ERRATUM: A PRESENTATION FOR HILDEN’S SUBGROUP OF
THE BRAID GROUP

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1. Introduction

After publication of [1] Allen Hatcher found a gap in the proof that the complex $X_n$ is simply connected. This complex is defined in terms of isotopy classes of discs, but the argument uses representatives of the isotopy classes. There was an implicit assumption that for an edge path in the complex there exists sufficiently nice representatives of each isotopy class. In Definition 2 of this paper the properties of these representatives will be made explicit. It is clear that such representatives exist for a path, the problem is that for a loop it is not obvious that the representative at the beginning and end can be chosen to coincide. This problem is addressed in Lemma 3 of this paper contains the complete proof that $X_n$ is simply connected, incorporating all of the necessary changes.

There were also small errors in Figure 7 and Figure 8 and the correct versions of these are included in Section 3.

2. The complex $X_n$

An embedded disc $D \subseteq H^3$ is said to cut out $a_i$ if the interior of $D$ is disjoint from $a_i$, the arc $a_i$ is contained in the boundary of $D$ and the boundary of $D$ lies in the union of the arc $a_i$ and the boundary of half-space, i.e. $a_i \subset \partial D$ and $\partial D \subset a_i \cup \partial H^3$. A cut system for $a_*$ is the isotopy class of $n$ pairwise disjoint discs $\langle D_1, D_2, \ldots D_n \rangle$ where each $D_i$ cuts out the arc $a_i$. Say that two cut systems differ by a simple $i$-move if there exist representatives $\langle D_1, D_2, \ldots D_n \rangle$ and $\langle E_1, E_2, \ldots E_n \rangle$ such that $D_i \cap E_i = a_i$ and $D_j = E_j$ for all $j \neq i$. If this is the case we will suppress the non-changing discs and write $\langle D_i \rangle \rightarrow \langle E_i \rangle$.

Definition 1. Define the cut system complex $X_n$ as follows. The set of all cut systems for $a_*$ forms the vertex set $X_n$. Two vertices are connected by a single edge iff they differ by a simple move. Finally, glue faces into every loop of the following form, giving triangular and rectangular faces.

Received by the editors September 7, 2009. Revision received July 27, 2010.
Define the basepoint to be \( v_0 = \langle d_1, d_2, \ldots, d_n \rangle \) where the \( d_i \) are vertical discs below the \( a_i \), see Figure 1. Sometimes it is convenient to think of the \( a_i \) and \( d_i \) rotated by a quarter turn.

![Figure 1. The arcs \( a_i \) and the discs \( d_i \)](image)

**Definition 2.** Given an edge path \( P = (v_1, v_2, \ldots, v_N) \) in \( \overline{X}_n \) we will say that a collection of discs \( \{ D_{i,k} \} \) form a realisation of \( P \) if for each vertex \( v_k \) of \( P \) we have
\[
v_k = \langle D_{1,k}, D_{2,k}, \ldots, D_{n,k} \rangle
\]
and if for each \( k \) there exists an \( i \) such that \( D_{j,k} = D_{j,k+1} \) for all \( j \neq i \) and either \( D_{i,k} = D_{i,k+1} \) or \( D_{i,k} \cap D_{i,k+1} = a_i \). In other words at most one disc changes as we go from \( k \) to \( k + 1 \).

**Lemma 3.** Every edge path \( P = (v_1, \ldots, v_N) \) admits a realisation and any loop (i.e. whenever \( v_1 = v_N \)) can be extended with stationary edges to
\[
P' = (v_1, v_2, \ldots, v_N, v_N, \ldots, v_1)
\]
so that \( P' \) admits a realisation for which the representatives of the first and last vertex are the same.

**Proof.** Suppose that we have the required discs for the first \( k \) vertices. If \( (v_k, v_{k+1}) \) is a stationary edge, i.e. \( v_k = v_{k+1} \), then let \( D_{i,k+1} = D_{i,k} \) for each \( i \). Otherwise, if \( (v_k, v_{k+1}) \) is an \( i \)-move then there exist discs \( D_1, \ldots, D_n, E \) such that \( v_k = \langle D_1, D_2, \ldots, D_n \rangle \) and \( v_{k+1} = \langle D_1, \ldots, E, \ldots, D_n \rangle \). Now as \( \langle D_1, D_2, \ldots, D_n \rangle = \langle D_{1,k}, D_{2,k}, \ldots, D_{n,k} \rangle \) there exists an isotopy \( F_t \) such that \( F_0 \) is the identity and \( F_1(D_j) = D_{j,k} \) for each \( j \). So let \( D_{j,k+1} = D_{j,k} \) for \( j \neq i \) and \( D_{i,k+1} = F_1(E) \).

Continuing in this way we can pick the required representative of each vertex, but if \( P \) is a loop the representatives of \( v_1 \) and \( v_N \) may differ.

