ASYMPTOTIC LINEARITY OF REGULARITY AND a\*-INARIANT OF POWERS OF IDEALS

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Abstract. Let \( X = \text{Proj} R \) be a projective scheme over a field \( k \), and let \( I \subseteq R \) be an ideal generated by forms of the same degree \( d \). Let \( \pi : \tilde{X} \to X \) be the blowing up of \( X \) along the subscheme defined by \( I \), and let \( \phi : \tilde{X} \to \bar{X} \) be the projection given by the divisor \( dE_0 - E \), where \( E \) is the exceptional divisor of \( \pi \) and \( E_0 \) is the pullback of a general hyperplane in \( X \). We investigate how the asymptotic linearity of the regularity and the \( a\*-\)invariant of \( I^q \) (for \( q \gg 0 \)) is related to invariants of fibers of \( \phi \).

1. Introduction

Let \( k \) be a field and let \( X = \text{Proj} R \subseteq \mathbb{P}^n \) be a projective scheme over \( k \). Let \( I \subseteq R \) be a homogeneous ideal. It is well known (cf. [1, 4, 6, 7, 9, 12, 18, 20]) that \( \text{reg}(I^q) = aq + b \), a linear function in \( q \), for \( q \gg 0 \). While the linear constant \( a \) is quite well understood from reduction theory (see [20]), the free constant \( b \) remains mysterious (see [10, 19] for partial results). Recently, Eisenbud and Harris [10] showed that when \( I \) is generated by general forms of the same degree, whose zeros set is empty in \( X \), \( b \) can be related to a set of local data, namely, the regularity of fibers of the projection map defined by the generators of \( I \). The aim of this paper is to exhibit a similar phenomenon in a more general situation, when \( I \) is generated by arbitrary forms of the same degree. In this case, the generators of \( I \) do not necessarily give a morphism. The projection map that we will examine is the map from the blowup of \( X \) along the subscheme defined by \( I \), considered as a bi-projective scheme, to its second coordinate.

Let \( I = (F_0, \ldots, F_m) \), where \( F_0, \ldots, F_m \) are homogeneous elements of degree \( d \) in \( R \). Let \( \pi : \tilde{X} \to X \) be the blowing up of \( X \) along the subscheme defined by \( I \). Let \( R = R[It] \) be the Rees algebra of \( I \). By letting \( \deg F_i t = (d, 1) \), the Rees algebra \( R \) is naturally bi-graded with \( R = \bigoplus_{p,q \in \mathbb{Z}} R_{(p,q)} \), where \( R_{(p,q)} = (I^q)_{p+qd} t^q \). Under this bi-gradation of \( R \), we can identify \( \tilde{X} \) with \( \text{Proj} R \subseteq \mathbb{P}^n \times \mathbb{P}^m \) (cf. [8, 15, 16]). Also, the projection \( \phi : \text{Proj} R \to \mathbb{P}^m \) is in fact the morphism given by the divisor \( D = dE_0 - E \), where \( E \) is the exceptional divisor of \( \pi \) and \( E_0 \) is the pullback of a general hyperplane in \( X \). For a close point \( \varphi \in \tilde{X} = \text{image}(\phi) \), let \( \tilde{X}_\varphi = \tilde{X} \times \text{Spec} \mathcal{O}_{\tilde{X}_\varphi} \) be the fiber of \( \phi \) over the affine neighborhood \( \text{Spec} \mathcal{O}_{\tilde{X}_\varphi} \) of \( \varphi \). Then \( \tilde{X}_\varphi = \text{Proj} R_{(\varphi)} \), where \( R_{(\varphi)} \) is the homogeneous localization of \( R \) at \( \varphi \). We define the regularity of \( \tilde{X}_\varphi \), denoted by \( \text{reg}(\tilde{X}_\varphi) \), to be that of \( R_{(\varphi)} \). Inspired by the work of Eisenbud and Harris [10], we propose the following conjecture.

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**Conjecture 1.1.** Let $X = \text{Proj } R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $\text{reg}(\phi) = \max\{\text{reg}(\tilde{X}_\varphi) \mid \varphi \in \tilde{X}\}$. Then for $q \gg 0$, $$\text{reg}(I^q) = qd + \text{reg}(\phi).$$

We provide a strong evidence\(^1\) for Conjecture 1.1. More precisely, we prove a similar statement to Conjecture 1.1 for the $a^\ast$-invariant, a closely related variant of the regularity. For a closed point $\varphi \in \bar{X}$, we define the $a^\ast$-invariant of $\tilde{X}_\varphi$, denoted by $a^\ast(\tilde{X}_\varphi)$, to be the $a^\ast$-invariant of its homogeneous coordinate ring $\mathcal{R}_{(\varphi)}$. Our first main result is stated as follows.

