REALITY AND TRANSVERSALITY FOR SCHUBERT CALCULUS IN OG(n, 2n+1)

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Abstract. We prove an analogue of the Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) for the maximal type B_{n} orthogonal Grassmannian OG(n, 2n+1).

1. The Mukhin-Tarasov-Varchenko Theorem

For any non-negative integer k, let \( C_k[z] \) denote the \((k+1)\)-dimensional complex vector space of polynomials of degree at most k:
\[
C_k[z] := \{ f(z) \in \mathbb{F}[z] \mid \deg f(z) \leq k \}.
\]
Fix integers \( 0 \leq d \leq m \), and consider the Grassmannian \( X = \text{Gr}(d, C_{m-1}[z]) \), the variety of all \( d \)-dimensional linear subspaces of the \( m \)-dimensional vector space \( C_{m-1}[z] \). A point \( x \in X \) is \emph{real} if \( x \) is spanned by polynomials in \( \mathbb{R}_{m-1}[z] \); a subset of \( S \subset X \) is real if every point in \( S \) is real.

The Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) asserts that any zero-dimensional intersection of Schubert varieties in \( X \), relative a special family of flags in \( C_{m-1}[z] \), is transverse and real. This theorem is remarkable for two immediate reasons: first, it is a rare example of an algebraic geometry problem in which the solutions are always provably real; second, the usual arguments to prove transversality involve Kleiman's transversality theorem [5], which requires that the Schubert varieties be defined relative to generic flags. We recall the most relevant statements here, and refer the reader to the survey article [14] for a discussion of the history, context, reformulations and applications of this theorem.

To begin, we define a full flag in \( C_{m-1}[z] \), for each \( a \in \mathbb{C}P^1 \):
\[
F_\bullet(a) : \{0\} \subset F_1(a) \subset \cdots \subset F_{m-1}(a) \subset C_{m-1}[z].
\]
If \( a \in \mathbb{C} \),
\[
F_i(a) := (z + a)^{m-i} C[z] \cap C_{m-1}[z]
\]
is the set of polynomials in \( C_{m-1}[z] \) divisible by \((z + a)^{m-i}\). For \( a = \infty \), we set \( F_i(\infty) := C_{i-1}[z] = \lim_{a \to \infty} F_i(a) \). The flag \( F_\bullet(a) \) is often described as the flag osculating the rational normal curve \( \gamma : \mathbb{C}P^1 \to \mathbb{P}(C_{m-1}[z]), \gamma(t) = (z + t)^{m-1} \), which simply means that \( F_i(a) \) is the span of \( \{ \gamma(a), \gamma'(a), \ldots, \gamma^{(i-1)}(a) \} \).

Let \( \Lambda = \Lambda_{d,m} \) be the set of all partitions \( \lambda : (\lambda^1 \geq \cdots \geq \lambda^d) \), where \( \lambda^1 \leq m - d \) and \( \lambda^d \geq 0 \). We say \( \lambda \) is a partition of \( k \) and write \( \lambda \vdash k \) or \( |\lambda| = k \) if \( k = \lambda^1 + \cdots + \lambda^d \).
For every $\lambda \in \Lambda$, the Schubert variety in $X$ relative to the flag $F_{\bullet}(a)$ is

$$X_{\lambda}(a) := \{ x \in X \mid \dim (x \cap F_{n-d-\lambda'+i}(a)) \geq i, \text{ for } i = 1, \ldots, d \}. $$

The codimension of $X_{\lambda}(a)$ in $X$ is $|\lambda|$.

**Theorem 1** (Mukhin-Tarasov-Varchenko [6, 7]). If $a_1, \ldots, a_s \in \mathbb{R}P^1$ are distinct real points, and $\lambda_1, \ldots, \lambda_s \in \Lambda$ are partitions with $|\lambda_1| + \cdots + |\lambda_s| = \dim X$, then the intersection

$$X_{\lambda_1}(a_1) \cap \cdots \cap X_{\lambda_s}(a_s)$$

is finite, transverse, and real.

In [13], Sottile conjectured an analogue of Theorem 1 for $\text{OG}(n, 2n+1)$, the maximal orthogonal Grassmannian in type $B_n$. In Section 2 of this note, we give a proof of this conjecture (our Theorem 3). We discuss some of its consequences in Section 3; in particular, we note that Theorem 3 should yield a geometric proof of the Littlewood-Richardson rule for $\text{OG}(n, 2n+1)$.