Suppose that the representatives of the first and last vertex are as follows.
\[
v_1 = \langle D_1, D_2, \ldots, D_n \rangle \quad v_N = \langle E_1, E_2, \ldots, E_n \rangle
\]
As \( v_1 = v_n \) there exists an isotopy \( F : H^3 \times I \to H^3 \) with \( F_0 = \text{Id}_{H^3} \), \( F_1 \) the identity on \( a_i \) for all \( t \in I \) and for each \( i \), and \( F_1(E_i) = D_i \) for each \( i \).

Now, for each \( t \in I \) there exists an open neighbourhood \( U \) of \( t \) such that there exist closed neighbourhoods \( \phi_i \) of each disc \( F_t(D_i) \) which satisfy the following. Each \( \phi_i \) is homeomorphic to a ball and intersects the boundary of half-space in a disc. These neighbourhoods are pairwise disjoint, i.e. \( \phi_i \cap \phi_j = \emptyset \) for \( i \neq j \). And as we move \( F_s(E_i) \), for \( s \in U \), the discs remains in \( \phi_i \), i.e. \( F(E_i \times U) \subset \phi_i \).
Therefore, there exists a partition $0 = t_0 < t_1 < \cdots < t_K = 1$ such that for each $k$ we have the following. Each disc $F_i(t_i)$ has a closed neighbourhood $\phi_i$ homeomorphic to a ball. Each of these balls intersect the boundary of half-space in a disc. These neighbourhoods are pairwise disjoint. And each disc remains in the same ball as we move from $t_k$ to $t_{k+1}$, i.e. $F(E_i \times [t_k, t_{k+1}]) \subset \phi_i$ for each $i$.

Within each $\phi_i$ there is only one isotopy class of discs cutting out $a_i$.

We may assume that $F_i(t_i)$ and $F_{i+1}(t_i)$ intersect transversely and so intersect in a collection of arcs and circles. If this intersection is not just $a_i$ then we can carry out the following.

Each arc or circle of this intersection separates $F_i(t_d)$ into two pieces. Say that one of these pieces is minimal if it contains no arcs or circles of this intersection. Now pick a minimal piece $A \subset F_i(t_d)$ which comes from an arc or circle $\alpha$ of the intersection. Similarly each arc or circle cuts $F_i(t_d)$ into two pieces. Hence we can cut $F_i(t_d)$ along $\alpha$ giving two pieces $B_1$ and $B_2$. So we have that $F_i(t_d) = B_1 \cup B_2$ and $\alpha = B_1 \cap B_2$.

For one and only one of the $B_p$ we have $B_p \cap a_i = A \cap a_i$. So we can perform the following surgery on $F_i(t_d)$: discard $B_p$, glue in $A$ and push off slightly. This gives a new disc that will have at least one fewer arc or circle of intersection with $F_i(t_d)$ and which only intersects $F_i(t_d)$ along $a_i$.

Repeating in this way gives a sequence of discs from $F_i(t_d)$ to $F_{i+1}(t_d)$ each intersecting the next only along the arc $a_i$ and, as each one is contained in $\phi_i$, each isotopic to $F_i(t_d)$ rel $H^3 \setminus \phi_i$. Therefore, by changing one disc at a time, we have a realisation of a stationary path from $v_N$ to $v_1$.

From now on we will assume we have already chosen a realisation for each path. Furthermore we will assume that for loops sufficiently many stationary edges have been added at the end and that the representatives of the first and last vertex are the same.

**Definition 4.** A triple $(v, D, D^*)$, where $v = (D_1, D_2, \ldots D_n)$ is a vertex of $X_n$ with a choice of representative discs, $D$ and $D^*$ are two discs cutting out the arc $a_i$ with $D \cap D^* = a_i$, forms a substitution if we can replace any occurrence of $D$ with $D^*$. Note that we include the possibility that $D \neq D_i$. In other words, either $D = D_i$ and for all $j \neq i$ we have that $D_j \cap D^* = \emptyset$ or $D \neq D_i$.

So if $(v, D, D^*)$ is a substitution then there is a (possibly stationary) edge $(v, v^*)$ where

$$v^* = \begin{cases} v & \text{if } D_i \neq D, \\ \langle D^* \rangle & \text{if } D_i = D. \end{cases}$$

Similarly, for any edge path $P = (v_1, v_2, \ldots, v_N)$ with a choice of realisation by discs, we say $(P, D, D^*)$ forms a substitution iff for each vertex $v$ of $P$ the tuple $(v, D, D^*)$ forms a substitution and for each edge $(u, v)$ of $P$ there is a (possibly stationary) edge $(u^*, v^*)$.

If $(P, D, D^*)$ forms a substitution then we can replace each vertex $v$ with $v^*$, giving a new path $P^* = (v_1^*, v_2^*, \ldots, v_N^*)$ whose realisation is given by replacing each occurrence of the disc $D$ with the disc $D^*$.
Lemma 5. Suppose $P$ is a path with a given realisation. If $(P, D, D^*)$ forms a substitution, where $P = (v_1, \ldots, v_N)$, then the loop

$$
\begin{array}{c}
 v_1 \\
\uparrow \\
 P \\
\downarrow \\
v_N \\
\end{array}
\quad
\begin{array}{c}
 v_1^* \\
\uparrow \\
 P^* \\
\downarrow \\
v_N^* \\
\end{array}
$$

is null homotopic. Moreover, if $P$ is a loop then so is $P^*$ and they are homotopic as loops.