**Theorem 1.2** (Theorems 2.6). Let $X = \text{Proj } R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $a^\ast(\phi) = \max\{a^\ast(\tilde{X}_\varphi) \mid \varphi \in \tilde{X}\}$. Then for $q \gg 0$, we have $$a^\ast(I^q) = qd + a^\ast(\phi).$$

As a consequence of Theorem 1.2, we obtain in Theorem 3.1 an upper and a lower bounds for the asymptotic linear function $\text{reg}(I^q)$. We prove that for $q \gg 0$, $$qd + a^\ast(\phi) \leq \text{reg}(I^q) \leq qd + a^\ast(\phi) + \dim R.$$ This, in particular, allows us to settle Conjecture 1.1 in an important case. A fiber $\tilde{X}_\varphi$ is said to be arithmetically Cohen-Macaulay if its homogeneous coordinate ring $\mathcal{R}_{\varphi}$ is Cohen-Macaulay. Our next result shows that Conjecture 1.1 holds under the additional condition that each fiber $\tilde{X}_\varphi$ is arithmetically Cohen-Macaulay. This hypothesis is satisfied, for instance, when the Rees algebra $\mathcal{R}$ is a Cohen-Macaulay ring.

**Theorem 1.3** (Theorem 3.2). Let $X = \text{Proj } R \subseteq \mathbb{P}^n$ be an irreducible projective scheme of dimension at least 1, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $\text{reg}(\phi) = \max\{\text{reg}(\tilde{X}_\varphi) \mid \varphi \in \tilde{X}\}$. Assume that each fiber $\tilde{X}_\varphi$ is an arithmetically Cohen-Macaulay scheme. Then for $q \gg 0$, we have $$\text{reg}(I^q) = qd + \text{reg}(\phi).$$

Our method in proving Theorem 1.2, and subsequently Theorem 1.3, is based upon investigating different graded structures of the Rees algebra $\mathcal{R}$. More precisely, beside the natural bi-graded structure mentioned above, $\mathcal{R}$ possesses two other $\mathbb{N}$-graded structures; namely

$$\mathcal{R} = \bigoplus_{q \in \mathbb{Z}} \mathcal{R}^1_q,$$ where $\mathcal{R}^1_q = \bigoplus_{p \in \mathbb{Z}} \mathcal{R}_{(p,q)}$, and

$$\mathcal{R} = \bigoplus_{p \in \mathbb{Z}} \mathcal{R}^2_p,$$ where $\mathcal{R}^2_p = \bigoplus_{q \in \mathbb{Z}} \mathcal{R}_{(p,q)}$.

Under these $\mathbb{N}$-graded structures, it can be seen that $R = \mathcal{R}^1_0 \hookrightarrow \mathcal{R}$, $\mathcal{R}^1_q$ is a graded $R$-modules for any $q \in \mathbb{Z}$, $S = \mathcal{R}^2_0 \hookrightarrow \mathcal{R}$, and $\mathcal{R}^2_p$ is a graded $S$-modules for any $p \in \mathbb{Z}$.

Let $\mathcal{R}^1_q$ be the coherent sheaf associated to $\mathcal{R}^1_q$ on $X$, and let $\mathcal{R}^2_p$ be the coherent sheaf.

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\(1\)Marc Chardin in a recent preprint [5] has proved that Conjecture 1.1 holds in general.
sheaf associated to $R_p$ on $\tilde{X}$. Observe further that $R_p^q = \bigoplus_{p+q \in \mathbb{Z}} [I_p^q]$, the module $I^q$ shifted by $qd$. As a consequence, for any $p, q \in \mathbb{Z}$ we have

$$\widetilde{R_p^q}(p) = \tilde{I}(p+qd).$$

Thus, to study the regularity of $I^q$, we examine sheaf cohomology groups of $\widetilde{R_p^q}(p)$. Our results are obtained by investigating how these sheaf cohomology groups behave by pulling back via the blowup map $\pi$ and pushing forward through the projection map $\phi$.

Our paper is outlined as follows. In the next section, we consider $\tilde{X}$ as a biprojective scheme and prove a similar statement to Conjecture 1.1 for the $a^*$-invariant. In the last section, we prove an important case of Conjectures 1.1.