### 2. The theorem for $\text{OG}(n, 2n+1)$

Fix a positive integer $n$, and consider the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the $(2n+1)$-dimensional vector space $\mathbb{C}_{2n}[z]$ given by

$$\left\langle \sum_{k=0}^{2n} a_k z^k, \sum_{\ell=0}^{2n} b_\ell z^\ell \right\rangle = \sum_{m=0}^{2n} (-1)^m a_m b_{2n-m}.$$

Let $Y = \text{OG}(n, \mathbb{C}_{2n}[z])$ be the orthogonal Grassmannian in $\mathbb{C}_{2n}[z]$, which is the variety of all $n$-dimensional isotropic subspaces of $\mathbb{C}_{2n}[z]$. The dimension of $Y$ is $\frac{n(n+1)}{2}$.

The definition of a Schubert variety in $Y$ requires our reference flags to be orthogonal flags. As explained in the next proposition, the bilinear form on $\mathbb{C}_{2n}[z]$ has been chosen so that this is true for the flags $F_{\bullet}(a)$.

**Proposition 2.** For $a \in \mathbb{C}P^1$, then the flag $F_{\bullet}(a)$ is an orthogonal flag; that is, $F_i(a)^\perp = F_{2n+1-i}(a)$, for $i = 0, \ldots, 2n+1$.

**Proof.** For $a = 0, \infty$, this is straightforward to verify. We deduce the result for all other $a$ by showing that $\langle f(z), g(z) \rangle = \langle f(z+a), g(z+a) \rangle$.

To see this, note that $\langle \frac{dz}{dz} (\frac{z}{z'}), \frac{dz}{dz} (\frac{z'}{z'}) \rangle = -\langle \frac{z'}{z}, \frac{z}{z'} \rangle$, so $\frac{dz}{dz}$ is a skew-symmetric operator on $\mathbb{C}_{2n}[z]$. It follows that $\exp(a \frac{dz}{dz})$ is an orthogonal operator on $\mathbb{C}_{2n}[z]$ and so $\langle f(z+a), g(z+a) \rangle = \langle \exp(a \frac{dz}{dz}) f(z), \exp(a \frac{dz}{dz}) g(z) \rangle = \langle f(z), g(z) \rangle$. □

The Schubert varieties in $Y$ are indexed by the set $\Sigma$ of all strict partitions $\sigma : (\sigma_1 > \sigma_2 > \cdots > \sigma_k)$, with $\sigma_1 \leq n$, $\sigma_k > 0$, $k \leq n$. For convenience, we put $\sigma_j = 0$ for $j > k$. We associate to $\sigma$ a decreasing sequence of integers, $\mathbf{\sigma}^1 > \cdots > \mathbf{\sigma}^n$, such that $\mathbf{\sigma}^j = \sigma^i$ if $\sigma^i > 0$, and $\{ |\sigma^1|, \ldots, |\sigma^n| \} = \{ 1, \ldots, n \}$. It is not hard to see that $\mathbf{\sigma}^j$ is given explicitly by the formula

$$\mathbf{\sigma}^j = \sigma^i - i + \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^j \}.$$

For $\sigma \in \Sigma$, the Schubert variety in $Y$ relative to the flag $F_{\bullet}(a)$ is defined to be

$$Y_{\sigma}(a) := \{ y \in Y \mid \dim (y \cap F_{1+n-\mathbf{\sigma}}(a)) \geq i, \text{ for } i = 1, \ldots, n \}. $$
The codimension of $Y_\sigma(a)$ in $Y$ is $|\sigma|$. We refer the reader to [2, 12] for further details.

**Theorem 3.** If $a_1, \ldots, a_s \in \mathbb{R}P^1$ are distinct real points, and $\sigma_1, \ldots, \sigma_s \in \Sigma$, with $|\sigma_1| + \cdots + |\sigma_s| = \text{dim } Y$, then the intersection

$$Y_{\sigma_1}(a_1) \cap \cdots \cap Y_{\sigma_s}(a_s)$$

is finite, transverse, and real.

**Proof.** Let $X = \text{Gr}(n, \mathbb{C}^{2n}[z])$, and let $\Lambda = \Lambda_{n, 2n+1}$. We prove this result by viewing $Y$ as a subvariety of $X$, and the Schubert varieties $Y_\sigma$ as the intersections of Schubert varieties in $X$ with $Y$. Note that $\text{dim } X = 2 \text{dim } Y = n(n + 1)$.

