)\) defined for \(|t| < \mu \). The main theorem follows from the Moser Lemma [25].

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