### 2. Bi-projective schemes and $a^*$-invariants

The goal of this section is to give a similar statement to Conjecture 1.1 for the $a^*$-invariant of powers of an ideal. We first recall the definition of regularity and $a^*$-invariant.

**Definition 2.1.** For any $\mathbb{N}$-graded algebra $T$, let $T_+$ denote its irrelevant ideal. For $i \geq 0$, let $a^i(T) = \max \{l \mid [H^i_{T_+}(T)]_l \neq 0 \}$ (if $H^i_{T_+}(T) = 0$ then take $a^i(T) = -\infty$).

The $a^*$-**invariant** and the **regularity** of $T$ are defined to be

$$a^*(T) = \max_{i \geq 0} \{a^i(T)\} \text{ and } \text{reg}(T) = \max_{i \geq 0} \{a^i(T) + i\}.$$

Note that $H^i_{T_+}(T) = 0$ for $i > \dim T$, so $a^*(T)$ and $\text{reg}(T)$ are well-defined and finite invariants.

Let $S$ denote the homogeneous coordinate ring of $\tilde{X} \subseteq \mathbb{P}^m$. For each closed point $\varphi \in \tilde{X}$, i.e., $\varphi$ is a homogeneous prime ideal in $S$, let $R_\varphi$ be the localization of $R$ at $\varphi$; that is, $R_\varphi = R \otimes_S S_\varphi$. The homogeneous localization of $R$ at $\varphi$, denoted by $R_\varphi$, is the collection of all element of degree 0 (in $t$) of $R_\varphi$. Observe that homogeneous localization at $\varphi$ annihilates the grading with respect to powers of $t$. Thus, inheriting from the bi-graded structure of $R$, the homogeneous localization $R_\varphi$ is a $\mathbb{N}$-graded ring. The regularity and $a^*$-invariant of $R_\varphi$ are therefore defined as usual.

Associated to $\phi : \tilde{X} \to \tilde{X}$, let

$$a^i(\phi) = \max \{a^i(R_{\varphi}) \mid \varphi \in \tilde{X} \} \text{ for } i \geq 0,$$

$$a^*(\phi) = \max \{a^*(R_{\varphi}) \mid \varphi \in \tilde{X}) \},$$

and

$$\text{reg}(\phi) = \max \{\text{reg}(R_{\varphi}) \mid \varphi \in \tilde{X}\}.$$

**Remark 2.2.** By definition, $a^*(\phi) = \max_{i \geq 0} \{a^i(\phi)\}$ and $\text{reg}(\phi) = \max_{i \geq 0} \{a^i(\phi) + i\}$. Note that $H^i_{R_{\varphi}}(R_{\varphi}) = [H^i_{\widetilde{R_{\varphi}}}][R_{\varphi}]$, where on the right hand side we view $R$ under its $\mathbb{N}$-graded structure $\tilde{R} = \bigoplus_{p \in \mathbb{Z}} R_p$, which induces the embedding $S \hookrightarrow R$. Thus, $a^i(\phi)$ is a well-defined and finite invariant for any $i \geq 0$. As a consequence, $a^*(\phi)$ and $\text{reg}(\phi)$ are well-defined and finite invariants. These invariants are defined in a similar fashion to the projective $a^*$-**invariant** that was introduced in [16]. We shall also let $r_\phi$ denote the smallest integer $r$ such that

$$a^*(\phi) = a^r(\phi).$$
Recall further that the Rees algebra $R = R[It]$ of $I$ is naturally bi-graded with $R_{(p,q)} = (I^t)^{p+qd}t^q$, and we identify $\tilde{X}$ with $\text{Proj} R \subseteq \mathbb{P}^n \times \mathbb{P}^m$. It can also be seen that $\tilde{X}$ and $X$ can be realized as the (closure of the) graph and the (closure of the) image of the rational map $\varphi : X \dashrightarrow \mathbb{P}^m$ given by $P \mapsto [F_0(P) : \cdots : F_m(P)]$ (cf. [8, 15, 16]). Under this identification, $\pi$ and $\phi$ are restrictions on $\tilde{X}$ of natural projections $\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$. We have the following diagram:

\[
\begin{array}{ccc}
\pi & \nearrow & \phi \\
X & \rightarrow & \tilde{X} \\
\end{array}
\]

Let $I$ be the ideal sheaf of $I$, and let $L = IO_{\tilde{X}} = O_{\tilde{X}}(0,1)$.