For a strict partition $\sigma \in \Sigma$, let

$$\tilde{\sigma}^i := \sigma^i + i = \sigma^i + \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^i \}.$$

Observe that $\tilde{\sigma}^i - \tilde{\sigma}^{i+1} = \sigma^i - \tilde{\sigma}^{i+1} - 1 \geq 0$, and $\tilde{\sigma}^1 \leq \sigma^1 + 1 \leq n + 1$; hence we see that

$$\tilde{\sigma} : (\tilde{\sigma}^1 \geq \tilde{\sigma}^2 \geq \cdots \geq \tilde{\sigma}^n)$$

is a partition in $\Lambda$.

It follows directly from the definitions of Schubert varieties in $X$ and $Y$ that

$$X_{\tilde{\sigma}}(a) \cap Y = Y_\sigma(a).$$

Moreover, we have,

$$|\tilde{\sigma}| = |\sigma| + \sum_{i \geq 1} \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^j \}$$

$$= |\sigma| + \sum_{j \geq 1} \# \{ i \in \mathbb{N} \mid i \leq j < i + \sigma^j \}$$

$$= |\sigma| + \sum_{j \geq 1} \sigma^j = 2|\sigma|.$$ 

Thus, if $|\sigma_1| + \cdots + |\sigma_s| = \text{dim } Y$, then $|\tilde{\sigma}_1| + \cdots + |\tilde{\sigma}_s| = 2 \text{dim } Y = \text{dim } X$, and so by Theorem 1 the intersection

$$X_{\tilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\tilde{\sigma}_s}(a_s)$$

is finite, transverse, and real; in particular this intersection is a zero-dimensional reduced scheme. It follows immediately that

$$Y_{\sigma_1}(a_1) \cap \cdots \cap Y_{\sigma_s}(a_s) = Y \cap X_{\tilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\tilde{\sigma}_s}(a_s)$$

is finite and real. To see that the intersection on the left hand side is also transverse, note that it is proper, so it suffices to show that it is scheme-theoretically reduced. But this is immediate from the fact that the right hand side is the intersection of $Y$ with a zero-dimensional reduced scheme. \qed
3. Consequences

Let $0 \leq d \leq m$, $X = \text{Gr}(d, \mathbb{C}_{m-1}[z])$, be as in Section 1. We can consider the Wronskian of $d$ polynomials $f_1(z), \ldots, f_d(z) \in \mathbb{C}_{m-1}[z]$:

$$\text{Wr}_{f_1, \ldots, f_d}(z) := \begin{vmatrix}
    f_1(z) & \cdots & f_d(z) \\
    f'_1(z) & \cdots & f'_d(z) \\
    \vdots & \ddots & \vdots \\
    f^{(d-1)}_1(z) & \cdots & f^{(d-1)}_d(z)
\end{vmatrix}.$$ 

This is a polynomial of degree at most $\dim X = d(n - d)$. If $f_1, \ldots, f_d$ are linearly dependent, the Wronskian is zero; otherwise up to a constant multiple, $\text{Wr}_{f_1, \ldots, f_d}(z)$ depends only on the linear span of $f_1(z), \ldots, f_d(z)$ in $\mathbb{C}_{m-1}[z]$. Thus the Wronskian gives us a well defined morphism of schemes $\text{Wr} : X \rightarrow \mathbb{P}(\mathbb{C}_{d(n-d)}[z])$, called the Wronski map. This morphism is flat and finite [1]. For $x \in X$ we will write $\text{Wr}(x; z)$ for any representative of $\text{Wr}(x)$ in $\mathbb{C}_{d(n-d)}[z]$.

The Wronski map has a deep connection to the Schubert varieties on $X$ relative to the flags $F^i_x(a)$, $a \in \mathbb{C}P^1$. A proof of the following classical result may be found in [1, 9, 14].

**Theorem 4.** The Wronskian $\text{Wr}(x; z)$ is divisible by $(z + a)^k$ if and only if $x \in X_\lambda(a)$ for some partition $\lambda \vdash k$. Also, $x \in X_{\mu}(\infty)$ for some $\mu \vdash (\dim X - \deg \text{Wr}(x; z))$.

For $X = \text{Gr}(n, \mathbb{C}_{2n}[z])$, and $Y = \text{OG}(n, \mathbb{C}_{2n}[z])$ we deduce the following analogue:

**Theorem 5.** If $y \in Y$ then $\text{Wr}(y; z) = P(y; z)^2$ for some polynomial $P(y; z) \in \mathbb{C}_{n(n+1)/2}[z]$. $P(y; z)$ is divisible by $(z + a)^k$ if and only if $y \in Y_{\sigma}(a)$ for some strict partition $\sigma \vdash k$ in $\Sigma$. Also, $y \in Y_{\tau}(\infty)$ for some strict partition $\tau \vdash (\dim Y - \deg P(y; z))$.