Proof. For each edge $(u, v)$ in $P$ we have the following rectangle where some edges could be degenerate stationary edges.

$$
\begin{array}{c}
u \\
\downarrow \\
u^* \\
\end{array}
\quad
\begin{array}{c}
v \\
\downarrow \\
v^* \\
\end{array}
$$

If one edge is degenerate then this loop is contained in the boundary of a triangular face. If more than one edge is degenerate then this loop is contained within an edge.

So suppose that none of the edges are degenerate and that $u$ and $v$ have representatives as follows.

$$
u = (D_1, D_2, \ldots, D_n) \quad v = (D'_1, D'_2, \ldots, D'_n)
$$

With $D_i \neq D'_i$ and $D_j = D'_j$ for all $j \neq i$. Suppose that $D \cap D^* = a_k$. In other words that $u$ and $v$ differ by a simple $i$–move and $u$ and $v^*$ differ by a simple $k$–move.

As $v \neq v^*$ we must have that $D$ is one of the discs representing $v$. Similarly, as $u \neq u^*$ we must have that $D$ is one of the discs representing $u$. Hence $k \neq i$ and $D_k = D'_k = D$. Therefore this loop is the boundary of the following rectangular face.

$$
\begin{array}{c}
\langle D, D_i \rangle \\
\downarrow \\
\langle D^*, D_i \rangle \\
\end{array}
\quad
\begin{array}{c}
\langle D, D'_i \rangle \\
\downarrow \\
\langle D^*, D'_i \rangle \\
\end{array}
$$

□

Theorem 6. The complex $X_n$ is connected and simply connected.

Proof. It suffices to show that any loop is homotopic to the constant loop at $v_0$. Given a loop in $X_n$, it is homotopic to an edge path $P$. Now by Lemma 3 after adding sufficiently many stationary points, we can pick a collection of discs realising $P$. We can make this choice such that the intersection of any two representative discs and between any representative disc and any of the $d_i$ are transverse. We shall write $D \in P$ if $D$ is one of the discs chosen as a representative of some vertex of $P$.

Claim. The path $P$ is homotopic to a path with a realisation whose discs intersect the discs $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$. 

Assuming that the intersection of the discs $D \in P$ with $d_1 \cup d_2 \cup \ldots \cup d_n$ isn’t only $a_1, a_2, \ldots, a_n$ we can carry out the following procedure.

For some $i$ the union of the discs in $P$ intersects $d_i$ in a non-trivial collection of arcs and circles. Each arc or circle of this intersection separates $d_i$ into two pieces. Say that one of these pieces is minimal if it contains no other complete arc or circle of this intersection.

Now pick a minimal piece $A \subset d_i$ which comes from an arc or circle $\alpha$ on its boundary. The arc $\alpha$ is a subset of some $D \in P$, i.e. $\alpha \subset D \cap d_i$. We can cut $D$ along $\alpha$ giving two pieces $B_1$ and $B_2$. So we have that $D = B_1 \cup B_2$ and $\alpha = B_1 \cap B_2$.

For one and only one of the $B_k$ we have $B_k \cap a_i = A \cap a_i$. So we can perform the following surgery on $D$: discard $B_k$, glue in $A$ and push off slightly. This gives a new disc $D^*$. The new disc will intersect $d_i$ in at least one less arc or circle.

Any disc $E \in P$ for which $E \cap D = a_j$ or $\emptyset$ also has $E \cap D^* = a_j$ or $\emptyset$ respectively; if not $E$ must intersect $D^*$ in the section parallel to $d_i$ and this contradicts the condition that $A$ contains no complete arc or circle of $E \cap d_i$. Therefore the triple $(P, D, D^*)$ forms a substitution and, by Lemma 5 we can replace $D$ with $D^*$ to get a new homotopic loop $P^*$.

We now have a homotopic loop $P^*$ that has fewer intersections with $d_1 \cup d_2 \cup \ldots \cup d_n$. So by induction on the number of intersections we have proved the claim.

So we may assume that the representatives of $P$ meets $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$. Therefore, for each $D \in P$ cutting out the arc $a_i$, the triple $(P, D, d_i)$ forms a substitution and so by in turn replacing each $D \in P$ with the corresponding $d_i$ we see that $P$ is homotopic to the constant path $(v_0)$. The connectedness of $X_n$ follows by taking $P$ to be a constant loop.

3. Other errors

Unfortunately Figure 7 and Figure 8 were incorrect in [1]. In Figure 7 the diagram for $s_i$ is the same as the one for $p_i$. This is clearly incorrect; two of the crossings need to be changed. In Figure 8 the diagrams for the generators are the same as for their inverses, and this is also incorrect. The correct versions of these figures are as follows.

\begin{figure}[h]
\centering
\begin{align*}
  s_i &= \quad p_i &= \\
  t_i &=
\end{align*}
\caption{Generators of $H_{2n}$}
\end{figure}
Figure 8. Pictorial representation of the $p$, $s$, $t$ and their inverses

References


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