**Lemma 2.3.** With notations as above.

1. The homogeneous coordinate ring of $\tilde{X}$ is $S \cong k[F_0, \ldots, F_m]$.
2. $\phi^* O_X(q) = L^q \otimes \pi^* O_X(qd)$ $\forall q \in \mathbb{Z}$.
3. $O_{\tilde{X}}(p,q) = \pi^* O_X(p) \otimes \phi^* O_X(q) \cong L^q \otimes \pi^* O_X(p + qd)$ $\forall p, q \in \mathbb{Z}$.

**Proof.** (1) follows from the construction of $\varphi$. (2) and (3) follow from the graded structures of $R, R$ and $S$. \qed

The next few lemmas exhibit how the $\alpha^*$-invariant of fibers of $\phi$ governs sheaf cohomology groups via a push forward along $\phi$.

**Lemma 2.4.** Let $p > \alpha^*(\phi)$. Then

1. $\phi_* O_{\tilde{X}}(p,q) = \widetilde{R^2 p}(q)$ and $R^j \phi_* O_{\tilde{X}}(p,q) = 0$ for any $j > 0$ and any $q \in \mathbb{Z}$,
2. $H^i(\tilde{X}, O_{\tilde{X}}(p,q)) = 0$ for $i > 0$ and $q \gg 0$.

**Proof.** By Lemma 2.3 and the projection formula we have

$\phi_* O_{\tilde{X}}(p,q) = \phi_* O_X(p,0) \otimes O_{\tilde{X}}(q)$ and $R^j \phi_* O_{\tilde{X}}(p,q) = R^j \phi_* O_X(p,0) \otimes O_{\tilde{X}}(q)$.

Thus, to show (1) it suffices to prove the assertion for $q = 0$.

Let $\wp$ be any closed point of $\tilde{X}$, and consider the restriction $\phi_{\wp} : \tilde{X}_{\wp} = \text{Proj} R_{(\wp)} \rightarrow \text{Spec} O_{X,\wp}$ of $\phi$ over an open affine neighborhood $\text{Spec} O_{X,\wp}$ of $\wp$. We have

\[(2.1) \quad R^j \phi_{\wp} O_{\tilde{X}}(p,0) \mid_{\text{Spec} O_{X,\wp}} = R^j \phi_{\wp} \left( \widetilde{R^2 \wp} (p) \right) = H^j(\tilde{X}_{\wp}, \widetilde{R^2 \wp}(p)) \quad \forall j \geq 0.\]

For any $j \geq 0$ and any $\wp \in \tilde{X}$, we have $p > \alpha^*(\phi) \geq \alpha^j(R_{(\wp)})$; and thus, $[H^j_{R_{(\wp)}}, (R_{(\wp)})]_p = 0$. Moreover, the Serre-Grothendieck correspondence give us an exact sequence

\[
0 \rightarrow [H^{j+1}_{R_{(\wp)}}, (R_{(\wp)})]_p \rightarrow [R_{(\wp)}]_p = (R^2_{(\wp)})_p \\
\rightarrow H^0(\tilde{X}_{\wp}, \widetilde{R^2 \wp}(p)) \rightarrow [H^j_{R_{(\wp)}}, (R_{(\wp)})]_p \rightarrow 0
\]

and isomorphisms

$H^1(\tilde{X}_{\wp}, \widetilde{R^2 \wp}(p)) \cong [H^{j+1}_{R_{(\wp)}}, (R_{(\wp)})]_p$ for $i > 0$. 

Therefore, for any \( j \geq 0 \) and any \( \varphi \in \bar{X} \),
\[
R^j \phi_* \mathcal{O}_{\bar{X}}(p, 0) \big|_{\text{Spec} \mathcal{O}_{\bar{X}, \varphi}} = H^j(\bar{X}_\varphi, \mathcal{R}_q(p)) = \begin{cases} 
(\mathcal{R}_q^2)_{(\varphi)} & \text{for } j = 0 \\
0 & \text{for } j > 0.
\end{cases}
\]

This is true for any \( \varphi \in \bar{X} \), and so (1) is proved.

Now, it follows from (1) that the Leray spectral sequence \( H^i(\bar{X}, R^j \phi_* \mathcal{O}_{\bar{X}}(p, q)) \Rightarrow H^{i+j}(\bar{X}, \mathcal{O}_{\bar{X}}(p, q)) \) degenerates. Thus, for any \( j \geq 0 \),
\[
H^j(\bar{X}, \mathcal{O}_{\bar{X}}(p, q)) = H^j(\bar{X}, \mathcal{R}_p^2(q)).
\]

Moreover, since \( \mathcal{O}_{\bar{X}}(1) \) is a very ample divisor, we have \( H^j(\bar{X}, \mathcal{R}_p^2(q)) = 0 \) for all \( q \gg 0 \), and (2) is proved.