**Proof.** Let $y \in Y$, and let $(z + a)^\ell$ be the largest power $(z + a)$ that divides $\text{Wr}(x; z)$. By Theorem 4, there exists a partition $\lambda \vdash \ell$ such that $y \in X_\lambda(a)$. Since $\ell$ is maximal, $y$ is in the Schubert cell

$$X_\lambda^i(a) := \{ x \in X \mid \dim (x \cap F_{k}(a)) \geq i, \quad n+1-\lambda^i+i \leq k \leq n+1-\lambda^{i+1}+i, \quad 0 \leq i \leq n \}$$

$$= X_\lambda(a) \setminus \left( \bigcup_{|\mu|>|\lambda|} X_\mu(a) \right).$$

(Here, by convention, $\lambda^0 = n + 1$, $\lambda^{n+1} = 0$.) The Schubert cells in $Y$ are of the form

$$Y_{\sigma}^i(a) := \{ y \in Y \mid \dim (y \cap F_k(a)) \geq i, \quad n+1-\sigma^i \leq k \leq n-\sigma^{i+1}, \quad 0 \leq i \leq n \}$$

$$= X_\lambda^i(a) \cap Y.$$

(Here, by convention, $\sigma^0 = n + 1$, $\sigma^{n+1} = -n - 1$.) Now, the intersection $X_\lambda^i(a) \cap Y$ is nonempty, since it contains $y$, and is therefore a Schubert cell in $Y$. It follows that $\lambda = \kappa$ for some strict partition $\kappa \in \Sigma$. Thus $\ell = |\lambda| = 2|\kappa|$ is even, which proves that $\text{Wr}(y; z) = P(y; z)^2$ is a square.

We have shown that $(z + a)^{|\kappa|}$ is the largest power of $(z + a)$ that divides $P(y; z)$, and $y \in Y_{\sigma}^i(a)$. If $y \in Y_{\sigma}(a)$ then we must have $Y_{\sigma}(a) \supset Y_{\kappa}(a)$, which implies that $|\sigma| \leq |\kappa|$, and hence $(z + a)^k$ divides $P(y; z)$. Conversely, for any $k \leq |\kappa|$ there exists $\sigma \vdash k$ such that $Y_{\sigma}(a) \supset Y_{\kappa}(a)$, and so $y \in Y_{\sigma}(a)$. This proves the second
assertion. The third is proved by the same argument, taking \( \ell = \dim Y - \deg P(y; z) \) and \( a = \infty \).

If we write \( P(y) \) for the class of \( P(y; z) \) in projective space \( \mathbb{P}(\mathbb{C}_{n+1}/2[z]) \), then \( y \mapsto P(y) \) defines a morphism of schemes \( P : Y \to \mathbb{P}(\mathbb{C}_{n+1}/2[z]) \).

**Theorem 6.** \( P \) is a flat, finite morphism.

**Proof.** Let \( h(z) = (z + a_1)^{k_1} \cdots (z + a_s)^{k_s} \in \mathbb{C}_{n+1}/2[z] \). By Theorem 5,

\[
P^{-1}(h(z)) = \bigcap_{i=1}^s \left( \bigcup_{\sigma_j \in S_i} Y_{\sigma_i}(a_i) \right),
\]

which, by Theorem 3, is a finite set. Since \( P \) is a projective morphism, this implies that that \( P \) is flat and finite [4, Ch. III, Exer. 9.3(a)].

In [9] we showed that the properties of the Wronski map and Theorem 1 can be used to give geometric interpretations and proofs of several combinatorial theorems in the jeu de taquin theory, including the Littlewood-Richardson rule for Grassmannians in type \( A_n \). The map \( P \) and Theorem 3 are the appropriate analogues for \( OG(n, 2n+1) \).

With a few modifications, it should be possible to use the arguments in [9] to give geometric proofs of the analogous results in the theory of shifted tableaux, as developed in [3, 8, 10, 11, 15], including the Littlewood-Richardson rule for \( OG(n, 2n+1) \). The main ingredients required to adapt these proofs are Theorems 3, 5 and 6, and the Gel’fand-Tsetlin toric degeneration of \( OG(n, 2n+1) \), which can be also be computed by considering \( Y \subset X \). The complete details should be straightforward but somewhat lengthy, and we will not include them here.

**References**


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