**Lemma 2.5.** Let \( r_\phi \) be defined as above.

1. If \( r_\phi \leq 1 \) then \( H^0(\bar{X}, \mathcal{O}_{\bar{X}}(a^*(\phi), q)) \neq \mathcal{R}_{(a^*(\phi), q)} \) for \( q \gg 0 \).
2. If \( r_\phi \geq 2 \) then \( H^{s-1}(\bar{X}, \mathcal{O}_{\bar{X}}(a^*(\phi), q)) \neq 0 \) for \( q \gg 0 \).

**Proof.** For simplicity, let \( a = a^*(\phi) \). By the definition of \( r_\phi \), we have
\[
(2.2) \begin{cases}
[H_{\mathcal{R}_1(\phi)}(\mathcal{R}_q(q))]_a = 0 & \text{for } i < r_\phi \text{ and any } \varphi \in \bar{X} \\
[H_{\mathcal{R}_1(\phi)}(\mathcal{R}_q(q))]_a \neq 0 & \text{for some } q \in \bar{X}.
\end{cases}
\]

1. If \( r_\phi \leq 1 \) then it follows from (2.2) and the Serre-Grothendieck correspondence that \( H^0(\bar{X}_q, \mathcal{R}(q)(a)) \neq \mathcal{R}(q)(a) \). This and (2.1) imply that \( \phi_* \mathcal{O}_{\bar{X}}(a, 0) \neq \mathcal{R}_a^2(q) \), and so \( \phi_* \mathcal{O}_{\bar{X}}(a, q) \neq \mathcal{R}_a^2(q) \) for any \( q \in \mathbb{Z} \).

Since both \( \phi_* \mathcal{O}_{\bar{X}}(a, q) = \phi_* \mathcal{O}_{\bar{X}}(a, 0) \otimes \mathcal{O}_{\bar{X}}(q) \) (by Lemma 2.3 and the projection formula) and \( \mathcal{R}_a^2(q) \) are generated by global sections for \( q \gg 0 \), we must have
\[
H^0(\bar{X}, \phi_* \mathcal{O}_{\bar{X}}(a, q)) \neq H^0(\bar{X}, \mathcal{R}_a^2(q)) \forall q \gg 0.
\]

Moreover, \( H^0(\bar{X}, \mathcal{O}_{\bar{X}}(a, q)) = H^0(\bar{X}, \phi_* \mathcal{O}_{\bar{X}}(a, q)) \). Thus,
\[
H^0(\bar{X}, \mathcal{O}_{\bar{X}}(a, q)) \neq \mathcal{R}_{(a, q)} \text{ for } q \gg 0.
\]

2. If \( r_\phi \geq 2 \), then it follows from (2.2) and (2.1) that
\[
(2.3) \begin{cases}
R^j \phi_* \mathcal{O}_{\bar{X}}(a, q) = 0 & \text{for } 0 < j < r_\phi - 1, \\
R^{s-1} \phi_* \mathcal{O}_{\bar{X}}(a, q) \neq 0.
\end{cases}
\]

By Lemma 2.3 and the projection formula, \( \phi_* \mathcal{O}_{\bar{X}}(a, q) = \phi_* \mathcal{O}_{\bar{X}}(a, 0) \otimes \mathcal{O}_{\bar{X}}(q) \). Thus, for \( q \gg 0 \) we have \( H^{s-1}(\bar{X}, \phi_* \mathcal{O}_{\bar{X}}(a, q)) = 0 \). From this, together with (2.3) and the Leray spectral sequence \( H^i(\bar{X}, R^j \phi_* \mathcal{O}_{\bar{X}}(a, q)) \Rightarrow H^{i+j}(\bar{X}, \mathcal{O}_{\bar{X}}(a, q)) \), we can deduce that
\[
H^{s-1}(\bar{X}, \mathcal{O}_{\bar{X}}(a, q)) = 0 \Rightarrow H^{s-1}(\bar{X}, \phi_* \mathcal{O}_{\bar{X}}(a, q)) \neq 0 \text{ for } q \gg 0.
\]

It then follows, since \( R^{s-1} \phi_* \mathcal{O}_{\bar{X}}(a, q) = R^{s-1} \phi_* \mathcal{O}_{\bar{X}}(a, 0) \otimes \mathcal{O}_{\bar{X}}(q) \) is globally generated for \( q \gg 0 \), that
\[
H^{s-1}(\bar{X}, \mathcal{O}_{\bar{X}}(a, q)) \neq 0 \text{ for } q \gg 0.
\]
Our first main result is a similar statement to Conjecture 1.1 for the $a^*$-invariant.

**Theorem 2.6.** Let $X = \text{Proj } R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $a^*(\phi) = \max\{a^*(\tilde{X}, p) \mid \phi \in \tilde{X}\}$. Then for $q \gg 0$, we have

$$a^*(I^q) = qd + a^*(\phi).$$

**Proof.** By a similar argument as in Lemma 2.4, considering $\pi_*$ instead of $\phi_*$, we can show that for $q \gg 0$,

$$(2.4) \quad \pi_* \mathcal{O}_{\tilde{X}}(p, q) = \tilde{R}^1_q(p) = \tilde{I}^q(p + qd) \text{ and } R^j \pi_* \mathcal{O}_{\tilde{X}}(p, q) = 0 \text{ for } j > 0.$$

This implies that for $q \gg 0$, the Leray spectral sequence $H^i(X, R^j \pi_* \mathcal{O}_{\tilde{X}}(p, q)) \Rightarrow H^{i+j}(\tilde{X}, \mathcal{O}_{\tilde{X}}(p, q))$ degenerates and we have

$$H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}(p, q)) = H^j(X, \tilde{I}^q(p + qd)) \gg j \geq 0.$$

Therefore, for $j > 0$, $q \gg 0$ and $p > a^*(\phi)$, it follows from Lemma 2.4 that $H^j(X, \tilde{I}^q(p + qd)) = 0$. That is,

$$(2.5) \quad \left[H^{j+1}(I^q)\right]_{p+qd} = 0.$$

Furthermore, for $j = 0$ and $q \gg 0$, we have $H^0(X, \tilde{R}^2_q(q)) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(p, q)) = H^0(\tilde{X}, \tilde{I}^q(p + qd))$, where the first equality follows from Lemma 2.4. On the other hand, for $q \gg 0$, $H^0(\tilde{X}, \tilde{R}^2_q(q)) = (\tilde{R}^2_q)_q = \mathcal{R}_q = [I^q]_{p+qd}$. Thus, for $q \gg 0$, $H^0(X, \tilde{I}^q(p + qd)) = [I^q]_{p+qd}$. This and (2.5) imply that for $q \gg 0$,

$$a^*(I^q) \leq qd + a^*(\phi).$$

To prove the other inequality, let $r_\phi$ be defined as in Remark 2.2. We consider two cases: $r_\phi \leq 1$ and $r_\phi \geq 2$. If $r_\phi \leq 1$ then by Lemma 2.5, $H^0(X, \pi_* \mathcal{O}_{\tilde{X}}(a^*(\phi), q)) \neq \mathcal{R}_{a^*(\phi), q}$ for all $q \gg 0$. This implies that $H^0(X, \pi_* \mathcal{O}_{\tilde{X}}(a^*(\phi), q)) \neq \mathcal{R}_{a^*(\phi), q}$ for $q \gg 0$. That is,

$$H^0(X, \tilde{I}^q(a^*(\phi) + qd)) \neq [I^q]_{a^*(\phi)} \gg q \gg 0.$$

By the Serre-Grothendieck correspondence, for $q \gg 0$, we have either

$$\left[H^0_{R^+}(I^q)\right]_{a^*(\phi) + qd} \neq 0 \text{ or } \left[H^1_{R^+}(I^q)\right]_{a^*(\phi) + qd} \neq 0.$$

It then follows that $a^*(I^q) \geq qd + a^*(\phi)$ for $q \gg 0$.

If $r_\phi \geq 2$, then by Lemma 2.5, $H^{a-1}(X, \mathcal{O}_{\tilde{X}}(a^*(\phi), q)) \neq 0$ for $q \gg 0$. Moreover, for $q \gg 0$, it follows from (2.4) that the Leray spectral sequence

$$H^i(X, R^j \pi_* \mathcal{O}_{\tilde{X}}(p, q)) \Rightarrow H^{i+j}(\tilde{X}, \mathcal{O}_{\tilde{X}}(p, q))$$

degenerates. Thus, for $q \gg 0$, we have $H^{a-1}(X, \tilde{I}^q(a^*(\phi) + qd)) \neq 0$. By the Serre-Grothendieck correspondence, we have $\left[H^i_{R^+}(I^q)\right]_{a^*(\phi) + qd} \neq 0$ for $q \gg 0$. This implies that $a^*(I^q) \geq qd + a^*(\phi)$ for $q \gg 0$. \qed
Example 2.7. Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay standard graded domain, and let $A = (a_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ be an $r \times s$ matrix ($r \leq s$) of entries in $R_1$. Let $I_t(A)$ denote the ideal generated by $t \times t$ minors of $A$, and let $I = I_r(A)$. Assume that for any $1 \leq t \leq r$, $\text{ht} I_t(A) \geq (r - t + 1)(s - r) + 1$. Let $t(\omega_R)$ be the least generating degree of $\omega_R$, the canonical module of $R$. Then for $q \gg 0$,

$$a^*(I^q) = qr - t(\omega_R).$$

Indeed, let $S = k[I_t]$ denote the homogeneous coordinate ring of $\tilde{X}$, let $\varphi$ be any point in $\tilde{X}$, and let $T = R_{(\varphi)}$. By [11, Theorem 3.5], the Rees algebra $R$ is Cohen-Macaulay. Thus, $R_{(\varphi)}$ is Cohen-Macaulay. This implies that

$$a^*(T) = a^\dim T(T) = -\min\{s \mid [\omega_T]_s \neq 0\}.$$ 

Furthermore, by [17, Example 3.8],

$$\omega_R = \omega_R(1, t)g^{-2} = \omega_R \oplus \omega_R t \oplus \cdots \oplus \omega_R t^{g-2} \oplus \omega_R t^{g-1} \oplus \cdots,$$

where $g = \text{ht} I$. Hence, by localizing at $\varphi$, we obtain

$$\omega_T = (\omega_R)_{(\varphi)} = (\omega_R(1, t)g^{-2})_{(\varphi)}.$$

Observe that the homogeneous localization at $\varphi$ annihilates the grading inherited from powers of $t$, so it follows that the degrees of $\omega_T$ arise from the degrees of $\omega_R$. That is, $t(\omega_T) = t(\omega_R)$, and the conclusion follows from Theorem 2.6.

### 3. Regularity of powers of ideals

In this section, we investigate the asymptotic linearity of regularity and prove a special case of Conjecture 1.1.

We start by giving an upper and a lower bound for the free constant of $\text{reg}(I^q)$ in terms of $a^*(\varphi)$.

**Theorem 3.1.** Let $X = \text{Proj} R \subseteq \mathbb{P}^n$ be a projective scheme, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $a^*(\varphi) = \max\{a^*(\tilde{X}_\varphi) \mid \varphi \in \tilde{X}\}$. Then there exists an integer $0 \leq r \leq \dim R$ such that for $q \gg 0$, we have $\text{reg}(I^q) =qd + a^*(\varphi) + r$. In particular, for $q \gg 0$,

$$qd + a^*(\varphi) \leq \text{reg}(I^q) \leq qd + a^*(\varphi) + \dim R.$$

**Proof.** Suppose $\text{reg}(I^q) =aq + b$ for $q \gg 0$. It can be easily seen from the definition of the regularity and $a^*$-invariant of graded $R$-modules that $a^*(I^q) \leq \text{reg}(I^q) \leq a^*(I^q) + \dim R$ for any $q$. This and Theorem 2.6 imply that $a = d$; that is, $\text{reg}(I^q) = qd + b$ for $q \gg 0$. Let $r = b - a^*(\varphi)$. Then $\text{reg}(I^q) = qd + a^*(\varphi) + r$, and since $a^*(I^q) \leq \text{reg}(I^q) \leq a^*(I^q) + \dim R$, we have $0 \leq r \leq \dim R$. 

Our next result shows that Conjecture 1.1 holds under an extra condition that each fiber $\tilde{X}_\varphi$ is an arithmetically Cohen-Macaulay scheme.

**Theorem 3.2.** Let $X = \text{Proj} R \subseteq \mathbb{P}^n$ be an irreducible projective scheme of dimension at least 1, and let $I \subseteq R$ be a homogeneous ideal generated by forms of degree $d$. Let $\text{reg}(\varphi) = \max\{\text{reg}(\tilde{X}_\varphi) \mid \varphi \in \tilde{X}\}$. Assume that each fiber $\tilde{X}_\varphi$ is an arithmetically Cohen-Macaulay scheme. Then for $q \gg 0$, we have

$$\text{reg}(I^q) = qd + \text{reg}(\varphi).$$
Proof. Let \( l = \dim X \geq 1 \). Since \( X \) is irreducible, \( \tilde{X} \) is also irreducible. Moreover, for any point \( \wp \in \tilde{X} \), \( \text{Spec} \mathcal{O}_{\tilde{X}, \wp} \) is an open neighborhood of \( \wp \), and so \( \tilde{X}_\wp = \phi^{-1}(\text{Spec} \mathcal{O}_{X, \wp}) \) is an open subset in \( \tilde{X} \). Thus, \( \dim \tilde{X}_\wp = \dim \tilde{X} = \dim X \).

By the hypothesis, for each \( \wp \in \tilde{X} \), \( \mathcal{R}(\wp) \) is a Cohen-Macaulay ring of dimension \( \dim \tilde{X}_\wp + 1 = l + 1 \). This implies that \( a^*(\mathcal{R}(\wp)) = a^{l+1}(\mathcal{R}(\wp)) \) and \( \text{reg}(\mathcal{R}(\wp)) = a^{l+1}(\mathcal{R}(\wp)) + (l + 1) \). Therefore,

\[
\begin{align*}
(3.1) & \quad a^*(\wp) = a^{l+1}(\wp), \\
(3.2) & \quad \text{reg}(\wp) = a^*(\wp) + l + 1.
\end{align*}
\]

It follows from (3.1) that \( r_\wp = l + 1 \geq 2 \). By the same arguments as the last part of the proof of Theorem 2.6, we have that for \( q \gg 0 \), \( \text{reg}(I^q) \geq qd + a^*(\wp) + r_\wp = qd + a^*(\wp) + \dim R \). This, together with Theorem 3.1, implies that for \( q \gg 0 \), \( \text{reg}(I^q) = qd + a^*(\wp) + \dim R \). The conclusion now follows from (3.2). \( \square \)

**Corollary 3.3.** Let \( X = \text{Proj} R \subseteq \mathbb{P}^n \) be an irreducible projective scheme of dimension at least 1, and let \( I \subseteq R \) be a homogeneous ideal generated by forms of degree \( d \). Assume that \( \mathcal{R} \) is a Cohen-Macaulay ring. Then for \( q \gg 0 \),

\[
\text{reg}(I^q) = qd + \text{reg}(\wp).
\]

**Proof.** Since \( \mathcal{R} \) is Cohen-Macaulay, so is \( \mathcal{R}(\wp) \) for any \( \wp \in \tilde{X} \). Thus, each fiber \( \tilde{X}_\wp \) is arithmetically Cohen-Macaulay. The conclusion follows from Theorem 3.2. \( \square \)

We shall end the paper with a number of examples in which the hypotheses of Corollary 3.3 are satisfied.

**Example 3.4.** Let \( R \) and \( I \) be as in Example 2.7. In this case, \( I \) is generated in degree \( r \). As noted before, the Rees algebra \( \mathcal{R} \) is Cohen-Macaulay. Notice further that \( X = \text{Proj} R \) is an irreducible projective scheme. Thus, by Corollary 3.3, we have

\[
\text{reg}(I^q) = qr + \text{reg}(\wp) \quad \forall \ q \gg 0.
\]

**Example 3.5.** Let \( R = k[x_{ij}]_{1 \leq i \leq r, 1 \leq j \leq s} \) and let \( I \) be the ideal generated by \( t \times t \) minors of \( M = (x_{ij})_{1 \leq i \leq r, 1 \leq j \leq s} \) for some \( 1 \leq t \leq \min\{r, s\} \). By [11, Theorem 3.5] and [3, Corollary 3.3], the Rees algebra \( \mathcal{R} \) of \( I \) is Cohen-Macaulay. Also, \( X = \text{Proj} R \) is an irreducible projective scheme. It follows from Corollary 3.3 that

\[
\text{reg}(I^q) = qt + \text{reg}(\wp) \quad \forall \ q \gg 0.
\]

**Example 3.6.** Let \( R \) be a Cohen-Macaulay graded domain of dimension at least 2. Let \( I \) be either a complete intersection, or an almost complete intersection that is also generically a complete intersection. Assume that \( I \) is generated in degree \( d \). Then the Rees algebra \( \mathcal{R} \) of \( I \) is Cohen-Macaulay (cf. [2, 21]). By Corollary 3.3, we have

\[
\text{reg}(I^q) = qd + \text{reg}(\wp) \quad \forall \ q \gg 0.
\]

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